O-minimal Structures - Assignment 3

Exercise 1 *(4 points)*

In this exercise, let X and Y be nonempty topological spaces and assume that Y is compact. We equip $X \times Y$ with the product topology.

(a) *(2 points)* Let $\pi : X \times Y \to X$ denote the projection map from $X \times Y$ onto X. Prove that the image of a closed set under π is again closed.

(b) *(2 points)* Suppose that the graph $\Gamma(f)$ of the function $f : X \to Y$ is closed. Prove that f is continuous.

Exercise 2 *(5 points)*

In this exercise, let F be an ordered field.

(a) *(3 points)* Let $f = f(X_1, \dots, X_m) \in F[X_1, \dots, X_m]$, and let $d_1, \dots, d_m \in \mathbb{N}$ be such that $\deg_{X_i}(f) \leq d_i$ for $1 \leq i \leq m$. We put $\deg_{X_i}(0) = -\infty$ by convention. Furthermore, let $A_1 \times \cdots \times A_m \subset F^m$, with $|A_1| > d_1, \dots, |A_m| > d_m$. Prove that if the restriction of f to $A_1 \times \cdots \times A_m$ is identically zero, then $f = 0$.

(b) *(2 points)* Let $f = f(X_1, \dots, X_m) \in F[X_1, \dots, X_m]$, with $f \neq 0$. Prove that the zero set $\mathrm{Z}(f) = \{a \in F^m : f(a) = 0\}$ is a closed subset of F^m with empty interior.

Grade = total points + 1