exercise 1. Let $(R, <, \mathcal{S})$ be an o-minimal structure. Prove:

1. For every definable subset $X \subseteq R$, if $X \neq \emptyset$ and $X \neq R$ then either X has an upper (resp. lower) bound in R, or $R \setminus X$ has an upper (resp. lower) bound in R.

2. For every $X \subseteq Y \subseteq R$ with X and Y definable, if X is dense in Y then X is open in Y. (A set X is dense in Y iff for every nonempty set U open in Y, $U \cap X \neq \emptyset$.)

exercise 2. Let $(R, <)$ be a dense linear order without end points, and let $\mathcal{S}$ be a structure on R containing $<$ as well as all intervals and singletons. Assume:

1. For every non-empty definable subset $A \subseteq R$, $\inf(A)$ and $\sup(A)$ exist in $R \cup \{-\infty, +\infty\}$.

2. For every definable subset $A \subseteq R$, if $A \neq \emptyset$ and $A \neq R$ then either A has an upper (resp. lower) bound in R or $R \setminus A$ has an upper (resp. lower) bound in R.

3. For every infinite definable subset $A \subseteq R$, the interior $\text{int}(A)$ is non-empty.

4. For every two definable subsets $A, B \subseteq R$, if A is dense in B, then A is open in B.

Prove that $(R, <, \mathcal{S})$ is o-minimal.
(Hint: prove for every definable set $A \subseteq R$ that the boundary $\text{bd}(A) = \text{cl}(A) \setminus \text{int}(A)$ is finite. Find $-\infty = b_0 < b_1 < \ldots < b_k < b_{k+1} = +\infty$ such that for every i, either $(b_i, b_{i+1}) \subseteq A$ or $(b_i, b_{i+1}) \subseteq R \setminus A$.)