Topos Theory, Fall 2018
Hand-In Exercises

Jaap van Oosten

September–December 2018

1 Exercises

Exercise 1 (To be handed in October 1) This is Exercise 3 of Chapter 1 of Johnstone's *Topos Theory*.

Let Ω be a subobject classifier in a topos, and $\alpha : \Omega \to \Omega$ a monomorphism. By considering the subobject $m : U \to \Omega$ classified by α, and then the subobject V of U classified by m, prove that the composition $\alpha\alpha : \Omega \to \Omega$ is equal to the identity on Ω.

Exercise 2 (To be handed in October 22. Deadline extended to October 29.) In this exercise we prove Corollary 1.37 of *Topos Theory*: if we have a logical functor between toposes and this functor has a left adjoint, then it also has a right adjoint.

a) Let T be a monad on a category $\mathcal{C}$ and let $\mathcal{C}^T$ the category of algebras for T; let $F^T : \mathcal{C} \to \mathcal{C}^T$ be the free algebra functor (i.e. the left adjoint to the forgetful functor). Show that for every T-algebra $h : TX \to X$ there is a reflexive pair of arrows

$$\begin{tikzcd} F^T(Y) \arrow[r, shift left=1.5] \arrow[r, shift right=1.5] & F^T(Z) \arrow[l] \end{tikzcd}$$

with coequalizer $F^T(Z) \to h$. Formulate in what sense this construction is functorial in h.

b) Let T and S be monads on categories $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose we have a commutative diagram of functors

$$\begin{tikzcd} \mathcal{C}^T \arrow[r, "\bar{F}"] \arrow[d, "U^T"'] & \mathcal{D}^S \arrow[d, "U^S"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

where U^T, U^S are the forgetful functors. Suppose F has a left adjoint L. Moreover, assume that the category $\mathcal{C}^T$ has coequalizers of reflexive pairs.

Show that $\bar{F}$ also has a left adjoint. [Hint: consider that, if $\bar{L} : \mathcal{D}^S \to \mathcal{C}^T$ is to be a left adjoint to $\bar{F}$, then we must have that the functors $F^T \circ L$ and $\bar{L} \circ F^S$ are naturally isomorphic. Moreover, $\bar{L}$ should preserve colimits.]

c) Prove Corollary 1.37, using part b). [Hint: apply the Monadicity Theorem.]

Exercise 3 (To be handed in November 12) Let $\mathcal{G}$ be a group, and $\widehat{\mathcal{G}}$ the topos of right $\mathcal{G}$-sets (presheaves on $\mathcal{G}$). Characterize the points of $\widehat{\mathcal{G}}$, i.e. the geometric morphisms $\mathrm{Set} \to \widehat{\mathcal{G}}$. Give the inverse and direct image functors explicitly.

Exercise 4 (To be handed in November 26) Let j be a Lawvere-Tierney topology in a topos $\mathcal{E}$, and let X be an object of $\mathcal{E}$. By $c_{X\times X}(\delta)$ we denote the j-closure of the diagonal $\delta : X \times X \to X$ as subobject of $X \times X$.

a) Prove that, for a pair of maps $f, g : Z \to X$, the morphism $\langle f, g\rangle : Z \to X \times X$ factors through $c_{X\times X}(\delta)$ if and only if the equalizer of f, g is a j-dense subobject of Z.

b) Deduce that $c_{X\times X}(\delta)$ is an equivalence relation on X.

c) Let MX be the coequalizer of $c_{X\times X}(\delta) \rightrightarrows X$. Show that every map $X \to L$, where L is j-separated, factors uniquely through MX; and hence, that $M(-)$ is the object part of a functor which is left adjoint to the inclusion $\mathrm{Sep}_j(\mathcal{E}) \to \mathcal{E}$, where $\mathrm{Sep}_j(\mathcal{E})$ is the full subcategory of $\mathcal{E}$ on the j-separated objects.

Exercise 5 (To be handed in December 10) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories and denote the induced geometric morphism $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ also by F. Find (and prove) the factorization of F as a surjection followed by an embedding.

Exercise 6 (To be handed in (digitally) January 14, 2019) This is an open-ended exercise. That is: it is not altogether clear beforehand what the best answer is, and indeed I don't know the answer yet:

Give a presentation of a classifying topos for posets, as simple and concrete as possible.

2 Solutions

Exercise 1 We follow the hint. Let $m : U \to \Omega$ be the subobject classified by α, and let $n : V \to U$ be the subobject classified by m. We have that $U \to 1$ is mono, $V \to U$ is mono and the arrow $V \to 1$ factors through $U \to 1$. Consider the diagram

$$\begin{tikzcd}
V \arrow[r, "\mathrm{id}"] \arrow[d, "n"'] & V \arrow[r, "n"] \arrow[d] & U \arrow[r] \arrow[d, "m"'] & 1 \arrow[d, "t"] \\
U \arrow[r, "!_U"'] & 1 \arrow[r, "t"'] & \Omega \arrow[r, "\alpha"'] & \Omega
\end{tikzcd}$$

The left hand square is a pullback because $V \to 1$ factors through $U \to 1$; the other two are pullbacks by definition. We conclude that the composite $\alpha t!_U$ classifies the mono $V \to U$, and hence that $\alpha t!_U = m$.

Next, consider the grid of pullbacks:

$$\begin{tikzcd}
U \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & U \arrow[r, "\mathrm{id}"] \arrow[d] & U \arrow[d, "!_U"] \\
U \arrow[r] \arrow[d, "m"'] & 1 \arrow[r, "\mathrm{id}"] \arrow[d, "t"'] & 1 \arrow[d, "t"] \\
\Omega \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"'] & \Omega \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \Omega \arrow[d, "\alpha"] \\
\Omega \arrow[r, "\alpha"'] & \Omega \arrow[r, "\alpha"'] & \Omega
\end{tikzcd}$$

Note that these are all pullbacks; for example, the one in the lower right-hand corner is a pullback because α is mono.

Taking the outer square of this grid, and using that $\alpha t!_U = m$, we get a pullback

$$\begin{tikzcd}
U \arrow[r, "\mathrm{id}"] \arrow[d, "m"'] & U \arrow[d, "m"] \\
\Omega \arrow[r, "\alpha^2"'] & \Omega
\end{tikzcd}$$

Composing this with the pullback defining m, we see that α^3 classifies m. It follows that $\alpha(\alpha^2) = \alpha^3 = \alpha = \alpha \mathrm{id}_\Omega$. Since α is mono, we obtain $\alpha^2 = \mathrm{id}_\Omega$, as desired.

Exercise 2. This exercise (in particular part b)) was a bit too hard for a normal hand-in exercise. I have been very lenient in the grading of part b), and there were two bonus points to be earned if you managed to come up with substantial parts of the solution.

a): For a T-algebra $TX \stackrel{h}{\to} X$, consider the parallel pair

$$T^2X \underset{\mu_X}{\overset{Th}{\rightrightarrows}} TX$$

where μ is the multiplication of the monad T. This is a diagram of algebra maps $F^T(TX) \to F^T(X)$: $\mu_X T^2 h = Th\mu_{TX}$ by naturality of μ, and $\mu_X\mu_{TX} =$

$\mu_X T(\mu_X)$ by associativity of μ. The two arrows have a common splitting $T(\eta_X)$ (η being the unit of the monad T) which is also an algebra map since it is $F^T(\eta_X)$. That is: we have a reflexive pair in T-Alg. It is easy to see that $h : TX \to X$ coequalizes this pair: this is the associativity of h as an algebra. If $a : F^T(X) \to (\xi : TY \to Y)$ is an algebra map which coequalizes our reflexive pair then a factors through $h : F^T(X) \to (h : TX \to X)$ by $a\eta_X : (TX \overset{h}{\to} X)_{\to}(TY \overset{\xi}{\to} Y))$ and the factorization is unique because the arrow h is split epi in $\mathcal{C}$.

This construction is functorial. Given a T-algebra map $f : (TX \overset{h}{\to} X) \to (TY \overset{k}{\to} Y)$ the diagram

$$\begin{tikzcd}
T^2X \arrow[r, shift left, "\mu_X"] \arrow[r, shift right, "Th"'] \arrow[d, "T^2f"'] & TX \arrow[d, "Tf"] \\
T^2Y \arrow[r, shift left, "\mu_Y"] \arrow[r, shift right, "Tk"'] & TY
\end{tikzcd}$$

commutes serially (i.e., $Tf\mu_X = \mu_Y T^2 f$ and $TfTh = TkT^2f$). So, we have a functor R from T-Alg to the category of diagrams of shape $\circ \rightrightarrows \circ$ in T-Alg, with the properties:

- i) The vertices of $R(h)$ are free algebras.
- ii) $R(h)$ is always a reflexive pair.
- iii) The colimit of $R(h)$ is h.

b) Since $\bar{F}$ is a lifting of F ($U^S\bar{F} = FU^T$) there is a natural transformation $\lambda : SF \to FT$ constructed as follows. Consider $F(\eta) : F \to FT = FU^TF^T = U^S\bar{F}F^T$ and let $\tilde{\lambda} : F^SF \to \bar{F}F^T$ be its transpose along $F^S \dashv U^S$. Define λ as the composite

$$SF = U^SF^SF \overset{U^S\tilde{\lambda}}{\longrightarrow} U^S\bar{F}F^T = FU^TF^T = FT.$$

Claim: The natural transformation λ makes the following diagram commute, where ι and ν are, respectively, the unit and multiplication of the monad S:

$$\begin{tikzcd}
F \arrow[r, "\iota_F"] \arrow[dr, "F(\eta"'] & SF \arrow[d, "\lambda"'] & S^2F \arrow[l, "\nu_F"'] \arrow[d, "S\lambda"] \\
 & FT & SFT \arrow[d, "\lambda T"] \\
 & & FT^2 \arrow[ul, "F\mu"]
\end{tikzcd}$$

Definition of $\bar{L}$ on objects: if $\bar{L}$ is going to be left adjoint to $\bar{F}$ then, by uniqueness of adjoints and the fact that adjoints compose, $\bar{L}F^S = F^TL$, so we know what $\bar{L}$ should do on free S-algebras F^SY. Now by part a), every S-algebra

$\xi : SY \to Y$ is coequalizer of a reflexive pair of arrows between free S-algebras, and as a left adjoint, $\bar{L}$ should preserve coequalizers. Therefore we expect $\bar{L}(\xi)$ to be coequalizer of a reflexive pair

$$F^T LSY = \bar{L}F^S(SY) \underset{g_\xi}{\overset{f_\xi}{\rightrightarrows}} \bar{L}F^S(Y) = F^T LY$$

between free T-algebras. It is now our task to determine f_ξ and g_ξ.

By a) we have a coequalizer

$$F^S(SY) \underset{\nu_Y}{\overset{S\xi}{\rightrightarrows}} F^S Y \xrightarrow{\xi} (\xi)$$

and the topmost arrow of the reflexive pair is in the image of the functor F^S, so we can take $F^T L(\xi)$ for f_ξ. The other map – ν – is not in the image of F^S and needs a bit of doctoring using the adjunction $L \dashv F$ and the natural transformation λ we constructed. Let α be the unit of the adjunction $L \dashv F$. Consider the arrow

$$SY \xrightarrow{S(\alpha_Y} SFL(Y) \xrightarrow{\lambda_{L(Y)}} FTL(Y)$$

This transposes under $L \dashv F$ to a map $LS(Y) \to TL(Y) = U^T F^T L(Y)$, and this in turn transposes under $F^T \dashv U^T$ to a map

$$F^T LS(Y) \to F^T L(Y)$$

which we take as our g_ξ.

Note that the construction is natural in ξ, so if $k : \xi \to \zeta$ is a map of S-algebras, we obtain a natural transformation from the diagram of parallel arrows f_ξ, g_ξ to the diagram with parallel arrows f_ζ, g_ζ. Hence we also get a map from the coequalizer of the first diagram, which is $\bar{L}(\xi)$, to the coequalizer of the second one, which is $\bar{L}(\zeta)$.

There is still a lot to check. This is meticulously done in Volume 2 of Borceux's *Handbook of Categorical Algebra*, section 4.5. There the proof takes 10 pages!

c) Let $F : \mathcal{F} \to \mathcal{E}$ be a logical functor between toposes, which has a left adjoint. We apply part b) to the diagram

$$\begin{tikzcd}
\mathcal{F}^{\mathrm{op}} \arrow[r, "F^{\mathrm{op}}"] \arrow[d, "P"'] & \mathcal{E}^{\mathrm{op}} \arrow[d, "P"] \\
\mathcal{F} \arrow[r, "F"'] & \mathcal{E}
\end{tikzcd}$$

The diagram commutes because F is logical, and $\mathcal{F}^{\mathrm{op}}$ has coequalizers of reflexive pairs. By part b) we conclude that F^{op} has a left adjoint; but this means that F has a right adjoint.

Remark. There is a better theorem than the one we just partially proved: the *Adjoint Triangle Theorem*. It says that whenever we have functors $\mathcal{B} \xrightarrow{R} \mathcal{C} \xrightarrow{U} \mathcal{D}$ such that $\mathcal{B}$ has reflexive coequalizers and U is *of descent type* (that is: U has a left adjoint J and the comparison functor $K : \mathcal{C} \to UJ-\mathrm{Alg}$ is full and faithful), then UR has a left adjoint if and only if R has one.

Note, that given the diagram of the exercise, the diagram

$$\mathcal{C}^T \xrightarrow{\bar{F}} \mathcal{D}^S \xrightarrow{U^S} \mathcal{D}$$

satisfies the conditions of the Adjoint Triangle Theorem. Since the composition $U^S\bar{F}$, which is F^TL, has a left adjoint, we conclude that $\bar{F}$ has a left adjoint. Note in particular that we do not use that $\mathcal{C}^T$ is monadic.

Exercise 3. If p is a point of $\widehat{\mathcal{G}}$ then $p^* : \widehat{\mathcal{G}} \to \mathrm{Set}$ is given by $(-) \otimes_{\mathcal{G}} A$, for a flat functor $A : \mathcal{G} \to Set$. Such a functor A can be seen as a left $\mathcal{G}$-set, which we also denote by A; the category $\mathrm{Elts}(A)$ has as objects the elements of A, and an arrow $x \to y$ is an element $g \in \mathcal{G}$ such that $gx = y$. Now A is flat if and only if the category $\mathrm{Elts}(A)$ is filtering. Spelling out the definition, we obtain:

i) A is nonempty.

ii) For $x, y \in A$, there exist an element $z \in A$ and elements $g, h \in \mathcal{G}$ satisfying $gz = x$ and $hz = y$. Since $\mathcal{G}$ is a group, we conclude from this that for x, y in A there is some element $g \in \mathcal{G}$ such that $gx = y$: the action of $\mathcal{G}$ is transitive.

iii) For $x \in A$, $g, h \in \mathcal{G}$ satisfying $gx = hx$, there is an element $k \in \mathcal{G}$ and an element $y \in A$ such that $ky = x$ and $gk = hk$. Again using that $\mathcal{G}$ is a group, we get that whenever $gx = hx$, then $g = h$: the action of $\mathcal{G}$ is free.

There is, up to isomorphism, only one left $\mathcal{G}$-set which is transitive and free: the set $\mathcal{G}$ itself. We see that there is at most one point of the topos $\widehat{\mathcal{G}}$.

Now for an object X of $\widehat{\mathcal{G}}$, it is not hard to calculate that $X \otimes_{\mathcal{G}} \mathcal{G}$ is in natural bijective correspondence with the underlying set X; the functor $(-) \otimes_{\mathcal{G}} \mathcal{G}$ is isomorphic to the functor which takes the underlying set and forgets the $\mathcal{G}$-action. Clearly, this functor preserves finite limits. It also has a right adjoint: this is the functor which sends a set Y to the right $\mathcal{G}$-set $Y^{\mathcal{G}}$, with $\mathcal{G}$-action $\phi g(x) = \phi(xg)$. This describes inverse and direct image functors of the (up to isomorphism) unique point of $\widehat{\mathcal{G}}$ explicitly.

Exercise 4. a) Let $E_{fg} \to Z$ denote the equalizer of f, g. Consider the diagram:

$$\begin{tikzcd}
 & E' \arrow[dr] \arrow[dd] & \\
E_{fg} \arrow[ur] \arrow[rr] \arrow[dd] & & Z \arrow[dd, "\langle f,g\rangle"] \\
 & \overline{\delta} \arrow[dr] & \\
X \arrow[ur] \arrow[rr, "\delta"'] & & X \times X
\end{tikzcd}$$

where all the squares are pullbacks. We see that E' is the closure of E_{fg}, and we see that the map $\langle f, g\rangle$ factors through $\overline{\delta}$ if and only if $E' \to Z$ is an isomorphism, which holds if and only if E_{fg} is a dense subobject of Z.

b) We prove that for an arbitrary object Z of $\mathcal{E}$, the set of ordered pairs

$$\{(f, g) \in \mathcal{E}(Z, X)^2 \mid \langle f, g\rangle \text{ factors through } \overline{\delta}\}$$

is an equivalence relation on $\mathcal{E}(Z, X)$. Now reflexivity and symmetry are obvious, and using the notation above for equalizers we easily see that $E_{fg} \wedge E_{gh} \leq E_{fh}$. Since the meet of two dense subobjects is dense, we see that the relation is transitive.

c) We have to prove that any map $f : X \to L$ with L separated, coequalizes the parallel pair $r_0, r_1 : \overline{\delta} \to X$ which is the equivalence relation from part b). Now clearly for $f \times f : X \times X \to L \times L$, the composite $(f \times f)\circ\delta$ factors through the diagonal subobject $L \xrightarrow{\delta_L} L \times L$, so the composite $(f \times f)\circ\langle r_0, r_1\rangle$ factors through the closure of δ_L. But δ_L is closed, so $fr_0 = fr_1$ and f factors uniquely through $X \to MX$. The adjointness is also clear, provided we can show that MX is separated. Now δ is classified by $\Delta : X \times X \to \Omega$, which has as exponential transpose the map $\{\cdot\} : X \to \Omega^X$. So, δ is the kernel pair of $\{\cdot\}$. Now $\overline{\delta}$ is classified by $j\circ\Delta$, the exponential transpose of which is $j^X\circ\{\cdot\} : X \to \Omega_j^X$. And $\overline{\delta}$ is the kernel pair of $j^X\circ\{\cdot\}$. We see that, by the construction of epi-mono factorizations in a regular category, $X \to MX \to \Omega_j^X$ is an epi-mono factorization. So MX is a subobject of a sheaf, and therefore separated.

Exercise 5. The exercise wasn't too crisply formulated, so if you just rehashed the theory ("we have $F^* \dashv \prod_F$, so F factors through the category of $F^*\prod_F$-coalgebras…") this gave you 6 points. What I had in mind was a *concrete* presentation of the factorization.

Abandoning the notation of the exercise, given a functor $F : \mathcal{C} \to \mathcal{D}$ between small categories, let us denote the induced geometric morphism $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ by $\widehat{F}$.
Lemma 1. If the functor F is surjective on objects, then $\widehat{F}$ is a surjection.

Proof: the inverse image of $\widehat{F}$ is the functor F^* which is given by $F^*(Y)(C) = Y(FC)$. On morphisms: for $\mu : Y \to Y'$, $F^*(\mu)_C = \mu_{FC}$. Clearly, if F is surjective on objects and $F^*(\mu) = F^*(\nu)$ then $\mu_{FC} = \nu_{FC}$ for all objects C, hence $\mu_D = \nu_D$ for all objects D, so $\mu = \nu$. Hence F^* is faithful, and $\widehat{F}$ is a surjection.

Lemma 2. If the functor F is full and faithful, then $\widehat{F}$ is an embedding.
Proof: Now we consider also the direct image of $\widehat{F}$, which is the functor $\prod_F$ given by:

$$\prod_F(X)(D) = \widehat{\mathcal{C}}(F^*(y_D), X)$$

We calculate the composite $F^* \prod_F$. First, we observe that

$$F^*(y_{FC})C' = y_{FC}(FC') = \mathcal{D}(FC', FC) \simeq \mathcal{C}(C', C)$$

the last isomorphism because F is assumed full and faithful. We conclude that $F^*(y_{FC}) \simeq y_C$ naturally. Now we calculate $F^* \prod_F$:

$$F^*(\prod_F(Y))(C) = \prod_F(Y)(FC) = \widehat{\mathcal{C}}(F^*(y_{FC}), Y) \simeq \widehat{\mathcal{C}}(y_C, Y) \simeq Y(C)$$

so we conclude that $F^* \prod_F$ is isomorphic to the identity on $\widehat{\mathcal{C}}$. Therefore the counit is an isomorphism, and $\widehat{F}$ is an embedding.

Turning now to the exercise: given arbitrary $F : \mathcal{C} \to \mathcal{D}$, consider the factorization

$$\mathcal{C} \xrightarrow{P} \mathcal{E} \xrightarrow{Q} \mathcal{D}$$

where $\mathcal{E}$ is the full subcategory of $\mathcal{D}$ on objects of the form FC. Then P is surjective on objects and Q is full and faithful. Clearly, $\widehat{F} = \widehat{Q}\widehat{P}$, so we have a factorization of the desired kind.

Exercise 6. Of course one can define the syntactic category of the theory of posets, with an appropriate Grothendieck topology. This is not a very concrete presentation; what I had in mind was a solution similar to the example of rings. And this is possible. Many of you had (from the literature) correctly guessed that the classifying topos would be $\mathrm{Set}^{\mathrm{Pos}_f}$, where Pos_f is the category of finite posets and order-preserving maps.

Let us look at both a poset object in a category with finite limits and the dual notion, a *co-poset object* in a category with finite colimits.

A *poset object* in a category with finite limits consists of an object P and a monomorphism $\langle r_0, r_1 \rangle : R \to P \times P$, satisfying the conditions:

(R) Reflexifity: the diagonal $P \to P \times P$ factors through R.

(A) Antisymmetry: let R^{op} be the subobject $\langle r_1, r_0 \rangle : R \to X \times X$. Then the intersection of R and R^{op} is the diagonal $P \to P \times P$.

(T) Transitivity: let

$$\begin{tikzcd} R_1 \arrow[r, "s"] \arrow[d, "q"'] & R \arrow[d, "r_0"] \\ R \arrow[r, "r_1"'] & P \end{tikzcd}$$

be a pullback. Then the map $\langle r_0 q, r_1 s\rangle : R_1 \to P \times P$ factors through R.

Dually, a *co-poset object* in a category with finite colimits consists of an object P and an epimorphism $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : P + P \to S$, satisfying the conditions:

(co-R) Co-reflexivity: the codiagonal $\begin{bmatrix} \mathrm{id} \\ \mathrm{id} \end{bmatrix} : P+P \to P$ factors through $P+P \to S$.

(co-A) Co-antisymmetry: there is a pushout diagram

$$\begin{tikzcd} P + P \arrow[r, "\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}"] \arrow[d, "\begin{bmatrix} s_1 \\ s_0 \end{bmatrix}"'] & S \arrow[d] \\ S \arrow[r] & P \end{tikzcd}$$

where the composite $P + P \to P$ is the codiagonal.

(co-T) Co-transitivity: given a pushout diagram

$$\begin{tikzcd} P \arrow[r, "s_0"] \arrow[d, "s_1"'] & S \arrow[d, "\sigma"] \\ S \arrow[r, "\tau"'] & S_1 \end{tikzcd}$$

the map $\begin{bmatrix} \tau s_0 \\ \sigma s_1 \end{bmatrix} : P+P \to S_1$ factors through the map $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : P+P \to S$.

Now consider the category Pos_f of finite posets; this is a category with finite colimits. We have the posets $\mathbf{1} = \{*\}$ and $\mathbf{2} = \{a, b\}$ with $a < b$. We have the maps $s_0, s_1 : \mathbf{1} \to \mathbf{2}$ given by $s_0(*) = a$, $s_1(*) = b$. Clearly, the map $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : \mathbf{1} + \mathbf{1} \to \mathbf{2}$ is an epimorphism; we claim that this defines a co-poset structure on $\mathbf{1}$.

Clearly, co-reflexivity holds since $\mathbf{1}$ is terminal in Pos_f.

For co-antisymmetry, suppose the diagram

$$\begin{tikzcd}
\mathbf{1}+\mathbf{1} \arrow[r, "{\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} s_1 \\ s_0 \end{bmatrix}}"'] & \mathbf{2} \arrow[d, "f"] \\
\mathbf{2} \arrow[r, "g"'] & X
\end{tikzcd}$$

commutes. Let $\mathbf{1}+\mathbf{1} = \{x, y\}$ with $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}(x) = a$ and $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}(y) = b$.

Then we have the equations:

$$f(a) = f\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}(x) = g\begin{bmatrix} s_1 \\ s_0 \end{bmatrix}(x) = g(b)$$
$$f(b) = f\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}(y) = g\begin{bmatrix} s_1 \\ s_0 \end{bmatrix}(y) = g(a)$$

We conclude, by the monotonicity of f and g, that $f(a) \leq f(b) = g(a) \leq g(b) = f(a)$, so the diagram

$$\begin{tikzcd}
\mathbf{1}+\mathbf{1} \arrow[r, "{\begin{bmatrix} s_0 \\ s_1 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} s_1 \\ s_0 \end{bmatrix}}"'] & \mathbf{2} \arrow[d] \\
\mathbf{2} \arrow[r] & \mathbf{1}
\end{tikzcd}$$

is a pushout, and co-antisymmetry holds.

For co-transitivity, we see that in Pos_f the diagram

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "s_0"] \arrow[d, "s_1"'] & \mathbf{2} \arrow[d, "\sigma"] \\
\mathbf{2} \arrow[r, "\tau"'] & \mathbf{3}
\end{tikzcd}$$

is a pushout, where $\mathbf{3}$ is the poset $u < v < w$ and $\begin{bmatrix} \tau s_0 \\ \sigma s_1 \end{bmatrix} : \mathbf{1}+\mathbf{1} \to \mathbf{3}$ satisfies $\begin{bmatrix} \tau s_0 \\ \sigma s_1 \end{bmatrix}(x) = u$ and $\begin{bmatrix} \tau s_0 \\ \sigma s_1 \end{bmatrix}(y) = w$. By transitivity in $\mathbf{3}$ we have a map $\mathbf{2} \to \mathbf{3}$ (sending a to u and b to w), so that we have a factorization $\mathbf{1}+\mathbf{1} \to \mathbf{2} \to \mathbf{3}$, as required.

We conclude that we have a co-poset object in Pos_f. Moreover, every object of Pos_f is a finite colimit of a diagram of copies of $\mathbf{1}$ and $\mathbf{2}$. Therefore, we have:

> *The category* Pos_f *with the co-poset object* $\begin{bmatrix} s_0 \\ s_1 \end{bmatrix} : \mathbf{1}+\mathbf{1} \to \mathbf{2}$ *is the free category with finite colimits and a co-poset object.*

This means: for any category $\mathcal{C}$ with finite colimits and a co-poset object $P + P \to S$ there is an essentially unique functor $\mathrm{Pos}_f \to \mathcal{C}$ which preserves finite colimits and sends $\mathbf{1} + \mathbf{1} \to \mathbf{2}$ to $P + P \to S$.

Dually then, for every category $\mathcal{E}$ (in particular, a topos) with finite limits and a poset object $R \to P \times P$ we have an essentially unique functor from $\mathrm{Pos}_f^{\mathrm{op}}$ to $\mathcal{E}$ which preserves finite limits (hence is flat) and sends the poiset object $\mathbf{2} \to \mathbf{1} \times \mathbf{1}$ (product in $\mathrm{Pos}_f^{\mathrm{op}}$!) to $R \to P \times P$. Therefore, if $\mathcal{E}$ s a Grothendieck topos with poset object, we have an essentially unique geometric morphism $\mathcal{E} \xrightarrow{f} \mathrm{Set}^{\mathrm{Pos}_f}$, such that f^* sends the generic poset in $\mathrm{Set}^{\mathrm{Pos}_f}$ to the given one in $\mathcal{E}$. So $\mathrm{Set}^{\mathrm{Pos}_f}$ is the classifying topos for posets.