Topos Theory, Spring 2020
Exercises and Hand-In Exercises

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1 Exercises

Exercise 1 For a nonempty set A, let F_A be the following presheaf on the real numbers $\mathbb{R}$:

$$F_A(U) = \begin{cases} A & \text{if } 0 \in U \\ \{*\} & \text{else} \end{cases}$$

Show that F_A is a sheaf, and give a concrete presentation of the étale space corresponding to F_A.

Exercise 2 We denote the category of presheaves on a topological space X by $\widehat{\mathcal{O}(X)}$. Recall that Et/X is the full subcategory of the slice Top/X on the etale maps.

a) Show that Et/X is closed under the finite limits in Top/X.

b) Show that the construction which from a presheaf F on X makes an etale map $\pi : \coprod_{x\in X} G_x \to X$ is part of a functor $\widehat{\mathcal{O}(X)} \to \mathrm{Et}/X$ which preserves finite limits.

Exercise 3 Let $\Delta : X\times X \to \Omega$ be the classifying map of the diagonal subobject $X \stackrel{\delta}{\to} X \times X$; and let $\{\cdot\} : X \to \Omega^X$ be the exponential transpose of Δ. Prove that $\{\cdot\}$ is mono.

Exercise 4 (Hand-In Exercise 1, to be handed in Fenruary 27) Show that the functor $Y \mapsto \tilde{Y}$, which gives for each object Y the underlying object of the partial map classifier for Y, has the structure of a monad.

Exercise 5 Construct in the category $\widehat{\mathcal{C}}$ of presheaves on $\mathcal{C}$, the partial map classifier of an arbitrary presheaf F.

Exercise 6 Construct partial map classifiers in the category Et/X, for a space X.

Exercise 7 a) Prove that in a topos, an object is injective if and only if it is a retract of some object of the form Ω^Y.

b) Suppose $A \overset{G}{\to} B$ is a functor with left adjoint $B \overset{F}{\to} A$. Show that if F preserves monos, G preserves injective objects, and that the converse holds if A has enough injectives (that is, in A every object is a subobject of an injective object).

Exercise 8

Exercise 9

Exercise 10 Let F be a presheaf on a small category $\mathcal{C}$. Construct, for objects $G \overset{g}{\to} F, H \overset{h}{\to} F$ of the slice category $\widehat{\mathcal{C}}/F$, the exponential h^g.

Exercise 11 Prove that for objects X, Y of a topos $\mathcal{E}$ and their coproduct $X + Y$, the categories $\mathcal{E}/(X + Y)$ and $\mathcal{E}/X \times \mathcal{E}/Y$ are equivalent.

Exercise 12 (Hand-In Exercise 2, to be handed in March 12) Define for every morphism $f : X \to Y$, not just monomorphisms, an arrow $\exists_f : \Omega^X \to \Omega^Y$ in such a way that we obtain a functor $\Omega^{(-)} : \mathcal{E} \to \mathcal{E}$. Show also that the arrows $\{\cdot\} : X \to \Omega^X$ form a natural transformation from the identity on $\mathcal{E}$ to $\Omega^{(-)}$.