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Duistermaat, J. J. [Duistermaat, Johannes Jisse]; Kolk, J. A. C.
★**Distributions.**
Theory and applications.
Translated from the Dutch by J. P. van Braam Houckgeest.
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Birkhäuser Boston, Inc., Boston, MA, 2010. *xvi*+445 *pp.* $74.95.
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The aim of this book is to present the theory of distributions of Laurent Schwartz in a rigorous, accessible way, together with applications to linear partial differential equations, Fourier analysis, quantum mechanics and signal analysis. The text of Duistermaat and Kolk emphasizes applications to natural sciences like optics, quantum field theory, signal reconstruction or computer tomography. The exposition stresses applications and interactions with other parts of mathematics. This book evolved from lecture notes for courses taught by the authors at Utrecht University for more than 20 years to students of theoretical physics and mathematics. It is a useful text for students who want to learn the theory of distributions, for experts in the abstract theory who want to read an application-oriented presentation with many examples, and also for mathematicians in related areas or theoretical physicists who need a self-contained presentation of the theory. The book includes many clarifying examples, presented in full detail. Many exercises and problems, with different levels of difficulty, are included at the end of each chapter. A reader who tries to solve them will re-examine many aspects of analysis of several variables in the light of the theory of distributions. Complete solutions of 146 of the 281 problems are provided and hints are given for many others.

Schwartz's distributions are a powerful tool in modern mathematical analysis and especially in partial differential equations. However, the theory sometimes has been misunderstood or undervalued. A possible reason for this could be that the expositions of the theory require familiarity with measure theory and abstract functional analysis. The present authors succeed in presenting the theory in a self-contained way. The reader is only assumed to have a knowledge of linear algebra, analysis of several variables and elementary complex analysis. The authors give precise references to their books [*Multidimensional real analysis. I. Differentiation*, translated from the Dutch by J. P. van Braam Houckgeest, Cambridge Stud. Adv. Math., 86, Cambridge Univ. Press, Cambridge, 2004; MR2121976 (2005i:26001); *Multidimensional real analysis. II. Integration*, translated from the Dutch by J. P. van Braam Houckgeest, Cambridge Stud. Adv. Math., 87, Cambridge Univ. Press, Cambridge, 2004; MR2121977 (2005i:26002)].

We briefly describe the content of the book under review. Chapter 1 is very interesting. It presents several motivating examples of the theory of distribution, coming from different areas of mathematics and physics. The reader will have to go through the rest of the book and return to this first chapter to really enjoy it. The following eight chapters cover the basic theory of distributions: test functions, partitions of unity, the definition of distributions, differentiation and convergence of distributions, Taylor expansion in several variables, localization, distributions with compact support and multiplication by functions.

As the authors remark in their preface, one of the important characteristics of the present treatment of the theory is the systematic use of the operations of pullback and pushforward, presented in Chapter 10 and used consistently in the rest of the book, in particular in the study of convolution of distributions in Chapter 11. Convergence of distributions is defined only for sequences. Seminorms and the elementary theory of locally convex spaces are introduced in Chapter 8. It is important to remark that the amount of functional analysis that is needed is reduced to a minimum; it includes the uniform boundedness principle and the theorem of Hahn and Banach. The latter is mainly used to give alternative proofs.

Chapter 12 starts the study of fundamental solutions of linear partial differential operators with constant coefficients, including a few preliminary examples. Parametrices are also defined and results about the singular support that are needed in Chapter 17 are established. Chapter 13 is very original and contains material not easily found in other texts on distribution theory. It studies the complex powers of the differentiation operator, a concept elaborated by M. Riesz in the 1940's. Riesz's treatment of the wave equation and a fundamental solution of the wave equation are presented, too.

Chapter 14 studies the Fourier transform, the space $S(\mathbb{R}^n)$ of rapidly decreasing functions of Schwartz and tempered distributions. It is one of the longest of the book, and 61 problems are formulated at the end of it. Distribution kernels and their applications are treated in Chapter 15. This chapter also includes a proof of the kernel theorem of Schwartz using the Fourier inversion formula. The theory of Fourier series is reconstructed in Chapter 16 from the theory of Fourier transform. A proof of the Poisson summation formula is given.

The first part of Chapter 17 studies elliptic partial differential operators with constant coefficients. The main point of the chapter is the theorem of Malgrange and Ehrenpreis showing that every linear partial differential operator $P(D)$ with constant coefficients has a fundamental solution. It is presented in Theorem 17.13 with an efficient and explicit proof due to N. Ortner and P. Wagner (1996). We recommend to the reader the more recent, even simpler proof by Wagner in [Amer. Math. Monthly **116** (2009), no. 5, 457–462; MR2510844 (2010b:35007)]. Another proof of the Malgrange-Ehrenpreis theorem, based on the Paley-Wiener-Schwartz theorem for distributions with compact support, is given in Chapter 18. Several explicit formulas of fundamental solutions of certain operators are included in Chapter 17.

Chapter 18 discusses supports and the Fourier transform. It includes a study of hyperbolic operators, it gives the proof of the sampling theorem of Whittaker, Kotelnikov and Shannon and it finishes with some comments on Paley-Wiener spaces. A short introduction to Sobolev spaces $H^s(\mathbb{R}^n)$ is given in Chapter 19. A reader interested in Sobolev spaces would find here only an appetizer. In the appendix, Chapter 20, the authors survey the theory of Lebesgue integration with respect to a measure with the point of view of Daniell. This approach studies measures as linear forms and it is appropriate in the present context. The Riesz representation theorem is proved in detail.

In summary, this is a useful, well-written, self-contained book about the theory of distributions of Schwartz and applications to different areas of mathematics and physics. Although there are many good books that present an introduction to the theory of distribution of Schwartz, assuming different levels of knowledge of linear functional analysis, the text of Duistermaat and Kolk is a very welcome addition. In the opinion of this reviewer, this is a stimulating, informative book that invites further study, and I strongly recommend it to any readers interested in this topic. *José Bonet*