Seminar on Set Theory

Hand-in exercise 8

November 26, 2015

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1 Cardinality of Topological Basis

Let I be a set with cardinality $\aleph_\alpha$. Show that the family of sets of the form

$$\{f \in 2^I \mid f(i_1) = a_1, \ldots, f(i_n) = a_n\},$$

where $i_1, \ldots, i_n \in I$ and $a_1, \ldots, a_n \in 2$ also has cardinality $\aleph_\alpha$. *(4 pt.)*
Hint: You may use that if κ and λ are cardinals such that either κ or λ is infinite and both are non-zero, then $\kappa \times \lambda = \max\{\kappa, \lambda\}$.

2 Another Relative Consistency Result

Let κ be an infinite cardinal and $B = \mathrm{RO}(2^{\omega \times \omega_2})$. Denote the cardinality of B by λ.

(a) Use 1.48 and Lemma 1.52 to show that $V^{(B)} \models |\mathcal{P}\hat{\kappa}| \leq |\widehat{\lambda^\kappa}|$. *(2 pt.)*

(b) Assume the GCH. Use Corollary 2.11 and (a) to show that

$$V^{(B)} \models 2^{\aleph_1} = \aleph_2.$$

(4 pt.)

(c) Assume the GCH. In this exercise, you may use (without proof) that

$$V^{(B)} \models \forall \alpha \geq \hat{\lambda}\, (\mathrm{Card}(\alpha) \to 2^\alpha = \alpha^+).$$

Use this and Theorem 2.12 to show that

$$V^{(B)} \models 2^{\aleph_0} = \aleph_2 + \forall \kappa \geq \aleph_1\, (2^\kappa = \kappa^+).$$

(3 pt.)

(d) Prove that if ZF is consistent, then so is

$$\mathrm{ZFC} + 2^{\aleph_0} = \aleph_2 + \forall \kappa \geq \aleph_1\, (2^\kappa = \kappa^+).$$

Be explicit in your use of Theorem 1.19. *(2 pt.)*

3 Some details about an action on $V^{(B)}$

In this exercise you will prove some omitted details from the proof of Theorem 3.3. Define the map $\langle g, u\rangle \mapsto gu : G \times V^{(B)} \to V^{(B)}$ by recursion on the well-founded relation $y \in \mathrm{dom}(x)$ via

$$gu = \{\langle gx, g \cdot u(x)\rangle : x \in \mathrm{dom}(u)\}.$$

(a) Prove that

$$g \cdot [\![u \in v]\!] = [\![gu \in gv]\!]$$

and

$$g \cdot [\![u = v]\!] = [\![gu = gv]\!].$$

(2 pt.)

(b) Prove Theorem 3.3 (ii): $g\hat{v} = \hat{v}$ for any $v \in V$. *(3 pt.)*