Exam Topos Theory, June 16, 2022, 14:00–17:00

This exam consists of 4 exercises. Every exercise is worth 10 points; if an exercise consists of more than one part, it is indicated what each part is worth. The grade W for the written exam is your total number of points divided by 4. You final grade is the maximum of W and $\frac{7W+3H}{10}$ where H is your result from the homework exercises.

Advice: first do those exercises for which you see a solution right away. Then start thinking about the harder ones. Good luck!

Exercise 1 First, let us establish some terminology and notation. For an object A of a topos $\mathcal{E}$, the monomorphism $\in_A \rightarrow \Omega^A \times A$ is the subobject classified by the evaluation map $\Omega^A \times A \to \Omega$. For an object C, the map $\{\cdot\} : C \to \Omega^C$ is the exponential transpose of the map $\Delta : C \times C \to \Omega$ which classifies the diagonal $\delta : C \to C \times C$. For a subobject R of $B \times A$ we say that a map $r : B \to \Omega^A$ *names* R (or, *is a name for* R), if there is a pullback square

$$\begin{array}{ccc} R & \longrightarrow & \in_A \\ \downarrow & & \downarrow \\ B \times A & \xrightarrow[r \times \mathrm{id}_A]{} & \Omega^A \times A \end{array}$$

It can be used without proof that names are unique.

a) (4 pts) Let $g : A \to C$ be an arrow in $\mathcal{E}$. Show that the subobject $\langle \mathrm{id}_A, g\rangle : A \to A \times C$ is named by the composite arrow $\{\cdot\}g : A \to \Omega^C$.

b) (6 pts) Suppose that we have morphisms $f : A \to B$, $g : A \to C$ with f monic. Suppose $h : B \to \Omega^C$ names the subobject $\langle f, g\rangle : A \to B \times C$. Prove that the square

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \{\cdot\}} \\ B & \xrightarrow[h]{} & \Omega^C \end{array}$$

is a pullback. [Hint: in order to see that the diagram commutes, combine the square which testifies that h names $\langle f, g\rangle$ with the pullback

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle \langle \mathrm{id}_A, g\rangle}\downarrow & & \downarrow{\scriptstyle \langle f, g\rangle} \\ A \times C & \xrightarrow[f \times \mathrm{id}_C]{} & B \times C \end{array}$$

and conclude that $hf : A \to \Omega^C$ names the subobject $\langle \mathrm{id}_A, g\rangle : A \to A \times C$.]

Exercise 2 Recall that there is a map $\wedge : \Omega \times \Omega \to \Omega$ which is such that if A and B are subobjects of X, classified by $\phi, \psi : X \to \Omega$ respectively, then the intersection $A \cap B$ is classified by the composite

$$X \xrightarrow{\langle \phi,\psi\rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

Recall also that the order relation on Ω is the subobject Ω_1 of $\Omega \times \Omega$ which is the equalizer of $\wedge$ and the first projection.

Prove that there is a monic arrow $i_\Omega : \Omega \to \Omega_1$ which is a partial map classifier for Ω.

Exercise 3 a) (5 pts) Let $\mathcal{C}$ be a small category and let Cov be a Grothendieck topology on $\mathcal{C}$. Suppose $R \in \mathrm{Cov}(C)$ contains a split epimorphism $f : B \to C$. Prove: for any presheaf X on $\mathcal{C}$ and any compatible family $(x_g \,|\, g \in R)$ indexed by R (so: $x_g \in X(\mathrm{dom}(g))$ for all $g \in R$), there is a unique amalgamation in $X(C)$.

b) (5 pts) Use part a) to characterize all inclusions $\mathcal{E} \to \widehat{\mathcal{G}}$, where $\mathcal{E}$ is a Grothendieck topos and $\widehat{\mathcal{G}}$ is the topos of right $\mathcal{G}$-sets, for a group $\mathcal{G}$.

Exercise 4 A *filter of subterminals* in a topos $\mathcal{E}$ is a collection Φ of subobjects of 1, satisfying the following conditions:

i) if $U \in \Phi$ and $U \leq V$ then $V \in \Phi$

ii) $1 \in \Phi$

iii) if $U, V \in \Phi$ then $U \cap V \in \Phi$

a) (2 pts) Let Φ be a filter of subterminals in a topos $\mathcal{E}$. For objects A, B of $\mathcal{E}$ we define a Φ-*morphism from* A *to* B to be an equivalence class of arrows $A \times U \to B$ in $\mathcal{E}$ with $U \in \Phi$; two such arrows $f : A \times U \to B$ and $g : A \times V \to B$ are equivalent if there is a $W \in \Phi$ with $W \leq U \cap V$ and the square

$$\begin{array}{ccc} A \times W & \longrightarrow & A \times V \\ \downarrow & & \downarrow g \\ A \times U & \xrightarrow[f]{} & B \end{array}$$

commutes.

Show that there is a category $\mathcal{E}_\Phi$ which has the same objects as $\mathcal{E}$, and Φ-morphisms as arrows.

b) (3 pts) Characterize the monos in $\mathcal{E}_\Phi$.

c) (5 pts) Show that there is a functor $P_\Phi : \mathcal{E} \to \mathcal{E}_\Phi$ which preserves finite limits and exponentials. [Hint: given a finite diagram of Φ-morphisms, there is some $U \in \Phi$ such thet every Φ-morphism in the diagram is equivalent to one of the form $A \times U \to B$]

[Remark: in fact it can be shown that $\mathcal{E}_\Phi$ is a topos and the functor P_ϕ is logical.]