O-minimal Structures - Solution to assignment 3

Solution to exercise 1

(a) Let $A \subset X \times Y$ be closed. We show that $X \setminus \pi(A)$ is open. Let $x \in X \setminus \pi(A)$ and note that $\{x\} \times Y \subset (X \times Y) \setminus A$. Since Y is compact and $(X \times Y) \setminus A$ is open, by the Tube Lemma there exists $U \subset X$ open and containing x, such that $U \times Y \subset (X \times Y) \setminus A$. This implies that $U \subset X \setminus \pi(A)$. We conclude that $X \setminus \pi(A)$ is open. Hence $\pi(A)$ is closed.

(b) We show that the preimage of a closed set under f is closed. Let $A \subset X \times Y$ be closed. Verify that $f^{-1}(A) = \pi(\Gamma(f) \cap (X \times A))$, with π as in part **(a)**. Now $\Gamma(f) \cap (X \times A)$ is closed, as $\Gamma(f)$ and $(X \times A)$ are both closed. The result of part **(a)** then tells us that $\pi(\Gamma(f) \cap (X \times A))$ is closed, as desired. Hence f is continuous.

Solution to exercise 2

(a) We use induction over m to prove the statement in the exercise.

First set $m = 1$, so that $f = f(X_1) \in F[X_1]$. Suppose that $f \neq 0$. Then, because F is a field, the number of distinct zeros of $f(X_1)$ is at most $\deg_{X_1}(f)$. But then $\deg_{X_1}(f) \leq d_1 < |A_1| \leq \deg_{X_1}(f)$, which is a contradiction. We conclude that $f = 0$.

Now suppose that the statement in the exercise is true for $m = n$. Suppose also that $f = f(X_1, \dots, X_{n+1}) \in F[X_1, \dots, X_{n+1}]$; $d_1, \dots, d_{n+1} \in \mathbb{N}$ and $A_1, \dots, A_{n+1} \subset F$ satisfy the assumptions of the statement for $m = n+1$; that is, $\deg_{X_i}(f) \leq d_i$ for $1 \leq i \leq n+1$; $|A_1| > d_1, \dots, |A_{n+1}| > d_{n+1}$ and the restriction of f to $A_1 \times \cdots \times A_{n+1}$ is identically zero. Let $a \in A_{n+1}$. Then $f(X_1, \dots, X_n, a) \in F[X_1, \dots, X_n]$, $d_1, \dots, d_n \in \mathbb{N}$ and $A_1, \dots, A_n \subset F$ satisfy the assumptions of the statement for $m = n$. By our hypothesis, we must have that $f(X_1, \dots, X_n, a) \equiv 0$. We view $f(X_1, \dots, X_{n+1}) \in F[X_1, \dots, X_{n+1}]$ as $f(X_{n+1}) \in F[X_1, \dots, X_n][X_{n+1}]$. By the above, every $a \in A_{n+1}$ is a zero of $f(X_{n+1})$. Note that $F[X_1, \dots, X_n]$ is a domain, as F is a field. Suppose that $f \neq 0$. Then, because $F[X_1, \dots, X_n]$ is a domain, the number of distinct zeros of $f(X_{n+1})$ is at most $\deg_{X_{n+1}}(f)$. But then $\deg_{X_{n+1}}(f) \leq d_{n+1} < |A_{n+1}| \leq \deg_{X_{n+1}}(f)$, which is a contradiction. We conclude that $f = 0$. So by induction, the statement is true for every $m \geq 1$.

(b) Note that F is dense and without endpoints. Indeed for $a, b \in F$ with $a < b$, we have that $a - 1 < a < \frac{a+b}{2} < b < b+1$. This implies that every interval in F contains an infinite amount of elements. Suppose that $\mathrm{int}(\mathrm{Z}(f)) \neq \emptyset$ and let $x \in \mathrm{int}(\mathrm{Z}(f))$. Then there exist intervals $A_1, \dots, A_m \subset F$ such that $x \in A_1 \times \cdots \times A_m \subset \mathrm{int}(\mathrm{Z}(f))$. Note that the function f and the sets

$A_1, \ldots, A_m$ satisfy the conditions of part **(a)**. Hence $f = 0$. But this contradicts the assumption that $f \neq 0$. We must conclude that $\mathrm{int}(\mathrm{Z}(f)) = \emptyset$.

Lastly, f is continuous, as it is a polynomial, so $\mathrm{Z}(f) = f^{-1}(\{0\})$ is closed, as $\{0\}$ is closed.