Tame Topology and O-minimal Structures,
Euler Characteristic, homework set model solutions

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We take an o-minimal structure $(R, <, S)$.

1 Cell decomposition (5 points)

Take a cell $C \subset R^m$. This exercise tackles the similarity between the definition of a cell decomposition of R^m and the definition of a decomposition of a cell. The definition of a decomposition of a cell is given on page 70.

a. (2 points) Prove that that if $\mathbf{D}$ is a cell decomposition of R^m that partitions C, than $\mathbf{D}|C = \{E : E\epsilon\mathbf{D}, E \subseteq C\}$ is a decomposition of C.

We use induction on m. For $m = 1$ the result follows from the definitions.
Assume that the statement is correct for all dimensions below $m > 1$. Take C a cell in R^m and $\mathbf{D}$ a decomposition of R^m partitioning C. Take π to be the projection of R^m to it's first $m-1$ coordinates. Note that by the definition of decompositions, $\pi(\mathbf{D})$ is a decomposition of R^{m-1}, which by definition means it partitions R^{m-1}. So $\pi(\mathbf{D}|C)$ is precisely the elements in the partition $\pi(\mathbf{D})$ that originate form $\mathbf{D}|C$. So it is a partition of $\pi(C)$.
Now $\pi(\mathbf{D}|C) = \pi(\{A\epsilon\mathbf{D} : A \subset C\}) = \{\pi(A) : A\epsilon\mathbf{D}\&A \subset C\} = \{A : \pi(A)\epsilon\pi(\mathbf{D})\&A \subset \pi(C)\} = \pi(\mathbf{D})|\pi(C)$. Hence, we have that $\pi(\mathbf{D})$ is a decomposition of R^{m-1} which partitions $\pi(C)$. So by the induction hypothesis, $\pi(\mathbf{D})|\pi(C)$ is a decomposition of $\pi(C)$. Hence by the inductive definition of decompositions of cells we get that $\mathbf{D}|C$ is a decomposition of C.

b. (3 points) Prove that for any decomposition $\mathbf{D}$ of C, there is a cell decomposition of R^m that restricts to $\mathbf{D}$ on C.

We use induction on m.
For $m = 1$, take a cell C in R with a decomposition $\mathbf{D}$.
If C is a point $C = \{c\}$, then it's decomposition must be $\mathbf{D} = \{\{c\}\}$. So we can use the $\mathbf{E} := \{(-\infty, c), \{c\}, (c, \infty)\}$, which is a decomposition of R that restricts to $\mathbf{D}$ on C.
If C is an interval $C = (\alpha, \beta)$ with $\alpha, \beta\epsilon R_\infty$, we can use the decomposition of R given by $\mathbf{E} := \mathbf{D} \cup \{(-\infty, \alpha), \{\alpha\}\} \cup \{\{\beta\}, (\beta, \infty)\}$. The second part should be empty if $\alpha = -\infty$ and the third part if $\beta = \infty$. Either way, $\mathbf{E}$ is the desired decomposition.

Now assume for $m > 1$ we have that the statement is correct in all dimensions lower than m. Take C a cell in R^m and $\mathbf{D}$ a decomposition of C. Take π the projection of R^m to the first $m-1$ coordinates. $\pi(\mathbf{D})$ is a decomposition of $\pi(C)$. So we can use the induction hypothesis to find $\mathbf{E}$ a decomposition of R^{m-1} which partitions $\pi(C)$.
Case 1: C is a $(i_1, ..., i_{m-1}, 0)$-cell. So there is a continuous definable function $f : \pi(C) \to R$ such that $C = \Gamma(f)$. Now define $\mathbf{F} := \mathbf{D} \cup \{(-\infty, f)|F : F \epsilon \pi(\mathbf{D})\} \cup \{(f, \infty)|F : F \epsilon \pi(\mathbf{D})\} \cup \{F \times R : F \epsilon \mathbf{E} \& F \subset R^m \backslash \pi(C)\}$. A quick check yields $\pi(\mathbf{F}) = \mathbf{E}$, and that it partitions C.
Case 2: C is a $(i_1, ..., i_{m-1}, 0)$-cell. So there is are continuous definable functions $f, g : \pi(C) \to R$ s.t $C = (f, g)_{\pi(C)}$, $f < g$. Now we define $\mathbf{F} := \mathbf{D} \cup \{(-\infty, f)|F, \Gamma(f)|F, \Gamma(g)|F, (g, \infty)|F : F \epsilon \pi(\mathbf{D})\} \cup \{F \times R : F \epsilon \mathbf{E} \& F \subset R^m \backslash \pi(C)\}$. If f is the constant $-\infty$ map, all parts with f must be removed. Same for $g = \infty$. A quick check yields that this partition is a decomposition of R^m which partitions C and restricts to $\mathbf{D}$ on C.

Points:
1 for the induction basis
1 for the induction step in case that C is a $(..., 0)$-cell.
1 for the induction step in case that C is a $(..., 1)$-cell.

2 Closure (5 points)

Prove that the Euler characteristic of the closure of a bounded cell $C \subset R^m$ is always 1. Bounded means there is a box $B = [a_0, b_0] \times [a_1, b_1] \times ... \times [a_m, b_m]$ with $a_i, b_i \epsilon R$ for all i, such that $C \subset B$.

Hint: Use induction and consider the cases $i_m = 0$ and $i_m = 1$ separately. Use proposition 2.4.

We use induction on m. Let $B = [a_0, b_0] \times [a_1, b_1] \times ... \times [a_m, b_m]$ be the bounding box of C.
If $m = 1$, C is either a point $\{c\}$ or an interval (c_1, c_2) where c_1 and c_2 are in R since C is bounded. So its closure is either a point $\{c\}$ or a closed interval $[c_1, c_2] = \{c_1\} \cup (c_1, c_2) \cup \{c_2\}$. In both cases, the Euler Characteristic is 1.

Now take $m > 1$ and assume that for all dimension lower than an m, we have that all bounded cells have a closure with Euler Characteristic 1. Take $\pi : R^m \to R^{m-1}$, the projection to the first $m-1$ coordinates. First we prove that $\pi(cl(C)) \subset cl(\pi(C))$. The converse will be proven in the separate cases.

If $x \epsilon \pi(cl(C))$, then there is a $y \epsilon R$ such that $(x, y) \epsilon cl(C)$, hence for any open box in R^m containing (x, y) we have that it's intersection with C is non-empty. So for any open box U in R^{m-1} containing x, we have that $U \times R$ has a non-empty intersection with C, so $U = \pi(U \times R)$ has a non-empty intersection with $\pi(C)$. So $x \epsilon cl(\pi(C))$.

If C is an $(i_1, ..., i_{m-1}, 0)$-cell, then there is a continuous definable function $f : \pi(C) \to R$ s.t. $C = \Gamma(f)$. So $cl(C) = \Gamma(f)$. Take $x = (x_1, x_2, ..., x_{m-1}) \epsilon cl(\pi(\Gamma(f)))$, we want to define the fiber $cl(\Gamma(f))_x$ by using a limit of f to x using boxes around x. The limit of f to x is the limit of the images of increasingly smaller boxes, both the supremum and the infimum of those increasingly smaller boxes go to that limit. Define $b_x : B_x = (-\infty, x_1) \times ... \times (-\infty, x_{m-1}) \times (x_1, \infty) \times ... \times (x_{m-1}, \infty) \to \mathbf{P}(R^{m-1})$ by $b(d_1, .., d_{m-1}, e_1, ..., e_{m-1}) = (d_1, e_1) \times ... \times (d_{m-1}, e_{m-1})$. So the image of b_x is all open boxes containing x. Now define $s(x) := inf(sup(f(b_x(B_x) \cap \pi(C)))$, so the infimum of the supremum of the images of all open boxes containing x. This element is definable by construction. We also define $i(x) := sup(inf(f(b_x(B_x) \cap \pi(C)))$.
For $y \epsilon R$, if $(x, y) \epsilon cl(\Gamma(f))$, then any open box U around (x, y) has non-empty intersection with C. $\pi(U) \epsilon b_x(B_x)$. If $y > s(x)$, we can find an open box containing (x, y) disjoint from $cl(\Gamma(f))$, same for $y < i(x)$. Hence $y \epsilon [i(x), s(x)]$.
If $y \epsilon [i(x), s(x)]$, then y is in any image of a box containing x, so $(x, y) \epsilon cl(\Gamma(f))$. Hence $cl(\Gamma(f))_x = [i(x), s(x)]$.
So, for all $x \epsilon cl(\pi(\Gamma(f)))$, we have that the fiber $\Gamma(f)_x$ is a closed interval (or a point if $s(x) = i(x)$) and hence has Euler characteristic 1. Also note that we also have $cl(\pi(C)) \subset \pi(cl(C))$. So $cl(\pi(C)) = \pi(cl(C))$. Hence we can use Corollary 2.11 to conclude that with $E(cl(C)_x) = 1$ we get that $E(cl(C)) = E(\pi(cl(C))) * 1 = E(cl(\pi(C))) = 1$, where the last step follows from the induction hypothesis.

A different way to proof this case is to try and extend the map f to a map defined on the entirety of $cl(\pi(C))$, mapping it bijectively into $cl(C)$. This can be done by either using properties of the map π and using an inverse, or by calculating limits the same way as is done in the proof of proposition 2.13: Using monotonicity and coordinate permutation to study the behaviour of f near the edge and then defining the limit on the edge as either the supremum or infimum.
Once the map F has been constructed, one can use Proposition 2.4 to conclude that $1 = E(cl(\pi(C))) = E(\Gamma(F)) = E(cl(C))$

If C is an $(i_1, ..., i_{m-1}, 1)$-cell, then there are continuous definable functios $f, g : \pi(C) \to R$ with $f < g$ s.t. $C = (f, g)_{\pi(C)}$. Note that $[f, g]_{\pi(C)} \subset cl(C)$, hence $\Gamma(f) \subset cl(C)$ and $\Gamma(g) \subset cl(C)$. Also note that taking the closure of the closure does not change anything, so we get that $cl(\Gamma(f)) \subset cl(C)$ and $cl(\Gamma(g)) \subset cl(C)$. Now take $x \epsilon cl(\pi(C))$. The same way

as before, define $s(x) := inf(sup(g(b_x(B_x) \cap \pi(C)))$ and $i(x) := sup(inf(f(b_x(B_x) \cap \pi(C)))$.
These exist, so we at least get $cl(\pi(C)) \subset \pi(cl(C))$ hence $cl(\pi(C)) = \pi(cl(C))$.
More importantly, $cl(C)_x = [i(x), s(x)]$. Proof:
If $y \epsilon cl(C)_x$, then $(x, y) \epsilon cl(C)$. So any open box U containing (x, y) has non-empty intersection with C. So U contains a point (x_1, y_1) which is in C. So $g(x_1) < y_1 < f(x_1)$. So $y \leq sup(U \cap C) <= sup(g(\pi(U) \cap \pi(C))$ and the same way $y \geq if(f(\pi(U) \cap \pi(C))$. This is for all $U \epsilon b_x(B_x)$, so $i(x) \leq y \leq s(x)$.
If $y \epsilon [i(x), s(x)]$, then any box $U \epsilon b_x(B_x)$ has $y \epsilon [f, g]_U$, so $y \epsilon cl(C)_x$.
So $cl(C)_x = [i(x), s(x)]$.
We can conclude again that the fiber $cl(C)_x$ is a closed interval or a point, hence it has Euler characteristic 1. So the same way as before we can conclude that $E(cl(C)) = E(\pi(cl(C))) * 1 = E(cl(\pi(C))) = 1$.

Using the alternative route, we can extend both f and g to maps F and G on $cl(\pi(C))$. One can check that the closure of C is $C \cup \Gamma(F) \cup \Gamma(G) \cup (F, G)_{cl(\pi(C)) \setminus \pi(C)}$. This will depend on the amount of time F and G are equal. Calculating the Euler characteristic will yield a one.

Points:
1 for the induction basis
2 for the induction step in case that C is a $(..., 0)$-cell.
2 for the induction step in case that C is a $(..., 1)$-cell.

The 2 points in the cases are distributed differently for different proofs:
In the model proof it will be: 1 for $\pi(cl(C)) = cl(\pi(C))$, 1 for fiber has characteristic 1.