Resit Topos Theory, July 6, 2023, 14:00–17:00
with solutions

I recall the following definition. Given an object X of a topos $\mathcal{E}$, a *partial map classifier* for X is a monomorphism $\eta_X : X \to \tilde{X}$ with the property that for any diagram

$$\begin{array}{ccc} U & \xrightarrow{m} & Y \\ {\scriptstyle f}\downarrow & & \\ X & & \end{array}$$

with m mono, there is a unique map $\tilde{f} : Y \to \tilde{X}$ making the diagram

$$\begin{array}{ccc} U & \xrightarrow{m} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ X & \xrightarrow[\eta_X]{} & \tilde{X} \end{array}$$

a pullback.

Exercise 1 Let $\mathcal{C}$ be a small category; we work in the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves on $\mathcal{C}$. Let P be such a presheaf. We define a presheaf $\tilde{P}$ as follows: for an object C of $\mathcal{C}$, $\tilde{P}(C)$ consists of those subobjects α of $y_C \times P$ which satisfy the following condition: for all arrows $f : D \to C$, the set

$$\{y \in P(D) \,|\, (f, y) \in \alpha(D)\}$$

has at most one element.

a) (4 pts) Complete the definition of $\tilde{P}$ as a presheaf.

b) (6 pts) Show that there is a monic map $\eta_P : P \to \tilde{P}$ which is a partial map classifier for P.

Solution.a) Remark: for any object P in a topos $\mathcal{E}$, the partial map classifier $P \overset{\eta_P}{\to} \tilde{P}$ is the factorization through $\tilde{P}$ of the singleton map $\{\cdot\} : P \to \Omega^P$; so we define $\tilde{P}$ as a subpresheaf of Ω^P, which gives at once the presheaf structure. Concretely, elements of $\tilde{P}(C)$ are subpresheaves of $y_C \times P$. Given such a subpresheaf α, and an arrow $g : D \to C$ in $\mathcal{C}$ we let $g^*(\alpha) = \tilde{P}(g)(\alpha)$ be the subobject of $y_D \times P$ such that

$$\begin{array}{ccc} \alpha & \longrightarrow & y_C \times P \\ \uparrow & & \uparrow{\scriptstyle y_g \times \mathrm{id}_P} \\ g^*(\alpha) & \longrightarrow & y_D \times P \end{array}$$

is a pullback. So

$$g^*(\alpha)(D') = \{(\mu : D' \to D, \xi) \mid \xi \in P(D), (g\mu, \xi) \in \alpha(D')\}$$

b) The map $\eta : P \to \tilde{P}$ is given by $\eta_C(\xi) = (\mathrm{id}_C, \xi)$ for $C \in \mathcal{C}, \xi \in P(C)$.

To prove that η is a partial map classifier, suppose we have a diagram

$$\begin{array}{ccc} U & \xrightarrow{m} & Y \\ {\scriptstyle\phi}\downarrow & & \\ P & & \end{array}$$

with m mono. We complete it by defining $\tilde{\phi} : Y \to \tilde{P}$ as follows. $\tilde{\phi}_C(z)$ is the subpresheaf of $y_C \times P$ given by

$$\tilde{\phi}_C(z)(D) \;=\; \{(f : D \to C, \phi_D(u)) \mid m_D(u) = Y(f)(z)\}$$

Exercise 2 Recall that Ω_1 is the subobject of $\Omega \times \Omega$ defined by the equalizer diagram

$$\Omega_1 \longrightarrow \Omega \times \Omega \overset{p_0}{\underset{\wedge}{\rightrightarrows}} \Omega$$

Let $\theta : \Omega \to \Omega_1$ be the factorization through Ω_1 of the map $\langle \mathrm{id}, t \rangle$. Show that $\theta : \Omega \to \Omega_1$ is a partial map classifier for Ω.

Solution: Suppose we are given a diagram

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ {\scriptstyle f}\downarrow & & \\ \Omega & & \end{array}$$

with m monic. The map f classifies a subobject N of X, so we have subobjects $N \leq X \leq Y$. It is precisely this sort of "nested inclusions" that Ω_1 classifies.

Exercise 3 Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism; consider the universal closure operation on $\mathcal{E}$ induced by f.

a) (2 pts) For a subobject $A \stackrel{a}{\to} X$ in $\mathcal{E}$ we have: a is closed if and only if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & f_*f^*A \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f_*f^*(a)} \\ X & \xrightarrow[\eta]{} & f_*f^*X \end{array}$$

is a pullback (here, η denotes the unit of the adjunction $f^* \dashv f_*$).

b) (4 pts) Let $\alpha : f_*f^*\Omega \to \Omega$ classify the mono $1 \simeq f_*f^*1 \stackrel{f_*f^*(t)}{\to} f_*f^*\Omega$.

Show that, if the mono $a : A \to X$ is classified by $\phi : X \to \Omega$, then the closure of a is classified by the composite arrow

$$X \stackrel{\eta}{\to} f_*f^*X \stackrel{f_*f^*\phi}{\to} f_*f^*\Omega \stackrel{\alpha}{\to} \Omega$$

c) (4 pts) Let α be as in b).

Prove that the Lawvere-Tierney topology corresponding to our closure operation is the composite $\Omega \stackrel{\eta}{\to} f_*f^*\Omega \stackrel{\alpha}{\to} \Omega$.

Solution: a) Given a mono $a : A \to X$, the closure of A is the left hand vertical in the pullback

$$\begin{array}{ccc} c(A) & \longrightarrow & f_*f^*A \\ {\scriptstyle a'}\downarrow & & \downarrow \\ X & \xrightarrow[\eta_X]{} & f_*f^*X \end{array}$$

So a is closed if and only if the naturality square given in the exercise is a pullback.

b) Assuming that $a : A \to X$ is classified by $\phi : X \to \Omega$, we have pullbacks

$$(1)\quad \begin{array}{ccc} f_*f^*1 & \xrightarrow{f_*f^*t} & f_*f^*\Omega \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ 1 & \xrightarrow[t]{} & \Omega \end{array} \qquad (2)\quad \begin{array}{ccc} A & \xrightarrow{a} & X \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ 1 & \xrightarrow[t]{} & \Omega \end{array}$$

and a pullback diagram

$$(3)\quad \begin{array}{ccc} f_*f^*A & \xrightarrow{f_*f^*a} & f_*f^*X \\ \downarrow & & \downarrow{\scriptstyle f_*f^*\phi} \\ f_*f^*1 & \xrightarrow[f_*f^*t]{} & f_*f^*\Omega \end{array}$$

obtained by applying the functor f_*f^* (which preserves finite limits) to diagram (2). Combining (3) with (1) and the diagram defining the closure $c(A)$ of A from part a), we get

$$
\begin{array}{ccc}
c(A) & \xrightarrow{a'} & X \\
{\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta} \\
f_*f^*A & \xrightarrow{f_*f^*a} & f_*f^*X \\
\downarrow & & \downarrow{\scriptstyle f_*f^*\phi} \\
f_*f^*1 & \xrightarrow{f_*f^*t} & f_*f^*\Omega \\
\downarrow & & \downarrow{\scriptstyle \alpha} \\
1 & \longrightarrow & \Omega
\end{array}
$$

We see that $\alpha \circ f_*f^*\phi \circ \eta$ classifies the closure of a, as desired.

c) The Lawvere-Tierney topology corresponding to the universal closure operation is the classifying map of the closure of $1 \xrightarrow{t} \Omega$. The mono t is classified by the identity on Ω. Filling in id_Ω for ϕ in the expression obtained in b), we get $\alpha \circ \eta$ as desired.

Exercise 4 a) (3 pts) Let Set_f be the category of finite sets. Show that Set_f is the "free category with finite colimits generated by one object": there is an object I in Set_f such that for every finitely cocomplete category $\mathcal{C}$ and every object X of $\mathcal{C}$, there is an essentially unique finite-colimit-preserving functor $F : \mathrm{Set}_f \to \mathcal{C}$ sending I to X.

b) (4 pts) Formulate a similar universal property for the category $\mathrm{Set}_f^{\mathrm{op}}$.

c) (3 pts) Let $\mathcal{E}$ be a cocomplete topos. Show that there is a 1-1 correspondence between geometric morphisms $\mathcal{E} \to \mathrm{Set}^{\mathrm{Set}_f}$ and objects of $\mathcal{E}$. [The topos $\mathrm{Set}^{\mathrm{Set}_f}$ is called the "object classifier"]

Solution a) For any small category $\mathcal{C}$, we have that any functor $F : \mathcal{C} \to \mathcal{E}$ from $\mathcal{C}$ to a cocomplete category $\mathcal{E}$, admits an extension $\tilde{F} : \widehat{\mathcal{C}} \to \mathcal{E}$ which preserves all colimits. In fact, we can define $\tilde{F}(X)$ as the colimit of the diagram

$$(*) \quad y{\downarrow}X \longrightarrow \mathcal{C} \xrightarrow{F} \mathcal{E}$$

where y is the Yoneda embedding.

Now suppose $\mathcal{E}$ has finite colimits, $\mathcal{C}$ is the one-arrow category $\mathbb{I}$ and $X = 1$, the terminal presheaf; then the diagram $(*)$ is finite and it therefore shows that we have a *finite* colimit-preserving functor $\mathrm{Set}_f \to \mathcal{E}$, given any functor from $\mathbb{I}$ into $\mathcal{E}$, that is: given any object of $\mathcal{E}$.

b) The functor $\tilde{F} : \mathrm{Set}_f \to \mathcal{E}$ preserves finite colimits if and only if the functor $\tilde{F}^{\mathrm{op}} : \mathrm{Set}_f^{\mathrm{op}} \to \mathcal{E}^{\mathrm{op}}$ preserves finite limits. Therefore, given a finite limit category $\mathcal{F}$ and a functor $\mathbb{I} \to \mathcal{F}$, that is, again: an object of $\mathcal{F}$, we have an essentially unique extension $\mathrm{Set}_f^{\mathrm{op}} \to \mathcal{F}$ which preserves finite limits.

c) We have seen that for a cocomplete topos $\mathcal{E}$ we have a natural 1-1 correspondence between objects of $\mathcal{E}$ and finite-limit-preserving functors $\mathrm{Set}_f^{\mathrm{op}} \to \mathcal{E}$. Since $\mathrm{Set}_f^{\mathrm{op}}$ has finite limits, functors from it to a topos are flat if and only if they preserve finite limits; so objects of $\mathcal{E}$ correspond to flat functors $\mathrm{Set}_f^{\mathrm{op}} \to \mathcal{E}$, and by the theory of geometric morphisms, these correspond to geometric morphisms $\mathcal{E} \to \mathrm{Set}^{\mathrm{Set}_f}$.

Remark: it may appear to you that the category Set_f is not small. However, it is equivalent to a small category; so if we are only interested in functors out of Set_f we may replace it by an equivalent small category and therefore do as if Set_f itself is small.