# Exam Gödel's Incompleteness Theorems

May 26, 2010, 14.00–17.00
With solutions

THIS EXAM CONSISTS OF 4 PROBLEMS; SEE ALSO BACK SIDE
*Advice: first do those problems you can do right away; then, start thinking about the others. Good luck!*

**Exercise 1**. Define a variant of the Fibonacci function:

$$\begin{array}{rcl} F(0) & = & 1 \\ F(1) & = & 2 \\ F(n+2) & = & F(n)+F(n+1) \end{array}$$

a) Compute $F(n)$ for $0 \leq n \leq 5$.

b) Prove that $F$ is primitive recursive.

c) Show that there is a formula $\phi = \phi(x,y)$ such that for all $n \in \mathbb{N}$

$$\mathrm{PA} \vdash \phi(\overline{n}, \overline{F(n)}) \quad \mathrm{PA} \vdash \neg\phi(\overline{n}, \overline{F(n+1)}).$$

Can you write down such a $\phi$?

**Solution**:

a)

| $n$ | $F(n)$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 3 |
| 3 | 5 |
| 4 | 8 |
| 5 | 13 |

b) Let $\langle n, m\rangle$ be a primitive recursive pairing function with primitive recursive projections $(\langle n_0, n_1\rangle)_i = n_i$, for $0 \leq i \leq 1$. Define

$$G(n) = \langle F(n), F(n+1)\rangle.$$

Then

$$\begin{array}{rcl} G(0) & = & \langle 1, 2\rangle; \\ G(n+1) & = & \langle (G(n))_1, (G(n))_0 + (G(n))_1\rangle = H(G(n)). \end{array}$$

Since $H$ is primitive recursive, so is $G$. Finally

$$F(n) = (G(n))_0$$

and therefore $F$ is primitive recursive.

c) Let $F$ be represented numeralwise by $\phi$. Then

$$F(n) = m \quad \Rightarrow \quad \text{PA} \vdash \phi(\overline{n}, \overline{m}) \qquad (1)$$
$$\text{PA} \vdash \forall x \exists ! y . \phi(x, y) \qquad (2)$$

By (1) it follows that PA $\vdash \phi(\overline{n}, \overline{F(n)})$. By induction one can prove $F(n) > 0$, hence $F(n) < F(n+1)$. Therefore PA $\vdash \overline{F(n)} \neq \overline{F(n+1)}$. Then PA $\vdash \neg\phi(\overline{F(n)}, \overline{F(n+1)})$ by (2).
To give an explicit $\phi$, define

$$\begin{array}{rcll} \phi(n,m) & := & \exists x & [(x)_0 = 1 \ \wedge \ (x)_1 = 2 \wedge \\ & & & [\forall k \leq n . (x)_{k+2} = (x)_k + (x)_{k+1}] \\ & & & \wedge \ (x)_n = m.] \end{array}$$

**Exercise 2**.Define the set of terms $T_x$, with $x = v_0$ as follows.

$$T_x := 0 \mid 1 \mid x \mid T_x + T_x \mid T_x \cdot T_x$$

That is, $T_x$ is the smallest set of terms such that

$$\begin{array}{rcl} & & 0 \in T_x \\ & & 1 \in T_x \\ & & x \in T_x \\ t_1, t_2 \in T_x & \Rightarrow & (t_1 + t_2) \in T_x \\ t_1, t_2 \in T_x & \Rightarrow & (t_1 \cdot t_2) \in T_x \end{array}$$

Let $T$ be the set of all terms of PA and let $T_0$ be the set of closed terms of PA.

a) Show that there is a primitive recursive function $g$ such that for all $t \in T$

$$\begin{array}{rcll} g(\ulcorner t \urcorner) & = & 1, & \text{if } t \in T_x, \\ g(\ulcorner t \urcorner) & = & 0, & \text{if } t \in T - T_x. \end{array}$$

[Hint. There are primitive recursive functions $f^+, f_1^+, f_2^+, f^\cdot, f_1^\cdot, f_2^\cdot, K_T$ such that $n, m < f^+(n,m)$, $n, m < f^\cdot(n,m)$ and

$$\begin{array}{rcll} f_i^+(\ulcorner t_1 + t_2 \urcorner) & = & \ulcorner t_i \urcorner; & \\ f^+(\ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner) & = & \ulcorner t_1 + t_2 \urcorner; & \\ f_i^\cdot(\ulcorner t_1 \cdot t_2 \urcorner) & = & \ulcorner t_i \urcorner; & \\ f^\cdot(\ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner) & = & \ulcorner t_1 \cdot t_2 \urcorner; & \\ T^+(n) & = & 1, & \text{if } n = \ulcorner t_1 + t_2 \urcorner, \text{ for some } t_1, t_2 \in T; \\ & = & 0, & \text{otherwise;} \\ T^\cdot(n) & = & 1, & \text{if } n = \ulcorner t_1 \cdot t_2 \urcorner, \text{ for some } t_1, t_2 \in T; \\ & = & 0, & \text{otherwise.]} \end{array}$$

b) Show that there is a primitive recursive function $E$ such that for all $t \in T_x$ and $n \in \mathbb{N}$

$$e(\ulcorner t \urcorner, n) = (t[\overline{n}/x])^{\mathbb{N}}.$$

For example $e(\ulcorner(x.x)+1\urcorner, 3) = 10$. [Hint. Complete the following definition by cases.

$$\begin{array}{rcll} e(m,n) & = & 0, & \text{if } m = \ulcorner 0 \urcorner; \\ & = & \ldots, & \text{if } m = \ulcorner 1 \urcorner; \\ & = & \ldots, & \text{if } m = \ulcorner x \urcorner; \\ & = & \ldots, & \text{if } m = \ulcorner t_1 + t_2 \urcorner \text{ (use } e(f_i^+(m), n)); \\ & = & \ldots, & \text{if } m = \ulcorner t_1 \cdot t_2 \urcorner; \\ & = & 0, & \text{otherwise.} \end{array}$$

Give an argument why this is primitive recursive.]

c) Show that there is a formula $\psi = \psi(m,n)$ such that for all $t \in T_x$ and $n \in \mathbb{N}$ one has

$$\mathrm{PA} \vdash \psi(\overline{\ulcorner t \urcorner}, \overline{n}) \leftrightarrow (t[\overline{n}/x] = \overline{7}). \qquad (0)$$

[Hint. Let $e$ be numeralwise represented by $E$. Show that

$$\psi(m,n) := E(m,n,\overline{7})$$

works. Show first that for all $t \in T_0$

$$\mathrm{PA} \vdash \overline{t^{\mathbb{N}}} = t. \qquad (1)]$$

**Solution**.

a) Define by a course of value recursion the primitive recursive function

$$\begin{array}{rcll} g(n) & = & 1, & \text{if } n = \ulcorner 0 \urcorner,\ n = \ulcorner 1 \urcorner, \text{ or } n = \ulcorner x \urcorner; \\ & = & g(f_1^+(n)) \cdot g(f_2^+(n)), & \text{if } T^+(n) = 1; \\ & = & g(f_1^\cdot(n)) \cdot g(f_2^\cdot(n)), & \text{if } T^\cdot(n) = 1; \\ & = & 0, & \text{otherwise.} \end{array}$$

Then one can show by course of value induction that for all $t \in T$

$$\begin{array}{rcl} g(\ulcorner t \urcorner) & = & 1 \iff t \in T; \\ g(\ulcorner t \urcorner) & = & 0 \iff t \notin T. \end{array}$$

b) We can define $e$ by course of value primitive recursion

$$\begin{array}{rcll} e(m,n) & = & 0, & \text{if } m = \ulcorner 0 \urcorner; \\ & = & 1, & \text{if } m = \ulcorner 1 \urcorner; \\ & = & n, & \text{if } m = \ulcorner x \urcorner; \\ & = & e(f_1^+(m)) + e(f_2^+(m)), & \text{if } T^+(m); \\ & = & e(f_1^+(m)) \cdot e(f_2^+(m)), & \text{if } T^\cdot(m); \\ & = & 0, & \text{otherwise.} \end{array}$$

The use of $T^+, T^\cdot$ shows why $E$ is primitive recursive.

c) We have for all $m, n$

$$\begin{aligned}&\mathrm{PA} \vdash E(\overline{m}, \overline{n}, \overline{e(m,n)})\\&\mathrm{PA} \vdash \exists! z.E(\overline{m}, \overline{n}, z)\end{aligned}$$

In particular taking $m = \overline{t}$

$$\mathrm{PA} \vdash E(\overline{\ulcorner t \urcorner}, \overline{n}, \overline{e(\ulcorner t \urcorner, n)}) \tag{2.1}$$

$$\mathrm{PA} \vdash \exists! z.E(\overline{\ulcorner t \urcorner}, \overline{n}, z) \tag{2.2}$$

By (b) the following is provable in PA for all $t \in T$ and $n$

$$\begin{aligned}E(\overline{\ulcorner t \urcorner}, \overline{n}, \overline{e(\ulcorner t \urcorner, n)}) &\leftrightarrow E(\overline{\ulcorner t \urcorner}, \overline{n}, \overline{(t[\overline{n}/x])^{\mathbb{N}}})\\&\leftrightarrow E(\overline{\ulcorner t \urcorner}, \overline{n}, t[\overline{n}/x]), \qquad \text{by (1).}\end{aligned}$$

Therefore it follows by (2.1) that

$$\mathrm{PA} \vdash E(\overline{\ulcorner t \urcorner}, \overline{n}, t[\overline{n}/x]). \tag{3}$$

Now we prove (0). As to $\rightarrow$,

$$\begin{aligned}\psi(\overline{\ulcorner t \urcorner}, \ulcorner n \urcorner) &\rightarrow E(\overline{\ulcorner t \urcorner}, \overline{n}, \overline{7}), && \text{by definition,}\\&\phantom{\rightarrow} E(\overline{\ulcorner t \urcorner}, \overline{n}, t[\overline{n}/x]), && \text{by (3),}\\&\rightarrow t[\overline{n}/x] = \overline{7}, && \text{by (2.2).}\end{aligned}$$

As to $(\leftarrow)$,

$$\begin{aligned}t[\overline{n}/x] = \overline{7} &\rightarrow E(\overline{\ulcorner t \urcorner}, \overline{n}, \overline{7}), \quad \text{by (3),}\\&\rightarrow \psi(\overline{\ulcorner t \urcorner}, \ulcorner n \urcorner).\end{aligned}$$

**Exercise 3**. Recall that the notation $\Box\phi$ stands for $\exists x \mathrm{Prf}(x, \overline{\ulcorner \phi \urcorner})$ and that for $\Box$ the following three "derivability conditions" hold:

D1 $\mathrm{PA} \vdash \phi$ implies $\mathrm{PA} \vdash \Box\phi$

D2 $\mathrm{PA} \vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

D3 $\mathrm{PA} \vdash \Box\phi \rightarrow \Box\Box\phi$

Let $G$ be the Gödel sentence, which satisfies $\mathrm{PA} \vdash G \leftrightarrow \neg\Box G$. By the Diagonalisation Lemma let $H$ be a sentence such that

$$\mathrm{PA} \vdash H \leftrightarrow (G \rightarrow \neg\Box H)$$

a) Prove that $\mathrm{PA} \vdash H \leftrightarrow (\Box H \rightarrow \Box\bot)$

b) Prove that $\mathrm{PA} \vdash \Box\neg H \rightarrow H$

c) Show that $H$ is true in the standard model, but not provable in PA [Hint: consider whether or not $\Box H$ is true in the standard model].

d) Show that $G \to H$ is not provable in PA.

Solution: a) By Gödel's Second Incompleteness Theorem we have $\mathrm{PA} \vdash G \leftrightarrow \neg\Box\bot$; hence $\mathrm{PA} \vdash H \leftrightarrow (\neg\Box\bot \to \neg\Box H)$, from which the conclusion follows by contraposition.

b) By Logic, $\mathrm{PA} \vdash \neg H \leftrightarrow (H \to \bot)$. Using D2 and part a) we get

$$\mathrm{PA} \vdash \Box\neg H \leftrightarrow \Box(H \to \bot) \to (\Box H \to \Box\bot) \leftrightarrow H$$

c) Suppose $\Box H$ is true in the standard model. Then $H$ is provable in PA and hence true too. Also, $\neg\Box H$ is false and therefore $G \to \neg\Box H$ is false (since $G$ is true). But this last sentence is equivalent (in PA) to $H$; contradiction.

So $\Box H$ is false and $H$ is not provable in PA. So $\neg\Box H$ is true whence $G \to \neg\Box H$ is true, and therefore $H$ is true.

d) We have the following equivalences in PA:

$$\mathrm{PA} \vdash (G \to H) \leftrightarrow (G \to (G \to \neg\Box H)) \leftrightarrow (G \to \neg\Box H) \leftrightarrow H$$

So if $\mathrm{PA} \vdash G \to H$ then $\mathrm{PA} \vdash H$; quod non, by the previous part.

**Exercise 4**. We consider a nonstandard model $\mathcal{M}$ of PA. Let $F$ and $G$ be two primitive recursive functions, and $\phi_F(x, y)$, $\phi_G(x, y)$ formulas which represent $F$ and $G$ respectively in PA. Let $F^{\mathcal{M}}$ and $G^{\mathcal{M}}$ be the functions on $\mathcal{M}$ such that $\mathcal{M} \models \phi_F(a, F^{\mathcal{M}}(a)) \wedge \phi_G(a, G^{\mathcal{M}}(a))$ for all $a \in \mathcal{M}$.

We say that $F$ is *eventually dominated by* $G$ (notation: $F \preceq G$) if there is a natural number $n$ such that for every natural number $m > n$ we have $F(m) \leq G(m)$.

a) Show that there cannot exist an $L_{\mathrm{PA}}$-formula $\psi(x, y_1, \ldots, y_k)$ and elements $c_1, \ldots, c_k$ of $\mathcal{M}$ such that

$$\mathbb{N} \;=\; \{a \in \mathcal{M} \mid \mathcal{M} \models \psi(a, c_1, \ldots, c_k)\}$$

b) Show that for every $L_{\mathrm{PA}}$-formula $\psi(x, y_1, \ldots, y_k)$ and every $k$-tuple $c_1, \ldots, c_k$ of elements of $\mathcal{M}$ the following two statements are equivalent:

   i) For every standard element $n$ there is a standard element $m > n$ such that $\mathcal{M} \models \psi(m, c_1, \ldots, c_k)$

   ii) For every nonstandard $a \in \mathcal{M}$ there is a nonstandard $b < a$ in $\mathcal{M}$ such that $\mathcal{M} \models \psi(b, c_1, \ldots, c_k)$

c) Show that $F \preceq G$ holds precisely if there is a nonstandard element $c \in \mathcal{M}$ such that for every nonstandard $d < c$ in $\mathcal{M}$ we have $F^{\mathcal{M}}(d) \leq G^{\mathcal{M}}(d)$.

Solution: a) Suppose such $\psi$ and tuple $\vec{c}$ exist. Then $\mathcal{M} \models \psi(0, \vec{c})$ and $\mathcal{M} \models \forall x(\psi(x, \vec{c}) \to \psi(x+1, \vec{c}))$. Because $\mathcal{M}$ satisfies the induction axiom for $\psi$ (with arbitrary free variables!), it follows that $\mathcal{M} \models \forall x \psi(x, \vec{c})$. But this contradicts the assumption, since $\mathcal{M}$ is nonstandard.

Alternatively one might say: if $\psi(n, \vec{c})$ is true in $\mathcal{M}$ for all standard $n$, then by Overspill there must be a nonstandard $d \in \mathcal{M}$ such that $\psi(d, \vec{c})$; contradicting the assumption.

b) i) $\Rightarrow$ ii): suppose i) and, for contradiction, that for some nonstandard $c$ we have that $\mathcal{M} \models \neg\psi(d, \vec{c})$ for all nonstandard $d < c$. Then the formula

$$x < c \wedge \exists y(x < y < c \wedge \psi(y, \vec{c}))$$

defines the standard numbers, contradicting part a).

ii) $\Rightarrow$ i): suppose ii) and, for contradiction, that for some standard $n$ we have that $\mathcal{M} \models \neg\psi(m, \vec{c})$ for all standard $m > n$. Then the formula

$$x \leq n \vee (x > n \wedge \forall y(n < y \leq x \rightarrow \neg\psi(y, \vec{c})))$$

defines the standard numbers, contradicting part a).

c) Let $\psi(x)$ be the formula $\forall yz(\phi_F(x, y) \wedge \phi_G(x, z) \rightarrow y \leq z)$. Then the statement $F \not\preceq G$ is equivalent to: for every standard $n$ there is a standard $m > n$ such that $\mathcal{M} \models \neg\psi(m)$. By part b), this is equivalent to: for every nonstandard $a$ there is a nonstandard $b < a$ such that $\mathcal{M} \models \neg\psi(b)$.

Hence $F \preceq G$ is equivalent to: there is a nonstandard $c$ such that for all nonstandard $d < c$, $\mathcal{M} \models \psi(d)$. That is: there is a nonstandard $c$ such that for every nonstandard $d < c$, $F^{\mathcal{M}}(d) \leq G^{\mathcal{M}}(d)$, as required.