# Seminar O-minimal Structures

## hand-in exercise 8, due: 2014/12/19

We fix an o-minimal structure $(R, <, \mathcal{S})$.

**exercise 1.** In this exercise you must try to find definable sets for which $f_{\mathcal{C}}$ grows "at least as fast" as a given function.
**a.** In this exercise you show that the polynomial bounds $p_d(n)$ which we will obtain from the main theorem are "tight". Show that for every $d > 0$ there exists a definable relation $S \subseteq R^{d-1} \times R$ such that, if we put $\mathcal{C} = \{ S_x \mid x \in R^{d-1} \}$, then $f_{\mathcal{C}}(n) = p_d(n)$ for every $n$.
**b.** Assume for this part that $(R, <, \mathcal{S})$ expands an ordered abelian group $(R, <, +, 0)$. Show that for every natural number $c$ there exists a definable set $S \subseteq R \times R$ and a natural number $N$ such that, putting $\mathcal{C} = \{ S_x \mid x \in R \}$, we have $f_{\mathcal{C}}(n) > cn$ for every $n \geq N$.

**exercise 2.** Let $S \subseteq R \times R^q$. Again, put $\mathcal{C} = \{ S_x \mid x \in R \}$, and put $\mathcal{G} = \{ S^y \mid y \in R^q \} \subseteq \mathcal{P}(R)$. We have seen that there exists an $e \in \mathbb{N}$ such that $f^{\mathcal{G}}(n) \leq p_3(en)$. Therefore, $f_{\mathcal{C}}$ is of at most quadratic growth. This exercise asks you to improve on this.
**a.** Show that for any decomposition $\mathcal{D} = \{ E_1, \ldots, E_k \}$ of $R$, the atoms of the boolean algebra $B(E_1, \ldots, E_k)$ are precisely $E_1, \ldots, E_k$.
**b.** Assume $S_1, \ldots, S_k \subseteq R$ are definably connected. Show that there is a decomposition of $R$ into at most $4k + 1$ cells, partitioning each of the $S_i$.
**c.** Show that there exist natural numbers $c$ and $N$ such that if $F \subseteq R^q$ is any finite set with at least $n \geq N$ elements, then $|\mathcal{C} \cap F| \leq cn$.