solution to exercise 1

For part (1), let $A \subseteq R$ definable. Then A is a finite union of intervals and points: $A = \bigcup_{i=1}^{k}(p_i, q_i) \cup \{r_1, \ldots, r_l\}$, with $-\infty \leq p_i < q_i \leq +\infty$ for every i, and all $r_j \in R$. If for some i, $q_i = +\infty$, then for every $x > p_i$, $x \notin R \setminus A$. So p_i is an upper bound for $R \setminus A$. Otherwise, let $s = \max\{q_1, \ldots, q_k, r_1, \ldots, r_l\}$. Then clearly s is an upper bound for A. For the statement about lower bounds the argument is entirely similar.

For part (2), first observe that every definable infinite subset $A \subseteq R$ must contain an interval, for otherwise it would be a finite union of just points. Second, if X is dense in R, that is: for every p, q, if $p < q$ then there exists x with $p < x < q$, then X is cofinite. For, if $R \setminus X$ were infinite, then we would have an interval $(p, q) \subseteq R \setminus X$, contradicting X being dense in R. Now, if X is dense in Y, then $X \cup (R \setminus Y)$ is dense in R. For suppose $p < q$ are such that $(p, q) \cap (X \cup (R \setminus Y)) = \emptyset$, then $(p, q) \subseteq Y \setminus X$. But then (p, q) is open in Y and does not intersect X, contradicting X being dense in Y. So $X \cup (R \setminus Y)$ is cofinite. Its complement, $Y \setminus X$, is therefore finite. Since finite sets are closed in Y, we see that X is open in Y.

solution to exercise 2

First we prove: for every definable $A \subseteq R$, $\mathrm{bd}(A)$ is finite. Suppose not, then by assumption (3) there exist p, q such that $p < q$ and $(p, q) \subseteq \mathrm{bd}(A)$. In particular, $(p, q) \subseteq \mathrm{cl}(A)$, so $A \cap (p, q)$ is dense in (p, q) (since every non-empty interval $(p', q') \subseteq (p, q)$ is the neighborhood of a point in $\mathrm{cl}(A)$ and therefore intersects A). Then by assumption (4), $A \cap (p, q)$ is open in (p, q). Since $A \cap (p, q)$ is non-empty, we can find a non-empty interval $(p', q') \subseteq A \cap (p, q)$. But then $(p, q) \cap \mathrm{int}(A) \neq \emptyset$, contradicting the assumption that $(p, q) \subseteq \mathrm{bd}(A)$.

Now let A be an arbitrary definable subset of R. We claim that for every interval (a, b) such that $(a, b) \cap \mathrm{bd}(A) = \emptyset$, either $(a, b) \subseteq A$ or $(a, b) \subseteq R \setminus A$. We distinguish the case where one or both of the endpoints is $\pm\infty$ from the case where the endpoints are both in R.

Start with the case where at least one of the endpoints is $\pm\infty$. We can assume without loss of generality that $b = +\infty$. Note that the special case where $(a, b) = (-\infty, +\infty)$, that is: $\mathrm{bd}(A) = \emptyset$, also falls under this assumption. If $A = \emptyset$ or $A = R$ we are done. Otherwise, either A or $R \setminus A$ has an upper bound in R. Without loss of generality assume $R \setminus A$ has an upper bound and put $c = \sup(R \setminus A)$ by assumption (2). Observe that for every $p < c$ there exists $x \in R \setminus A$ such that $p < x < c$, since c is the *least* upper bound for $R \setminus A$. And for every x such that $c < x$, $x \in A$. So for every p, q such that $p < c < q$, both $(p, q) \cap (R \setminus A) \neq \emptyset$ and $(p, q) \cap A \neq \emptyset$. So $c \in \mathrm{bd}(A)$. Hence by assumption, $c \notin (a, +\infty)$. If $a = -\infty$, this constitutes a contradiction and we can conclude that $A = \emptyset$ or $A = R$ and we are done. Otherwise we must have $c \leq a$, hence for every $x > a$, $x \in A$. That is, $(a, +\infty) \subseteq A$, as required.

Now consider (a, b) where $a, b \in R$. If $\mathrm{int}(A) \cap (a, b) = \emptyset$, then $(R \setminus A) \cap (a, b)$ is dense in (a, b), hence open in (a, b). In that case, if $a < x < b$ and $x \in A$, then certainly $x \in \mathrm{cl}(A)$ and by assumption $x \notin \mathrm{int}(A)$. But then $x \in \mathrm{bd}(A)$ contradicting our assumption that $(a, b) \cap \mathrm{bd}(A) = \emptyset$. So such x cannot exist, and therefore $(a, b) \subseteq R \setminus A$. Similarly, if $\mathrm{int}(R \setminus A) \cap (a, b) = \emptyset$, then $(a, b) \subseteq A$. We are left with the case that there exist p, q such that $a < p < q < b$ and

$(p, q) \subseteq A$, and r, s such that $(r, s) \subseteq R \setminus A$. We will derive a contradiction. Clearly, either $q < r$ or $s < p$. Without loss of generality, assume $q < r$. Define $D = \{ x \in R \mid \forall y \in R \,.\, p < y < x \to y \in A \}$. Note that D is definable and $q \in D$, so we may define $c = \sup(D)$. Note that $q \leq c \leq r$ so $c \in (a, b)$. Now, for every $p' < c$ there exists x such that $\max(p, p') < x < c$ and therefore $x \in A$. So, clearly $c \in \mathrm{cl}(A)$. And $c \notin \mathrm{int}(A)$, for otherwise we could find $x \in D$ with $x > c$. But then $c \in \mathrm{bd}(A)$, and $c \in (a, b)$ contradicting our assumption.
Finally, using that the boundary $\mathrm{bd}(A) = \mathrm{bd}(R \setminus A)$ is finite, let it be enumerated in order by $b_1 < \ldots < b_k$, and in addition put $b_0 = -\infty$ and $b_{k+1} = +\infty$. Since for every $i \leq k$, either $(b_i, b_{i+1}) \subseteq A$ or $(b_i, b_{i+1}) \subseteq R \setminus A$, we can indeed write A as a finite union of intervals and points.