O-minimal Structures - Assignment 5
An Answermodel

November 26, 2014

1

Let $(R, <, S)$ be an o-minimal structure. Let A be a cell in R^{m+1}. Note that $\pi(A)$ is a cell in R^m, where $\pi : R^{m+1} \to R^m$ is the projection on the first m coordinates

a) Prove that $\pi^{-1}(\{x\}) \cap A$ is definably connected, for all $x \in \pi(A)$. Hint: use the definition.

   Proof. For the case that A is a $(1, \dots, 1, 0)$-cell, $\pi^{-1}(\{x\}) \cap A$ is a point. For the case that A is a $(1, \dots, 1, 1)$-cell, $\pi^{-1}(\{x\}) \cap A = \pi^{-1}(\{x\}) \cap \{(y, r) \in \pi(A) \times R : f(y) < r < f(y)\} = \{x\} \times (f(x), g(x))$ for some $f, g \in C(\pi(A))$. Since $\{x\} \times (f(x), g(x)) \cong (f(x), g(x))$, $\pi^{-1}(\{x\}) \cap A$ is definably connected. $\square$

b) Assume that A is definably disconnected, i.e. there exists nonempty definably open subsets $U, V \subset X$, such that $A = U \cup V$ and $U \cap V = \emptyset$. Show that $\pi^{-1}(\{x\}) \cap A \subset U$ or $\pi^{-1}(\{x\}) \cap A \subset V$, for all $x \in \pi(A)$.

   Proof. We prove by contradiction. Assume that it is not the case that $\pi^{-1}(\{x\}) \cap A \subset U$ or $\pi^{-1}(\{x\}) \cap A \subset V$, for all $x \in \pi(A)$. Then $\pi^{-1}(\{x\}) \cap A$ is definably disconnected for some $x \in \pi(A)$, namely by the definably open sets $\pi^{-1}(\{x\}) \cap V$ and $\pi^{-1}(\{x\}) \cap U$ (open in the subspace topology of $\pi^{-1}(\{x\}) \cap A$). $\square$

c) Let $C = \{x \in \pi(A) : \pi^{-1}(\{x\}) \cap A \subset U\}$ and $D = \{x \in \pi(A) : \pi^{-1}(\{x\}) \cap A \subset V\}$. Prove that C and D are open in $\pi(A)$. Hint: there is an obvious open mapping.

   Proof. Let $x \in C$. Then there is $r \in R$, such that $(x, r) \in U$, therefore in the view of (b) $\pi(U) \subset C$. Note that $C \subset \pi(U)$ and that $\pi(U)$ is open, because π is an open mapping. $\square$

d) Prove that if $\pi(A)$ is definably connected, then A is definably connected. And prove by using induction on m, that each cell is definably connected.

Proof. Assume that A is definably disconnected, then $\pi(A)$ is definably disconnected, because C and D are open (in view of (c)), definably, $\pi(A) = C \cup D$ and $C \cap D = \emptyset$. This is a contradiction. Thus, A is definably connected, if $\pi(A)$ is definably connected. We finish the proof by induction on $m \geq 0$. □

2

Let $(R, <, S)$ be an o-minimal structure. Let A be a cell in R^m and Y be a definable set in R^{m+1} and let $p : R^m \to R^k$ be defined as on page 51. Then $p(A)$ is an open cell in R^k. Let $id : R \to R$ be the identity map. Assume that there exists $n \in \mathbb{N}$, such that $|((p, id)Y)_{p(x)}| < n$ for all $x \in p(A)$. Prove that there exists $m \in \mathbb{N}$, such that $|Y_x| < m$ for all $x \in A$.

Proof. Let $(x, y) \in Y$. Then $(p(x), y) \in (p, id)Y$. Thus, $Y_x \subset ((p, id)Y)_{p(x)}$, for all $x \in A$. □

Note that you can also show that $((p, id)Y)_{p(x)} \subset Y_x$