Perhaps the intermediate value theorem and perhaps more

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Let $\mathcal{F}$ be the class of all continuous function f from $[0,1]$ to $\mathbb{R}$ such that $f(0) = 0$ and $f(1) = 1$. Let $\mathcal{I}_0$ be the subclass of $\mathcal{F}$ consisting of all functions f that assume the value $\frac{1}{2}$, that is, there (constructively) exists x in $[0,1]$ such that $f(x) = \frac{1}{2}$.

Constructive mathematicians sometimes are asked to give an example showing that the intermediate value theorem fails, that is, to provide an element of $\mathcal{F}$ that one can not prove to belong to $\mathcal{I}_0$. Very often, the example they offer has the following property:

> *There exists x in $[0,1]$ such that, if $f(x)$ is apart from $\frac{1}{2}$, then f assumes the value $\frac{1}{2}$.*

If f has this property, we want to say that f *perhaps* assumes the value $\frac{1}{2}$ and we denote the class of all such elements of $\mathcal{F}$ by $\mathcal{I}_1$, calling it the *first perhapsive extension* of the class $\mathcal{I}_0$.

We also want to say: *f perhaps perhaps assumes the value $\frac{1}{2}$* if and only if

> *there exists x in $[0,1]$ such that, if $f(x)$ is apart from $\frac{1}{2}$, then f perhaps assumes the value $\frac{1}{2}$.*

The class of all such functions is called $\mathcal{I}_2$, the *second perhapsive extension* of the class $\mathcal{I}_0$. $\mathcal{I}_1$ turns out to be a proper subclass of $\mathcal{I}_2$: there are functions that perhaps perhaps assume the value $\frac{1}{2}$ while we are unable to prove that they perhaps assume the value $\frac{1}{2}$.

We will explain how to continue this process, even into the transfinite, and prove that every perhapsive extension of $\mathcal{I}_0$ is a proper subclass of all later ones.

If time permits, we show that the phenomenon of *perhapsity* occurs in other contexts too. There are *perhaps-finite* sets of natural numbers, *perhaps-rational* real numbers, and *perhaps-countable* subsets of Baire space $\mathcal{N}$, and in all these contexts it makes sense to iterate perhaps.