More on Geometric Morphisms between Realizability Toposes

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In very recent years, renewed interest in realizability toposes:

- Benno van den Berg: the *Herbrand Topos* (2011)
- Theses by Wouter Stekelenburg and Jonas Frey (2013)
- Papers by Peter Johnstone (2013)

Starting point: the notion of a *Partial Combinatory Algebra* (pca).

A pca is a set A together with a partial binary map $(a, b \mapsto ab)$: $A \times A \rightarrow A$. We write $ab\downarrow$ for: *ab is defined*. We also write abc for $(ab)c$.

There should be elements k and s satisfying:

$$\mathsf{k}ab = a$$

$$\mathsf{s}ab\downarrow \text{ and, if } ac(bc)\downarrow, \text{ then } \mathsf{s}abc = ac(bc)$$

Prime example: $\mathcal{K}_1$, the set of natural numbers, where nm is the result (if defined) of the n-th computable function applied to m.

Peter Johnstone calls pcas *Schönfinkel Algebras*.

Every pca A admits:

- *pairing and unpairing combinators*: elements $\pi, \pi_0, \pi_1 \in A$ satisfying $\pi_0(\pi ab) = a$, $\pi_1(\pi ab) = b$
- *Booleans*: elements T and F and an element u satisfying $u\mathrm{T}ab = a$, $u\mathrm{F}ab = b$ (we can pronounce $uxyz$ as: if x then y else z)
- *Curry numerals*: elements $\overline{n}$ for every natural number n; for every computable function f there is an element $a_f \in A$ such that for every n, $a_f\overline{n} = \overline{f(n)}$

Every pca A gives rise to a category of *assemblies* $\mathrm{Ass}(A)$: an object of $\mathrm{Ass}(A)$ (an A-assembly) is a pair (X, E) where X is a set, and for each $x \in X$, $E(x)$ is a nonempty subset of A.

A morphism $(X, E) \to (Y, F)$ is a function $X \xrightarrow{f} Y$ which is *tracked* by some $b \in A$: for every $x \in X$ and every $a \in E(x)$, $ba \in F(f(x))$.

The category $\mathrm{Ass}(A)$ is a quasitopos with a natural numbers object.

The *Realizability Topos* on A, $\mathrm{RT}(A)$, is the exact completion of $\mathrm{Ass}(A)$ as regular category.

The realizability topos on $\mathcal{K}_1$ is Hyland's *Effective Topos*, Eff.

The category $\mathrm{Ass}(A)$ comes equipped with functors

$$\mathrm{Set} \underset{\nabla}{\overset{\Gamma}{\leftrightarrows}} \mathrm{Ass}(A)$$

where Γ is the global sections functor and ∇ sends a set X to the assembly $(X, \lambda x.A)$. We have $\Gamma \dashv \nabla$

A functor $\mathrm{Ass}(A) \to \mathrm{Ass}(B)$ is a Γ-functor if the diagram

$$\begin{array}{ccc} \mathrm{Ass}(A) & \longrightarrow & \mathrm{Ass}(B) \\ & {\scriptstyle\Gamma}\searrow \quad \swarrow{\scriptstyle\Gamma} & \\ & \mathrm{Set} & \end{array}$$

commutes.

Definition(J. Longley) Given pcas A and B, an *applicative morphism* $A \to B$ is a function γ which assigns to every $a \in A$ a nonempty subset $\gamma(a)$ of B, in such a way that for some element $r \in B$ (the *realizer* of γ) the following holds: whenever $ab{\downarrow}$ in A, $u \in \gamma(a)$, $v \in \gamma(b)$, we have $ruv \in \gamma(ab)$.
Composition is composition of relations.
Given two such applicative morphisms $\gamma, \delta : A \to B$, we say $\gamma \leq \delta$ if for some element $s \in B$: for all $a \in A$ and $u \in \gamma(a)$, $su \in \delta(a)$.
We obtain a preorder-enriched category PCA.
Every applicative morphism $\gamma : A \to B$ determines a regular Γ-functor $\gamma^* : \mathrm{Ass}(A) \to \mathrm{Ass}(B)$: it sends (X, E) to (X, F) where $F(x) = \bigcup_{a \in E(x)} \gamma(a)$.
Whenever $\gamma \leq \delta$ we have a natural transformation $\gamma^* \Rightarrow \delta^*$

Theorem(J. Longley) There is a biequivalence between the following two 2-categories:

- PCA
- The 2-category of categories of the form $\mathrm{Ass}(A)$, regular Γ-functors and natural transformations

Every regular Γ-functor $\mathrm{Ass}(A) \to \mathrm{Ass}(B)$ extends uniquely to a regular Γ-functor $\mathrm{RT}(A) \to \mathrm{RT}(B)$. We use the same notation for $\gamma^* : \mathrm{Ass}(A) \to \mathrm{Ass}(B)$ and its extension $\gamma^* : \mathrm{RT}(A) \to \mathrm{RT}(B)$.

Suppose $f : \mathrm{RT}(B) \to \mathrm{RT}(A)$ is a geometric morphism such that the inverse image functor f^* restricts to a functor $\mathrm{Ass}(A) \to \mathrm{Ass}(B)$. Then f^* is of the form γ^* for an essentially unique applicative morphism $\gamma : A \to B$.

Can we characterize those applicative morphisms γ for which γ^* has a right adjoint?

Hofstra-vO: these are the *computationally dense* applicative morphisms. The definition of “computationally dense” was rather complicated.

Two theorems of Peter Johnstone

Theorem 1 An applicative morphism $\gamma : A \to B$ is computationally dense if and only if there exists a function $f : B \to A$ such that $\gamma f \leq \mathrm{id}_B$
(i.e. there exists $r \in B$ such that for every $b \in B$ and $b' \in \gamma(f(b))$, $rb' = b$)

Theorem 2 *Every* geometric morphism $f : \mathrm{RT}(A) \to \mathrm{RT}(B)$ has the property that f^* restricts to a functor $\mathrm{Ass}(A) \to \mathrm{Ass}(B)$

A slight generalization of pcas:
An *order-pca* (opca) is a partially ordered set with a partial binary application function, such that:

If $ab{\downarrow}$, $a' \leq a$ and $b' \leq b$ then $a'b'{\downarrow}$ and $a'b' \leq ab$

there is an element k such that $\mathrm{k}ab \leq a$ for all a

there is an element s such that $\mathrm{s}ab{\downarrow}$, and whenever $ac(bc){\downarrow}$, $\mathrm{s}abc \leq ac(bc)$

Prime example: given a pca A, let $T(A)$ be the set of nonempty subsets of A. For $\alpha, \beta \in T(A)$ say $\alpha\beta{\downarrow}$ if for all $a \in \alpha, b \in \beta$, $ab{\downarrow}$ in A; then $\alpha\beta = \{ab \,|\, a \in \alpha, b \in \beta\}$

For opcas A, B, an *applicative morphism* $A \to B$ is a function $f : A \to B$ for which there exist elements $u, r \in B$ satisfying:

- whenever $a \leq b$ in A, $uf(a) \leq f(b)$ in B
- whenever $ab{\downarrow}$ in A, $rf(a)f(b) \leq f(ab)$ in B

Many things generalize:

- There is an order-enriched category OPCA of order-pcas, applicative morphisms and inequalities
- There is, for each order-pca A, a category of assemblies $\mathrm{Ass}(A)$: assemblies (X, E) now satisfy that $E(x)$ is a nonempty *downward closed* subset of A
- There is the realizability topos RT(A)

Moreover, if for an opca A we let $T(A)$ be the opca on the set of nonempty downwards closed subsets of A, them T extends to a 2-monad on the 2-category OPCA.

The category $\mathrm{Ass}(T(A))$ is the *regular completion* (in the sense of Carboni) of the category $\mathrm{Ass}(A)$.

Definition. Let A, B be pcas. A *proto-applicative morphism* $A \to B$ is an applicative morphism of opcas $T(A) \to T(B)$.

We have the following variation on Longleys result:

Theorem There is a biequivalence between the following two 2-categories:

- Pcas, proto-applicative morphisms and inequalities
- The 2-category of categories of the form $\mathrm{Ass}(A)$, finite-limit preserving Γ-functors and natural transformations

Note: every applicative morphism $\gamma : A \to B$ gives a proto-applicative morphism $\tilde{\gamma} : A \to B$

Corollary 1. The following are equivalent for an applicative morphism $\gamma : A \to B$:

- γ is computationally dense
- There is an applicative morphism $\delta : B \to A$ such that $\gamma\delta \leq \mathrm{id}_B$
- $\tilde{\gamma}$ has a right adjoint

Corollary 2. The following are equivalent:

- A geometric morphism $\mathrm{RT}(A) \to \mathrm{RT}(B)$
- An adjunction $\mathrm{Ass}(A) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Ass}(B)$, $f^* \dashv f_*$, such that f^* preserves finite limits

Geometric Inclusions

An applicative morphism $\gamma : A \to B$ induces an inclusion of toposes: $\mathrm{RT}(B) \to \mathrm{RT}(A)$ if and only if there is an applicative $\delta : B \to A$ such that $\gamma\delta \simeq \mathrm{id}_B$

Regular Geometric Morphisms

Call a geometric morphism *regular* if its direct image is a regular functor.

For a computationally dense applicative morphism $\gamma : A \to B$ the following are equivalent:

- the geometric morphism induced by γ is regular
- γ has a right adjoint in PCA
- γ is *projective*, that is: isomorphic to a single-valued relation.

This is because γ is projective iff γ^* preserves projective objects iff (given that categories of the form $\mathrm{Ass}(A)$ have enough projectives) the right adjoint to γ^* preserves regular epis (and is therefore induced by an applicative morphism)

Example. Consider $\mathcal{K}_2^{\text{rec}}$, this is a pca structure on the set of total recursive functions. There is a computationally dense applicative morphism

$$\gamma : \mathcal{K}_2^{\text{rec}} \to \mathcal{K}_1$$

where γ sends a recursive function to the set of its indices. This cannot be isomorphic to a single-valued relation, so γ is not projective and has no right adjoint in PCA.

Intermezzo: Total pcas.

A pca A is total if $ab{\downarrow}$ always. Call a pca *almost total* if for every a there is a' such that $a'b{\downarrow}$ always, and whenever $ab = c$, also $a'b = c$.

A pca is called *decidable* if there is $d \in A$ such that for all $a, b \in A$: $daa = \mathrm{T}$ and $dab = \mathrm{F}$ if $a \neq b$.

We know:

- (Johnstone, Robinson) Eff is not equivalent to RT(A) for A total
- (vO) Every total pca is isomorphic to a nontotal one
- (vO) Every RT(A) is covered by some RT(B) with B total

Furthermore:

- A decidable pca is never almost total
- A pca A is almost total iff there is $g \in A$ such that for all $a \in A$, $gab{\downarrow}$ always, and whenever $ab = c$ then $gab = c$
- A pca is almost total iff it is isomorphic to a total pca.

Decidable Applicative Morphisms

Definition (Longley) An applicative morphism $\gamma : A \to B$ is *decidable* iff γ^* preserves finite coproducts (equivalently, if γ^* preserves the NNO)

Clearly, every computationally dense morphism is decidable.

There is, for every pca A, exactly one decidable morphism $\mathcal{K}_1 \to A$: it sends n to $\bar{n}$, the n-th Curry numeral in A.

Definition. Let $\gamma : A \to B$ be applicative. A partial endofunction f on A is *representable w.r.t.* γ if there is an element b such that, whenever $f(a) = a'$ then $b\gamma(a) \subseteq \gamma(a')$

Construction. Given a pca A and a partial endofunction f on A, there is a universal decidable morphism $\iota_f : A \to A[f]$ w.r.t. which f is representable: whenever $\gamma : A \to B$ is decidable and f is representable w.r.t. γ, then γ factors uniquely through ι_f

The construction generalizes that of forming the pca of partial functions computable in an oracle.

The morphism ι_f is computationally dense and induces a geometric inclusion: $\mathrm{RT}(A[f]) \to \mathrm{RT}(A)$.

Theorem If A is such that $\mathrm{RT}(A)$ is a subtopos of Eff, then A is equivalent to $\mathcal{K}_1[f]$ for some partial function f on the natural numbers.

Local Operators in RT(A)

Let us write, for subsets U, V of A:

- $U \Rightarrow V \;=\; \{a \in A \,|\, \text{for all } b \in U, ab \in V\}$
- $U \times V \;=\; \{\pi ab \,|\, a \in U, b \in V\}$

A local operator in RT(A) is given by a map $J : \mathcal{P}(A) \to \mathcal{P}(A)$ such that the sets

- $\bigcap_{U \subseteq A} U \Rightarrow J(U)$
- $\bigcap_{U \subseteq A} J(J(U)) \Rightarrow J(U)$
- $\bigcap_{U,V \subseteq A} (U \Rightarrow V) \Rightarrow (J(U) \Rightarrow J(V))$

are all nonempty.

Example. $J(U) = \{a \in A \,|\, U \text{ is nonempty}\}$. This is the $\neg\neg$-operator, corresponding to the inclusion $\mathrm{Set} \to \mathrm{RT}(A)$.

Given any monomorphism m in a topos, there is a least local operator for which m is dense (i.e., the sheafification of m is an isomorphism).

Example. Consider a pca A and a partial function f on A, with domain $A' \subseteq A$. Consider the A-assemblies (A', E_1) and (A', E_2) where $E_1(a) = \{\pi a f(a)\}$ and $E_2(a) = \{a\}$. The identity on A' gives a monomorphism $(A', E_1) \to (A', E_2)$. The least local operator for which m is dense, corresponds to the inclusion of $\mathrm{RT}(A[f])$ in $\mathrm{RT}(A)$. This generalizes a result by Hyland for the effective topos.

Example. Consider, for a pca A, the A-assemblies $2 = 1 + 1$ and $\nabla(2)$. Explicitly: $2 = (\{0,1\}, E)$ with $E(0) = \{\mathsf{F}\}$ and $E(1) = \{\mathsf{T}\}$ and $\nabla(2) = (\{0,1\}, E)$ with $E(0) = E(1) = A$.
Theorem (Hyland) The least local operator in Eff for which the inclusion of 2 in $\nabla(2)$ is dense, is $\neg\neg$.
Let us look at this in arbitrary pcas.

Lemma 1 (Hyland-Pitts) Let J be a local operator in RT(A) with $J(\emptyset) = \emptyset$. Then $J = \neg\neg$ if and only if $\bigcap_{a \in A} J(\{a\}) \neq \emptyset$

Lemma 2 The least local operator in RT(A) for which $2 \to \nabla(2)$ is dense, can be given as

$$J(X) \ = \ (\{\mathrm{T}\} \times X) \cup (\{\mathrm{F}\} \times \bigcup_u (\{u\} \Rightarrow \{\mathrm{T}\} \times X))$$

where u runs over all coded finite sequences (in A) of F's and T's.

Since there is, in A, an A-definable bijection between such coded sequences and the natural numbers, we get

Lemma 3 The local operator J from Lemma 2 is equal to $\neg\neg$, if and only if there is an element $h \in A$ satisfying: for every $a \in A$ there is a natural number n such that $h\bar{n} = a$.

Theorem The least local operator in RT(A) making $2 \to \nabla(2)$ dense, is equal to $\neg\neg$ precisely if there exists a (necessarilty unique) geometric morphism from RT(A) to Eff.

Obviously, the condition that

for some $h \in A$, for all $a \in A$ there is n with $h\bar{n} = a$

can only hold for *countable* pcas.

Example. Let A be a countable nonstandard model of Peano Arithmetic. A is a pca, by putting $ab = c$ iff

$$A \models \exists y(T(a, b, y) \wedge U(y) = c)$$

where T and U are Kleene's symbols for 'computation' and 'result' respectively.

For each $h \in A$ we have the type

$$\{\forall y(T(h, n, y) \rightarrow U(y) \neq x) \,|\, n \in N\}$$

Every nonstandard model is saturated w.r.t. these types, so this type is satisfied in A. We conclude that there is no geometric morphism from RT(A) to Eff.

Assume J is a local operator on RT(A) satisfying:

$$J(\{a\}) \cap J(\{b\}) = \emptyset \text{ if } a \neq b$$

(This implies that the inclusion $2 \to \nabla(2)$ is not J-dense)
We have a partial binary function on A: say

$$a * b = c \text{ iff } ab \in J(\{c\})$$

Theorem. $(A, *)$ is a pca. Actually, $(A, *)$ is isomorphic to $A[f]$ where f is the partial function such that $a \in J(\{f(a)\})$.

We have $\mathrm{Sh}_J(\mathrm{RT}(A)) \to \mathrm{RT}(A[f]) \to \mathrm{RT}(A)$

Computable Functionals of Type 2

Recall that a partial function $A \xrightarrow{f} A$ is *representable w.r.t.* an applicative morphism $\gamma : A \to B$ if for some $b \in B$ we have: whenever $f(a) = a'$ then $b\gamma(a)\downarrow$ and $b\gamma(a) \subseteq \gamma(a')$. Say such a b represents f w.r.t. γ

Tot_γ is the set of total functions $A \to A$ that are representable w.r.t. γ.
For $f \in \mathrm{Tot}_\gamma$, let $I_1^\gamma(f)$ be the set of elements of B which represent f w.r.t. γ

Now look at a partial operation $A^A \xrightarrow{F} A$. We say F is a *computable functional of type 2 w.r.t. gamma* if for some $b \in B$ we have: whenever $f \in \mathrm{Tot}_\gamma$ and $F(f)$ is defined, then $bI_1^\gamma(f)\downarrow$ and $bI_1^\gamma(f) \subseteq \gamma(F(f))$.

Theorem. Let A be a pca, $F : A^A \to A$ a partial operation. There is a decidable applicative morphism $\iota_F : A \to A[F]$ with respect to which F is a computable functional of type 2, and which moreover is universal with this property: whenever $\gamma : A \to B$ is decidable and F is a computable functional of type 2 w.r.t. γ, then γ factors uniquely through ι_F

The pca $A[F]$ is actually a pca of the form $A[f]$ for some partial function $f : A \to A$. The morphism ι_F is computationally dense, and induces a geometric inclusion: $\mathrm{RT}(A[F]) \to \mathrm{RT}(A)$

S.C. Kleene has set up a theory of things 'computable in F' for partial operations $F : N^N \to N$; this is axiomatized by his famous clauses 'S1–S9'.

Theorem. If $F : N^N \to N$ is a partial operation, then a partial function $N \xrightarrow{f} N$ is computable in F in Kleene's sense, if and only if f is representable w.r.t. ι_F

Example Let E be the operation $N^N \to N$ given by

$$E(f) = \begin{cases} 0 & \text{if for some } n,\ f(n) = 0 \\ 1 & \text{else} \end{cases}$$

The E-computable functions are precisely the hyperarithmetical functions. For a local operator J on Eff defined by A. Pitts in his thesis, we proved earlier that the total functions $N \to N$ in $\mathrm{Sh}_J(\mathrm{Eff})$ are precisely the hyperarithmetical functions. Hence,

$$\mathrm{Sh}_J(\mathrm{Eff}) \subset \mathrm{RT}(\mathcal{K}_1[E]) \subset \mathrm{Eff}$$

Application

Our definition of pca was a little weaker than often seen. Most authors require s to satisfy:

(∗) $sab\downarrow$, and ($sabc\downarrow \Leftrightarrow ac(bc)\downarrow$) etc.

Let us call a pca satisfying (∗), *strict*.

Theorem. Every pca is isomorphic to a strict pca.