# Spectra of Compact Locally Symmetric Manifolds of Negative Curvature 

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## 1. Introduction

Let $S$ be a Riemannian symmetric space of noncompact type, and let $G$ be the group of motions of $S$. Then the algebra $\mathscr{L}_{\text {Diff }}(S)$ of $G$-invariant differential operators on $S$ is commutative, and its spectrum $A(S)$ can be canonically identified with $\mathscr{F} / \mathrm{w}$ where $\mathscr{F}$ is a complex vector space with dimension equal to the rank of $S$, and $\mathfrak{w}$ is a finite subgroup of $\operatorname{GL}(\mathscr{F})$ generated by reflexions. Let $\Gamma$ be a discrete subgroup of $G$ that acts freely on $S$ and let $X=\Gamma \backslash S$. Then the members of $\mathscr{L}_{\text {Diff }}(S)$ may be regarded as differential operators on $X$. Let us now assume that $X$ is compact and define the spectrum $\Lambda$ of $X$ as the set of those elements of $\Lambda(S)$ for which one can find a nonzero eigenfunction defined on $X$. In this paper we study the relationship of $A$ to the geometry of $X$ and determine the asymptotic growth of $\Lambda$ as a subset of $\Lambda(S)$. In subsequent papers we plan to study the asymptotic behaviour of the eigenfunctions and to examine the problem of obtaining improvements on the error estimates.

It is well-known that $G$, which is transitive on $S$, is a connected real semisimple Lie group with trivial center, and that the stabilizers in $G$ of the points of $S$ are the maximal compact subgroups of $G$. So we can take $S=G / K, X$ $=\Gamma \backslash G / K$, where $K$ is a fixed maximal compact subgroup of $G$, and $\Gamma$ is a discrete subgroup of $G$ containing no elliptic elements ( $=$ elements conjugate to an element of $K$ ) other than $e$, such that $\Gamma \backslash G$ is compact. Let $G=K A N$ be an Iwasawa decomposition of $G$; let $\mathfrak{a}$ be the Lie algebra of $A$; and let $\mathfrak{w}$ be the Weyl group of $(G, A)$. If we take $\mathscr{F}$ to be the dual of the complexification $\mathfrak{a}_{c}$ of $\mathfrak{a}$, then $\Lambda(S) \approx \mathscr{F} / \mathfrak{w}$ canonically. In what follows we shall commit an abuse of notation and identify $\Lambda(S)$ with $\mathscr{F}$, but with the proviso that points of $\mathscr{F}$ in the same w-orbit represent the same element of $\Lambda(S)$.

[^0]For the $L^{2}$-eigenvalue problem on $S$ determined by $\mathscr{L}_{\text {Diff }}(S)$ the spectrum is the $\mathbb{R}$-linear subspace $\mathscr{\mathscr { F }}_{I}=i \mathrm{a}^{*}$; the spectral multiplicity is always 1 ; and the spectral measure is of the form $\beta d v$, where $d v$ is a Lebesgue measure on $\mathscr{F}_{I}$, and $\beta$ is a nonnegative $\mathfrak{w}$-invariant smooth function on $\mathscr{\mathscr { F }}_{I}$, which, together with each of its derivatives, is of at most polynomial growth on $\mathscr{F}_{1}$. However, for the spectrum $\Lambda$ of $X$ we always have $\Lambda \nmid \mathscr{F}_{I}$; this leads to a natural splitting of $\Lambda$ as a union of the principal spectrum $\Lambda_{p}=\Lambda \cap \mathscr{F}_{I}$ and the complementary spectrum $\Lambda_{c}$ $=\Lambda \backslash \mathscr{F}_{I}$.

The main result of this paper may be divided naturally into two parts. The first part is concerned with the connection between $A$ and the geometry of $X$, and is treated in Sections 2-5. To every $\Gamma$-conjugacy class $c$ different from the class $[e]_{\Gamma}$ we associate in a natural manner a tempered $\mathfrak{w}$-invariant distribution $T_{c}$ on the vector group $A$; its construction is an application of Harish-Chandra's theory of harmonic analysis on semisimple Lie groups. We then prove (Theorem 5.1) that the distributions $T_{c}$ and the multiplicities $m(\lambda)(\lambda \in \Lambda)$ are related by the following identity of distributions:

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) \eta_{\lambda}=\operatorname{vol}(X)|\boldsymbol{w}|^{-1} \widehat{\beta}+\sum_{c \neq\left[[]_{\Gamma}\right.} v_{c} T_{c} . \tag{1.1}
\end{equation*}
$$

Here, $\eta_{\lambda}=e^{\lambda_{0} \log }, \log (A \rightarrow \mathfrak{a})$ being the inverse of $\exp (\mathfrak{a} \rightarrow A) ; \hat{\beta}$ is the Fourier transform of $\beta$, and $v_{\mathrm{c}}$ is the volume $\operatorname{vol}\left(\Gamma_{\gamma} \backslash G_{\gamma}\right.$ ), where $\gamma \in c$ and $G_{\gamma}$ (resp. $\Gamma_{\gamma}$ ) is the centralizer of $\gamma$ in $G$ (resp. $\Gamma$ ). It is clear that (1.1) is the analogue, for $X$, of the classical Poisson Summation formula; it is a consequence of the Selberg Trace formula and the Harish-Chandra theory of integrals over $G$-conjugacy classes.

To the $\Gamma$-conjugacy class $c \neq[e]_{\Gamma}$ corresponds a free homotopy class of loops in $X$. The closed geodesics in this class all have the same length $l(c)$ and form a compact manifold $F(c)$ in the tangent bundle $T(X) \cong T^{*}(X)$; the distribution $T_{c}$ is intimately related to $F(c)$. For instance, the support of $T_{\mathrm{c}}$ is contained in the union, of the images under $w$, of an affine subspace ${ }^{*} L(c)$ of $A$, at a distance $l(c)$ from 1 ; if the elements of $c$ are regular in $G, T_{c}$ is even a smooth function on $\bigcup_{s \in \boldsymbol{w}} s \cdot{ }^{*} L(c)$ (Theorem 5.2). When $c$ is such that the $G$-conjugacy class in which it
is contained passes through an element of $M A$ with component $h_{R}$ in $A$ which is regular in $A, F(c)$ reduces to a torus, and $T_{c}$ comes out to be equal to the sum of Dirac measures

$$
\begin{equation*}
|\mathfrak{w}|^{-1}\left|\operatorname{det}\left(I-P_{\gamma}\right)^{\#}\right|^{-\frac{1}{-1}} \sum_{s \in \mathfrak{w}} \delta_{\text {shr }}, \tag{1.2}
\end{equation*}
$$

where $P_{\gamma}$ is the linear Poincaré map along the element $\gamma$ of $c$ (Proposition 5.16).
It is clear from these remarks that the distributions $T_{c}$ are closely related to the singularities of the distribution $\hat{\sigma}$ studied by Duistermaat-Guillemin [6]. The precise contribution of the $T_{c}$ to the asymptotic expansions in Theorem 4.5 of Duistermaat-Guillemin (for the Laplace-Beltrami operator on $X$ ) can be determined easily once $F(c)$ and the singularities of $T_{c}$ are analysed in detail. In Section 5 we do this completely for the classes $c$ of the type mentioned above; for other classes we indicate some partial results. We hope to treat the general case in a subsequent article.

In the special case when $\operatorname{rk}(S)=1$, all classes $c \neq[e]_{\Gamma}$ are of the type mentioned earlier; then $F(c)$ reduces to a single geodesic of length $l(c)$, and the Poisson formula (1.1) then takes the form, with $A \simeq \mathbb{R}$,

$$
\begin{align*}
\sum_{\lambda \in A} m(\lambda) e^{\lambda}= & \frac{1}{2} \operatorname{vol}(X) \hat{\beta} \\
& +\frac{1}{2} \sum_{c \neq[c]_{\Gamma}} l_{0}(c)\left|\operatorname{det}\left(I-P_{\gamma}\right)^{\#}\right|^{-\frac{1}{2}}\left(\delta_{l(c)}+\delta_{-l(c)}\right), \tag{1.3}
\end{align*}
$$

here $l_{0}(c)$ is the primitive length corresponding to $c$ (Theorem 5.17). If we take $G$ $=S L(2, \mathbb{R})$ so that $S$ is the upper-half plane, this specializes to a formula of LaxPhillips [29] and Randol [37].

We now turn to a description of the second part of our results. Our aim here is a determination of the asymptotic growth of $\Lambda$; and the results that we get are in Sections 7 and 8. The starting point is the remark that there is some neighborhood of 1 in $A$ that does not meet the support of any of the distributions $T_{c}$. Consequently the Poisson formula (1.1) becomes

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) \hat{f}(\lambda)=\operatorname{vol}(X)|\mathfrak{w}|^{-1}\langle\beta, \hat{f}\rangle, \tag{1.4}
\end{equation*}
$$

for all $f \in C_{c}^{\infty}(V), V$ being a sufficiently small $\mathfrak{w}$-invariant neighborhood of 1 in $A$.

Roughly speaking, formula (1.4) says that the measure $\operatorname{vol}(X)|\mathfrak{w}|^{-1} \beta d v$ and the spectral measure which assigns to $\lambda \in A$ its multiplicity $m(\lambda)$, have "Fourier transforms" that coincide around the origin. General principles of Fourier analysis would then lead us to expect that these two measures should be asymptotically equal to each other at infinity on $\mathscr{F}_{I}$. However, it does not seem easy to deduce such a result from classical Tauberian theorems; not only are we dealing with a multidimensional case here, but the situation is further complicated by the presence of the complementary spectrum $\Lambda_{c}=\Lambda \backslash \mathscr{F}_{I}$.

A basic step in our treatment of this problem is to obtain an estimate for the number (counted with multiplicities) of points of $\Lambda$ lying in a ball of radius $t$ in $\mathscr{F}$ with center at a variable point $\mu \in \mathscr{F}_{I}$. We do this by using in (1.4) test functions $f$ whose Fourier transforms are $\geqq 0$ on $A$ and whose absolute values are $\geqq 1$ on large balls in $\mathscr{F}$. The resulting estimate is contained in Theorem 7.3 and asserts that the number in question is majorized by const $t^{N} \tilde{\beta}(\mu)$ where $\tilde{\beta}$ is a "smoothed out" version of $\beta$ defined by (6.11). Now, as $\|\mu\| \rightarrow+\infty, \tilde{\beta}(\mu)$ is $O\left(\|\mu\|^{n-r}\right)(n=\operatorname{dim} X, r=\operatorname{dim} A)$; but if $\mu$ varies only on a subspace of $\mathscr{\mathscr { F }}_{I}$ where $m$ positive roots vanish, $\tilde{\beta}(\mu)=O\left(\|\mu\|^{n-r-m}\right)$, basically because $\beta$ vanishes to the order $m$ at such $\mu$. Since the imaginary parts of the points of $\Lambda_{c}$ are of this type, it is not too difficult to argue that the number of points of $\Lambda_{c}$ in a ball in $\mathscr{F}$ around the origin of radius $t$ is $O\left(t^{n-d-1}\right)$, where $d$ is a certain integer $\geqq 1$. Without extra conditions, this estimate is sharp and it makes precise the heuristic remark that $\Lambda_{c}$ is negligible in comparison with $\Lambda_{p}$, because known results on the Laplace-Beltrami operator already imply that the number of points of $\Lambda$ in a ball in $\mathscr{F}$ of radius $t$ around the origin is $\sim$ const $t^{n}$.

One can now deal with the principal spectrum. Our main results are Theorems 8.5 and 8.8. Theorem 8.5 is a very general one which asserts that for
any bounded measurable subset $\Omega$ in $\mathscr{F}_{I}$ the number of spectral points $\lambda$ with $\lambda_{I} \in \Omega$ can be approximated by a constant $\times \int_{\Omega} \beta(v) \mathrm{d} v$, with an error that is majorized by the integral of $\beta$ over a neighborhood of the boundary of $\Omega$, of arbitrary but fixed size. We regard this result as a definitive formulation of the spectral information which is contained in formula (1.4). However, in this result the error term cannot be made small relative to the main term $\int_{\Omega} \beta(v) d v$, unless $\Omega$ is "big and fat".

Theorem 8.8 describes one particular result of this type. It follows from this theorem that if $\Omega$ is any bounded open subset of $\mathscr{F}_{I}$ with smooth boundary,

$$
\begin{equation*}
\sum_{\lambda \in A_{p} \cap(t \Omega)} m(\hat{\lambda})=\text { const } \int_{I \Omega} \beta(v) \mathrm{d} v+O\left(t^{n-1}\right) \quad(t \rightarrow+\infty) \tag{1.5}
\end{equation*}
$$

Here the constant, which depends only on $X$ and not on $\Omega$, can be explicitly determined; and the main term in (1.5) is $\sim$ const $t^{n}$.

In the last section we take a closer look at the case when $\mathrm{rk}(S)=1$. In this case, the simplicity of the Poisson formula (1.3) makes it possible to work with it rather than the truncated version (1.4). As a result we are able to show that the error term in (1.5) is even $O\left(t^{n-1} / \log t\right.$ ), when $t \rightarrow+\infty$ (Theorem 9.1) (cf. also Kolk [27, Proposition 5]). It must be remarked however that when $\operatorname{rk}(S)=1$, $\mathscr{L}_{\text {Diff }}(S)$ is the algebra generated by the Laplace-Bertrami operator $\Delta$, and the above result can be obtained also by putting together the results of Bérard [2], Hejhal [23] and Randol [38].

In order to carry out the proofs of the results described above it is necessary to make full use of the theory of harmonic analysis on semisimple Lie groups. Sections 2-4 describe briefly the aspects of this theory that are needed for our purposes, with some variations at suitable places.

We wish to point out that the suggestion for considering higher dimensional spectra of algebras of differential operators on compact locally symmetric spaces of negative curvature, and for studying it in the group theoretic framework, seems to have appeared first in Selberg's article [40] (p. 68), and later, in more detail, in the Stockholm address of Gel'fand [11]. This address also contained the indication that the spectrum of $X$ grew just like $\beta$ at infinity. The first systematic use of the Harish-Chandra theory in this problem goes back to Gangolli [7] who treated the asymptotics of the Laplace-Beltrami operator on $X$, but worked under the additional assumption that the group $G$ was actually complex. Since then various authors have taken up the group theoretic view, but always only for the Laplace operator, cf. De George-Wallach [5], Gangolli [9], Gangolli-Warner [10], Wallach [43], [44]. The first multidimensional treatment is due to Kolk in his Utrecht dissertation [28]. Kolk's results, which were announced in the note [27], form the point of departure for the present work. For multidimensional spectra in another context see the recent work of Colin de Verdière [4].

We have made a real effort throughout this paper to calculate explicitly all the constants that appear in the various formulae. In addition to providing various internal checks, this has made possible a very detailed comparison of our theory with other known results.

## 2. Eigenfunctions and Spectra for Locally Homogeneous Spaces

In this section and the next we introduce our basic framework. Almost all the results presented here are well-known and, in the context of semisimple Lie groups, due essentially to Harish-Chandra. However, as our point of view emphasizes distributions rather than representation theory, it will be convenient to describe these results in the context and form most suitable for us.
2.1. Generalities. For any $C^{\infty}$ manifold $M$ we topologize the spaces of test functions $C_{c}^{\infty}(M)$ and $C^{\infty}(M)$ in the usual manner. The space $\mathscr{D}^{\prime}(M)$ (resp. $\mathscr{E}^{\circ}(M)$ of distributions (resp. with compact supports) is the dual of $C_{c}^{\infty}(M)$ (resp. $C^{\infty}(M)$ ). The diffeomorphisms of $M$ act on these spaces by transport of structure.

Let $G$ be a Lie group with bi-invariant Haar measure $d x$. As usual we identify a locally integrable function $f$ on $G$ with the distribution $f d x$. Let $l(x)$ (resp. $r(x)$ ) be the left (resp. right) translation by $x \in G$ and let ${ }^{\vee}$ be the involution $x \mapsto x^{-1}(x \in G)$. For $f, g \in C_{c}^{\infty}(G), f * g$ denotes their convolution. We extend the convolution by duality to apply to (suitably restricted) distributions on $G$; in particular, $T * U$ is meaningful whenever either $T$ or $U$ belongs to $\mathscr{E}^{\prime}(G) . \mathscr{E}^{\prime}(G)$ is an algebra under $*$ with $\delta_{e}$, the Dirac measure at the identity $e$ of $G$, as its unit. If $H \subset G$ is a closed subgroup, we write $\mathscr{E}_{H}^{\prime}(G)$ for the subalgebra of distributions on $G$ whose supports are compact and contained in $H$.

We identify $\mathscr{E}^{\prime}(G)$ with the algebra $\mathscr{L}(G)$ of all continuous endomorphisms of $C^{\infty}(G)$ that commute with all left translations, by the correspondence $T \mapsto \Phi_{T}\left(\Phi_{T} f=f * \check{T}\right)$. This isomorphism maps $\mathscr{E}_{e}^{\prime}(G)$ onto the algebra of all left invariant differential operators on $G$. The latter algebra is identified as usual with the universal enveloping algebra $U\left(\mathfrak{g}_{c}\right)$ of $\mathfrak{g}_{c}$, the complexification of the Lie algebra $g$ of $G$.

For any closed subgroup $H \subset G$ we have the embedding $C^{\infty}(G / H) \hookrightarrow C^{\infty}(G)$; and if $H=K$ is compact, we also have the projection $P_{K}: C^{\infty}(G) \rightarrow C^{\infty}(G / K)$ given by $\left(P_{K} f\right)(x)=\int_{K} f(x k) d k(x \in G)$. Here $d k$ is the Haar measure on $K$ normalized by $\int_{K} d k=1$. Let $\mathscr{L}(G / K)$ be the algebra of continuous endomorphisms of $C^{\infty}(G / K)$ that commute with the action of $G$; then $\Psi \mapsto \Psi \circ P_{K}$ gives an embedding $\mathscr{L}(G / K) \rightarrow \mathscr{L}(G)$. Clearly

$$
\begin{equation*}
\mathscr{L}(G / K) \approx \mathscr{E}^{\prime}(G / / K), \tag{2.1}
\end{equation*}
$$

where $\mathscr{E}^{\prime}(G / / K)$ is the subring of all $T \in \mathscr{E}^{\mathscr{E}^{\prime}(G) \text { which are } K \text {-bi-invariant. } \mathscr{E}^{\prime}(G / / K), ~(2)}$ contains

$$
C_{c}^{\infty}(G / / K), \quad \text { resp. } \mathscr{E}_{K}(G / / K)
$$

which correspond to the subring in $\mathscr{L}(G / K)$ of smoothing operators, resp. the subalgebra of differential operators. Denoting the latter by $\mathscr{L}_{\text {Diff }}(G / K)$, there is a natural homomorphism

$$
\begin{equation*}
U\left(\mathfrak{g}_{c}\right)^{K} \rightarrow \mathscr{L}_{\text {Diff }}(G / K) \approx \mathscr{E}_{K}^{\prime}(G / / K) . \tag{2.3}
\end{equation*}
$$

It is actually surjective and, if $\mathfrak{f}$ denotes the Lie algebra of $K$, its kernel is

$$
\begin{equation*}
U\left(\mathfrak{g}_{c}\right)^{K} \cap\left(U\left(\mathfrak{g}_{c}\right) \mathfrak{f}\right)=U\left(\mathfrak{g}_{c}\right)^{K} \cap\left(\mathfrak{f} U\left(\mathfrak{g}_{c}\right)\right) . \tag{2.4}
\end{equation*}
$$

If $\mathfrak{s}$ is any $\operatorname{Ad}(K)$-stable subspace of $\mathfrak{g}$ complementary to $\mathfrak{f}$, and if $\lambda$ is the usual symmetrizer map that goes from the symmetric algebra $S\left(\mathfrak{g}_{c}\right)$ to $U\left(\mathfrak{g}_{c}\right)$, it is not difficult to verify that (2.3) is a linear bijection of $U\left(\mathfrak{g}_{\mathrm{c}}\right)^{K} \cap \lambda(S(\mathfrak{s}))$ onto $\mathscr{L}_{\text {Diff }}(G / K)$.

We make with Gel'fand the following classical assumption: $G$ has a closed Abelian subgroup $A$ and an involutive automorphism $\theta$ such that

$$
\begin{equation*}
G=K A K ; \quad \mathrm{A}^{\theta}=\mathrm{A} ; \quad a^{\theta}=a^{-1}(a \in A) \tag{2.5}
\end{equation*}
$$

Under these circumstances $\check{T}=T^{\theta}\left(T \in \mathscr{E}^{\prime}(G / / K)\right)$, and so $\mathscr{E}^{\prime}(G / / K)$ is a commutative algebra.
$\mathscr{E}^{\prime}(G / / K)$ acts on $\mathscr{D}^{\prime}(G / K)\left(\equiv \mathscr{D}^{\prime}(G)^{r(K)}\right)$ by convolution from the right. Observe that $\mathscr{L}_{\text {Diff }}(G / K)$ contains nonzero elliptic elements; in fact, if (.,.) is a positive definite $\operatorname{Ad}(K)$-invariant scalar product on $\mathfrak{s}$, then, for any orthonormal basis $\left\{Z_{1}\right\}_{1 \leqq i \leqq n}$ of $\mathfrak{s}$,

$$
\begin{equation*}
\omega_{S}=Z_{1}^{2}+\ldots+Z_{n}^{2} \in U\left(\mathfrak{g}_{c}\right)^{K} \tag{2.6}
\end{equation*}
$$

defines such an element via (2.3). $\omega_{s}$ induces the Laplace-Beltrami operator on $G / K$, assuming the Riemannian structure induced by (.,.).

Proposition 2.1. (i) Let $u \in \mathscr{D}^{\prime}(G / K)$ and $\chi$ a homomorphism $\mathscr{E}_{K}^{\prime}(G / / K) \rightarrow \mathbb{C}$ such that $u * \stackrel{T}{T}=\chi(T) u\left(T \in \mathscr{E}_{K}^{\prime}(G / / K)\right)$, then $u$ is an analytic function. (ii) Suppose that $A$ is connected. If $\chi_{K}: \mathscr{E}_{K}^{\prime}(G / / K) \rightarrow \mathbb{C}$ is any homomorphism, the subspace $F\left(\chi_{K}\right) \subset C^{\infty}(G / / K)$ of all $\varphi$ such that $\varphi * \check{T}=\chi_{K}(T) \varphi$ for all $T \in \mathscr{E}_{K}^{\prime}(G / / K)$ is at most onedimensional. It has dimension 1 if and only if $\chi_{K}$ extends to a continuous homomorphism $\chi: \mathscr{E}^{\prime}(G / / K) \rightarrow \mathbb{C}$; such an extension is necessarily unique, and the elements of $F\left(\chi_{X}\right)$ remain eigenfunctions for $\mathscr{E}^{\prime}(G / / K)$ with $\chi$ as the corresponding eigenhomomorphism.

Assertion (i) is clear using (2.6) and the classical Regularity Theorem. We note that the map $\varphi \rightarrow \varphi(e)$ is injective on $F\left(\chi_{K}\right)$ (cf. Varadarajan [42, II, Proposition 8.2 (ii)]). Suppose $\varphi \in F\left(\chi_{K}\right)$ and $\varphi(e)=1$, then $\chi_{K}(T)=(\varphi * \bar{T})(e)$ so that $\chi_{K}$ is continuous. If now $f \in C_{c}^{\infty}(G / / K), T \in \mathscr{E}_{K}^{\prime}(G / / K)$, then $\varphi * f^{\prime} * \widetilde{T}=\varphi * \widetilde{T} * \bar{f}$, which implies $\varphi * \hat{f} \in F\left(\chi_{K}\right)$. Hence $\varphi * f=\chi(f) \varphi$ for some $\chi(f) \in \mathbb{C}$. $\chi$ is continuous on $C_{c}^{\infty}(G / / K)$, and so extends to a continuous homomorphism $\chi$ : $\mathscr{E}^{\prime}(G / / K) \rightarrow \mathbb{C}$ which is an extension of $\chi_{K}$. Since $C^{\infty}(G / / K)$ is reflexive there is a unique $\psi \in C^{\infty}(G / / K)$ such that $\chi(T)=\langle T, \psi\rangle\left(T \in \mathscr{E}^{\prime}(G / / K)\right)$, while $\psi(e)=\langle d k, \psi\rangle$ $=\chi(d k)=1$. It is not hard to prove that $\varphi=\psi$, using the onedimensionality of $F\left(\chi_{K}\right)$. If $\chi^{\prime}$ is another continuous homomorphism of $\mathscr{E}^{\prime}(G / / K)$ into $\mathbb{C}$ that extends $\chi_{K}$ and $\psi^{\prime} \in C^{\infty}(G / / K)$ is such that $\chi^{\prime}(T)=\left\langle T, \psi^{\prime}\right\rangle\left(T \in \mathscr{E}^{\prime}(G / / K)\right)$, a similar argument gives $\psi^{\prime}=\psi$, and thus $\chi^{\prime}=\chi$.

Write $S=G / K$ and define the formal spherical spectrum $\Lambda(S)$ of $S$ (or $G$ ) by

$$
\begin{equation*}
A(S)=\left\{\chi: \chi \text { continuous homomorphism: } \mathscr{E}^{\prime}(G / / K) \rightarrow \mathbb{C}\right\} \tag{2.7}
\end{equation*}
$$

From Proposition 2.1 (ii) we get a bijection $\chi \leftrightarrow \varphi_{\chi}$ between $\Lambda(S)$ and the set of all $\varphi=\varphi_{\chi} \in C^{\infty}(G / / K)$ satisfying

$$
\begin{equation*}
\varphi(e)=1 ; \quad \chi(T)=\langle T, \varphi\rangle, \varphi * \check{T}=\chi(T) \varphi \quad\left(T \in \mathscr{E}^{\prime}(G / / K)\right) . \tag{2.8}
\end{equation*}
$$

2.2. Let $\Gamma \subset G$ be a discrete subgroup. We assume always that $\Gamma$ acts freely on $G / K$, so that

$$
\begin{equation*}
X=\Gamma \backslash G / K=\Gamma \backslash S \tag{2.9}
\end{equation*}
$$

is a manifold. The measures $d x$ on $G, d k$ on $K$ and the counting measure on $\Gamma$ induce a measure $d \bar{x}$ on $X$, and we write $L^{2}(X)$ for $L^{2}(X, d \bar{x})$. Since $\mathscr{D}^{\prime}(X) \approx \mathscr{D}^{\prime}(G / K)^{l(I)}, \mathscr{E}^{\prime}(G / / K)$ acts on $\mathscr{D}^{\prime}(X)$. From now on we assume that $X$ is compact, and we have the following well-known result.
Proposition 2.2. (i) For any $f \in C_{c}^{x}(G / / K), R(f): u \mapsto u * f\left(u \in L^{2}(X)\right)$, is an integral operator in $L^{2}(X)$ whose kernel $K_{f} \in C^{\infty}(X \times X)$ is given by

$$
\begin{equation*}
K_{f}\left(\bar{x}, \bar{x}^{\prime}\right)=\sum_{\gamma \in \Gamma} f\left(x^{-1} \gamma x^{\prime}\right)\left(x, x^{\prime} \in G, \bar{x}=\Gamma x K, \bar{x}^{\prime}=\Gamma x^{\prime} K\right) . \tag{2.10}
\end{equation*}
$$

(ii) All the $R(f)$ are operators of trace class, and

$$
\begin{equation*}
\operatorname{tr}(R(f))=\int_{X} K_{f}(\bar{x}, \bar{x}) d \bar{x} \quad\left(f \in C_{c}^{\infty}(G / / K)\right) \tag{2.11}
\end{equation*}
$$

(iii) If $\tilde{f}(x)=\tilde{f}(x)^{\operatorname{conj} j}$, then $R(\tilde{f})=R(f)^{\dagger}\left({ }^{\dagger}\right.$ denotes adjoint $)$. The identity operator lies in the strong closure of $R\left(C_{c}^{\infty}(G / / K)\right)$.

As before any eigendistribution on $X$ for $\mathscr{E}^{\prime}(G / / K)$ is an analytic function and the eigenhomomorphism is continuous. We put

$$
\begin{equation*}
\Lambda=\Lambda(X)=\{\chi: \chi \in \Lambda(S), \exists \text { nonzero eigenfunction on } X \text { for } \chi\} \tag{2.12}
\end{equation*}
$$

For any $\chi \in A$, let $C^{\infty}(X: \chi)$ be the corresponding space of eigenfunctions.
Proposition 2.3. (i) $\Lambda$ is nonempty, viz., if $\varepsilon(T)=\langle T, 1\rangle\left(T \in \mathscr{E}^{\prime}(G / / K)\right)$, then $\varepsilon \in \Lambda$ and $C^{\infty}(X: \varepsilon)=\mathbb{C} \cdot 1$. (ii) For each $\chi \in A$, the space $C^{\infty}(X: \chi)$ of $u \in C^{\infty}(X)$ satisfying $u * \check{T}=\chi(T) u\left(T \in \mathscr{E}^{\prime}(G / / K)\right)$ is finite dimensional. The subspaces $C^{\infty}(X: \chi)$ are mutually orthogonal in $L^{2}(X)$ and $L^{2}(X)$ is the orthogonal direct sum of the $C^{\infty}(X: \chi)$. In particular, $A$ is at most denumerable.

Assertion (i) is obvious. Ad(ii), we decompose $L^{2}(X)$ for the action of $C_{c}^{\infty}(G / / K)$ by means of Proposition 2.2 (iii). The eigenhomomorphisms $\chi: C_{c}^{\infty}(G / / K) \rightarrow \mathbb{C}$ are extendible to eigenhomomorphisms $\chi: \mathscr{E}^{\prime}(G / / K) \rightarrow \mathbb{C}$ by continuity, and they are continuous themselves.
Proposition 2.4. For any $\chi \in A$ we have: (i) $\chi(\tilde{T})=\chi(T)^{\mathrm{conj}}\left(T \in \mathscr{E}^{\prime}(G / / K), \tilde{T}=(\check{T})^{\mathrm{conj}}\right)$; (ii) $\varphi=\varphi_{\chi}$ (cf. (2.8)) is positive definite, i.e., $\langle T * \tilde{T}, \varphi\rangle \geqq 0\left(T \in \mathscr{E}^{\prime}(G)\right)$. In particular,

$$
\begin{equation*}
\tilde{\varphi}=\varphi ; \quad|\varphi(x)| \leqq 1 \quad(x \in G) \tag{2.13}
\end{equation*}
$$

For proving (ii) we go over to $L^{2}(\Gamma \backslash G)$. The action of $G$ on $L^{2}(\Gamma \backslash G)$ extends to an action of $C_{c}^{\infty}(G) \quad(f \mapsto \tilde{R}(f))$. Suppose now that $\chi \in \Lambda$ and choose $u \in C^{\infty}(X ; \chi)$ such that $(u, u)=1$, and regard $u$ as in $L^{2}(\Gamma \backslash G)$. Then

$$
\begin{equation*}
\tilde{R}(f) u, u)=\langle f, \varphi\rangle\left(f \in C_{c}^{\infty}(G)\right) . \tag{2.14}
\end{equation*}
$$

This follows from the easily established fact that the distribution $\Phi: f \mapsto\langle\tilde{R}(f) u, u\rangle$ on $G$ satisfies $\Phi * \check{h}=\chi(h) \Phi\left(h \in C_{c}^{\infty}(G / / K)\right)$. But (2.14) implies $\langle f * \tilde{f}, \varphi\rangle=\left(\tilde{R}(f) \tilde{R}(f)^{\dagger} u, u\right)=\left\|\tilde{R}(f)^{\dagger} u\right\|^{2} \geqq 0$, for all $f \in C_{c}^{\alpha c}(G)$.
2.3. A Trace Formula. In rough terms we can say that our interest is in studying the distribution of the points of $\Lambda$ (cf. (2.12)) within $\Lambda(S)$ (cf. (2.7)). Our basic tool for this purpose is the Selberg Trace Formula.

Let $f \in C_{c}^{\infty}(G / / K)$, and let $R(f)$ be the operator defined in Proposition 2.2 (i). Then $R(f)$ acts on $C^{\infty}(X: \chi)$ as the scalar $\chi(f)$. So, if

$$
\begin{equation*}
n(\chi)=\operatorname{dim}\left(C^{\infty}(X: \chi)\right) \quad(\chi \in \Lambda) \tag{2.15}
\end{equation*}
$$

then $\operatorname{tr}(R(f))=\sum_{\chi \in A} n(\chi) \cdot \chi(f)$, the series being absolutely convergent. On the other hand, using Proposition 2.2,

$$
\operatorname{tr}(R(f))=\int_{X} K_{f}(\bar{x}, \bar{x}) d \bar{x}=\int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f\left(x^{-1} \gamma x\right) d \tilde{x},
$$

where $d \tilde{x}$ is the $G$-invariant measure on $\Gamma \backslash G$. The integral on the right can be further simplified following Selberg's classical argument. First we have the following lemma (see Mostow [33, Lemma 8.1]).
Lemma 2.5. (i) For any $\gamma \in \Gamma$, the G-conjugacy class $[\gamma]_{G}$ is closed. More generally, for any subset $\Gamma^{\prime} \subset \Gamma, \bigcup_{\gamma \in \Gamma^{\prime}}[\gamma]_{G}$ is a closed subset of $G$. (ii) If, for $\gamma \in \Gamma$, we write $G_{\gamma}$ (resp. $\Gamma_{\gamma}$ ) for the centralizer of $\gamma$ in $G$ (resp. $\Gamma$ ), then $\Gamma_{\gamma} \backslash G_{\gamma}$ is compact. (iii) $A$ compactum in $G$ meets only finitely many $[\gamma]_{G}$.

Let us now make the assumption that for each $\gamma \in \Gamma, \mathbf{G}_{y}$ is unimodular. Since we have fixed a Haar measure $d x$ on $G$, it follows that once a Haar measure on $G_{\gamma}$ is chosen, we have a uniquely determined $G$-invariant measure $d_{\gamma} \bar{x}$ on $G_{\gamma} \backslash G$, and a $G_{\gamma}$-invariant measure on $\Gamma_{\gamma} \backslash G_{\gamma}$, the Haar measure on $\Gamma_{\gamma}$ being the counting measure. Define now, for $f \in C_{c}(G)$,

$$
\begin{equation*}
j_{\gamma}(f)=\operatorname{vol}\left(\Gamma_{\gamma} \backslash G_{\gamma}\right) \int_{G_{\gamma} \backslash G} f\left(x^{-1} \gamma x\right) d_{\gamma} \bar{x} . \tag{2.16}
\end{equation*}
$$

Since $[\gamma]_{G}$ is closed by Lemma 2.5 (i), the function $G_{\gamma} x \mapsto f\left(x^{-1} \gamma x\right)$ lies in $C_{c}\left(G_{\gamma} \backslash G\right)$ and so the right side of $(2.16)$ is well-defined; moreover, the remarks preceding (2.16) show that $j_{\gamma}(f)$ is independent of the choice of the Haar measure on $G_{\gamma}$. It follows from this that $j_{\gamma}(f)=j_{\gamma^{\prime} \gamma \gamma^{\prime}-1}(f)\left(\gamma^{\prime} \in \Gamma, f \in C_{c}(G)\right)$. Let us now define $\mathscr{C}(\Gamma)$ to be the set of all $\Gamma$-conjugacy classes of elements of $\Gamma$. Then these remarks make it clear that for any $c \in \mathscr{C}(\Gamma)$, the map $J_{c}: f \mapsto J_{c}(f)=j_{\gamma}(f)$ $(\gamma \in c)$ is a well defined Borel measure on $G$, invariant under the inner automorphisms of $G$. Selberg's formula, with the conjugacy class $[e]_{\Gamma}$ separated from the others as usual, can now be formulated as follows.

Proposition 2.6. Let $n(\chi)$ be defined by (2.15). Then, for all $f \in C_{c}^{\infty}(G / / K)$,

$$
\begin{equation*}
\sum_{\chi \in A} n(\chi) \chi(f)=\operatorname{vol}(\Gamma \backslash G) f(e)+\sum_{c \neq[e]_{r}} J_{c}(f), \tag{2.17}
\end{equation*}
$$

where

$$
\begin{equation*}
J_{c}(f)=\operatorname{vol}\left(\Gamma_{\gamma} \backslash G_{\gamma}\right) \int_{G_{\gamma} \backslash G} f\left(x^{-1} \gamma x\right) d_{\gamma} \bar{x}(\gamma \in c) . \tag{2.18}
\end{equation*}
$$

It is now interesting to note that to a first approximation we can ignore the $J_{c}, c \neq[e]_{\Gamma}$. We have
Lemma 2.7. There is an open nbhd $U$ of e satisfying:
(a) $U=U^{-1}=K U K$; (b) $U \cap[\gamma]_{G}=\emptyset$, if $\gamma \in \Gamma$ and $\gamma \neq e$.

Let $\Gamma^{\prime}=\Gamma \backslash\{e\}$. Then $\Omega=\bigcup_{\gamma \in I^{\prime}}[\gamma]_{G}$ is closed in $G$ by Lemma 2.5 (i); so $K \Omega K$ is closed in $G$. We claim that $e \notin K \Omega K$. For, if $e=k x \gamma x^{-1} k^{\prime}$, for some $k, k^{\prime} \in K$, $\gamma \in \Gamma^{\prime}, x \in G$, we see that $\gamma=x^{-1}\left(k^{-1} k^{-1}\right) x$ lies in $x^{-1} K x \cap \Gamma$. So $\gamma$ fixes the coset $x^{-1} K \in G / K$, showing $\gamma=e$, since $\Gamma$ acts freely on $G / K$; contradiction. So we can find an open nbhd $V=V^{-1}$ of $e$ such that $V \cap K \Omega K=\emptyset$. We set now $U=K V K$.

Proposition 2.8. Let $U$ be any open nbhd of $e$ with properties (a) and (b) of Lemma 2.7. Then, for all $f \in C_{c}^{\infty}(G / / K)$ such that $\operatorname{supp}(f) \subset U$, we get

$$
\begin{equation*}
\sum_{\chi \in A} n(\chi) \chi(f)=\operatorname{vol}(\Gamma \backslash G) f(e) . \tag{2.19}
\end{equation*}
$$

This is obvious now.
2.4. Eigenfunction Expansions in $L^{2}(G / / K)$. The Plancherel Measure. Let us now define the transorm $\hat{f}$ of any $f \in L^{1}(G / / K)$ by

$$
\begin{equation*}
\hat{f}(\chi)=\chi(f)=\left\langle f, \varphi_{\chi}\right\rangle \quad(\chi \in A(S)) \tag{2.20}
\end{equation*}
$$

Since $A(S)$ is a closed subset of the separable Fréchet space $C^{x}(G / / K)$ it may be regarded as a standard Borel space (Mackey [30]) and the functions $\hat{f}$ are Borel on $\Lambda(S)$. If $\Lambda^{+}(S)$ is the subset of $\Lambda(S)$ of all $\chi$ for which $\varphi_{\chi}$ is positive definite (cf. Proposition 2.4), then $\Lambda^{+}(S)$ is a Borel subset of $\Lambda(S)$ and

$$
\begin{equation*}
|\widehat{f}(\chi)| \leqq\|f\|_{1} \quad\left(\chi \in \Lambda(S), f \in L^{1}(G / / K)\right) . \tag{2.21}
\end{equation*}
$$

Let $\lambda$ be the $*$-representation, of the commutative Banach algebra $L^{1}(G / / K)$ with involution $\sim$, in the Hilbert space $L^{2}(G / / K)$, the action being via left convolution. It is then possible to prove the existence of a unique projection valued measure $\mathbb{P}$ on $\Lambda^{+}(S)$ (in $L^{2}(G / / K)$ ) such that

$$
\begin{equation*}
\lambda(f)=\int_{A^{+}(S)} \hat{f}(\chi) d \mathbb{P}(\chi) \quad\left(f \in L^{1}(G / / K)\right) \tag{2.22}
\end{equation*}
$$

One can moreover prove the existence of a unique $\sigma$-finite measure $\omega$ on $\Lambda^{+}(S)$, the so-called Plancherel measure, such that

$$
\begin{equation*}
(\mathbb{P}(E) g, g)=\int_{E}|\hat{g}(\chi)|^{2} d \omega(\chi) \quad\left(g \in C_{c}(G / / K)\right) \tag{2.23}
\end{equation*}
$$

for all Borel sets $E \subset \Lambda^{+}(S)$. Taking $E=\Lambda^{+}(S)$ we get the Plancherel formula, valid for all $g \in C_{c}(G / / K)$ :

$$
\begin{equation*}
\int_{G}|g(x)|^{2} d x=\int_{A^{+}(S)}|\hat{g}(\chi)|^{2} d \omega(\chi) \tag{2.24}
\end{equation*}
$$

Further, one has the inversion formula, valid for $g \in C_{f}(G / / K) * C_{c}(G / / K)$ :

$$
\begin{equation*}
g(x)=\int_{\Lambda^{+}(S)} \hat{g}(\chi) \overline{\varphi_{\chi}(x)} d \omega(\chi) \quad(x \in G) \tag{2.25}
\end{equation*}
$$

If $\mathfrak{Y l}$ is the algebra which is the closure of $\lambda\left(L^{1}(G / / K)\right)$ and $\Sigma$ is the Gel'fand spectrum of $\mathfrak{Q}$, the above results can be established on $\Sigma$, with the Gel'fand transform $\dot{\lambda}(f)$ in place of $\hat{f}$. But now there is a canonical map $\sigma \mapsto \chi_{\sigma}$ of $\Sigma$ into $\Lambda^{+}(S)$, which is a Borel isomorphism of $\Sigma$ onto a Borel subset $\Lambda_{p}(S)$ of $\Lambda^{+}(S)$, such that $\overline{\lambda(f)}(\sigma)=\hat{f}\left(\chi_{\sigma}\right)$ for all $\sigma \in \Sigma, f \in L^{1}(G / / K)$; the results (2.22)-(2.25) are then obtained by transferring from $\Sigma$ to $\Lambda^{+}(S)$. In particular this gives

$$
\begin{equation*}
\int_{A^{+}(S) \backslash A_{p}(S)} d \mathbb{P}=0, \quad \int_{A^{+}(S) \backslash A_{p}(S)} d \omega=0 . \tag{2.26}
\end{equation*}
$$

Let $\mathfrak{Q}_{0}$ be the $\mathbb{R}$-algebra of all formally self-adjoint elements of $\mathfrak{Q}_{\text {Diff }}(G / K)$. Then, for any $\chi \in \Lambda(S)$ we may regard $\varphi_{\chi}$ as the unique solution to the eigenvalue problem

$$
\begin{equation*}
u \varphi=\chi(u) \varphi\left(u \in \mathcal{Q}_{0}\right), \quad \varphi \in C^{\infty}(G / / K), \quad \varphi(e)=1 \tag{2.27}
\end{equation*}
$$

Now, for any $u \in \mathfrak{I}_{0}$, it is well known that $u$ is an essentially self-adjoint operator on the (Gårding) subspace $C_{c}^{\infty}(G / / K) * L^{2}(G / / K)$; if $\lambda(u)$ denotes the unique selfadjoint operator in $L^{2}(G / / K)$ thus obtained, it can be shown that

$$
\begin{equation*}
\lambda(u)=\int_{\Lambda^{+}(S)} \chi(u) d \mathbb{P}(\chi) \quad\left(u \in \mathfrak{Q}_{0}\right) \tag{2.28}
\end{equation*}
$$

We may therefore interpret $\mathbb{P}$ as the spectral measure of the commuting system of self-adjoint differential operators $\lambda(u), u \in \mathscr{E}_{0} ;$ (2.24) and (2.25) are then respectively the Plancherel and eigenfunction expansion formulae for the problem (2.26).

The $\mathbb{R}$-algebra $\mathfrak{Q}_{0}$ is finitely generated. If $u_{j}(1 \leqq j \leqq l)$ is a system of generators, the map $\chi \mapsto\left(\chi\left(u_{1}\right), \ldots, \chi\left(u_{i}\right)\right)$ sets up a Borel isomorphism of $\Lambda(S)$ with a Borel subset of $\mathbb{R}^{l}$. The eigenvalue problem (2.27) as well as the measures $\mathbb{P}$ and $\omega$ may then be transferred to $\mathbb{R}^{l}$ to give the Plancherel and eigenfunction expansion formulae for the simultaneous eigenvalue problem

$$
\begin{equation*}
u_{j} \varphi=t_{j} \varphi(1 \leqq j \leqq l), \quad \varphi \in C^{\infty}(G / / K), \varphi(e)=1 \tag{2.29}
\end{equation*}
$$

Let us now consider the spectrum $\Lambda=\Lambda(X)$ of $X$. Since it is possible that $\Lambda_{p}(S) \neq \Lambda^{+}(S)$ and $\Lambda \notin \Lambda_{p}(S)$, we may define the principal and complementary spectra $\Lambda_{p}$ and $\Lambda_{c}$ of $X$ by $\Lambda_{p}=\Lambda \cap \Lambda_{p}(S), \Lambda_{c}=\Lambda \backslash A_{p}$. The trace formula (2.19) now becomes

$$
\begin{equation*}
\sum_{\chi \in A} n(\chi) \hat{f}(\chi)=\operatorname{vol}(\Gamma \backslash G) \int_{A_{p}(S)} \hat{f}(\chi) d \omega(\chi) \tag{2.30}
\end{equation*}
$$

valid for all $f \in C_{c}^{\infty}(G / / K)$ with $\operatorname{supp}(f) \subset U$. It is this relation that suggests that, asymptotically, $\Lambda_{p}$ grows like $\omega$ and $\Lambda_{c}$ is negligible in comparison with $\Lambda_{p}$.

The results mentioned in this paragraph are all essentially known; they may be proved by suitably adapting to our context the arguments of Harish-Chandra ([14]).

## 3. In Which $G$ is Semisimple

3.1. Notation. From now on we shall assume that $G$ is connected semisimple and that it has a finite center; other notation is as in Section 2; so $d x$ is a fixed Haar measure on G. $K$ is a maximal compact subgroup of $G$ and is the set of fixed points of an involutive automorphism $\theta$ of $G$. We have $\mathfrak{g}=\mathfrak{f} \oplus \mathfrak{s}$ where $\mathfrak{s}$ is the orthogonal complement of 1 with respect to the Killing form; $\langle\cdot, \cdot\rangle$ is positive definite on $\mathfrak{s}$ a is a maximal Abelian subspace of $\mathfrak{s} ; \Delta$ the set of roots of $(\mathfrak{g}, \mathfrak{a})$. We choose a positive system of roots $\Delta^{+}$and write $\mathfrak{g}=\ddagger \oplus \mathfrak{a} \oplus \mathfrak{n}, G=K A N$, for the corresponding Iwasawa decompositions; here, as usual, $A=\exp \mathfrak{a}, N$ $=\exp \mathfrak{n}$. Let $\mathfrak{w}$ be the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. For $\alpha \in \Delta, \mathfrak{g}_{\alpha}$ is the corresponding root space. We put $\rho=\frac{1}{2} \sum_{\alpha \in \mathcal{A}^{+}} \operatorname{dim}\left(\mathfrak{g}_{\alpha}\right) \alpha, \rho(H)=\frac{1}{2} \operatorname{tr}\left(\left.\operatorname{ad} H\right|_{\mathbf{n}}\right)(H \in \mathfrak{a})$. Let $\mathscr{F}$ be the dual of the complexification $\mathfrak{a}_{c}$ of $\mathfrak{a}$. The Weyl group $w$ acts naturally on $\mathscr{F}$. We denote by $\mathscr{F}_{I}$ (resp. $\mathscr{F}_{R}$ ) the $\mathbb{R}$-linear subspace of $\mathscr{F}$ of all elements that take only purely imaginary (resp. real) values on $\mathfrak{a} . \mathscr{F}_{I}$ and $\mathscr{F}_{R}$ are $\mathfrak{w}$-stable. Then $\mathscr{F}=\mathscr{F}_{I} \oplus \mathscr{F}_{R}$; for any $\xi=\xi_{I}+\xi_{R}, \xi \mapsto \xi^{\mathrm{conj}}=-\xi_{I}+\xi_{R}$ is then the conjugation in $\mathscr{F}$ induced by $\mathscr{F}_{R}$. The Killing form restricted to $\mathfrak{a} \times \mathfrak{a}$ is nondegenerate and positive definite. We extend it to a complex bilinear form $\langle.,$.$\rangle on \mathfrak{a}_{c} \times \mathfrak{a}_{c}$. Using the isomorphism $\mathfrak{a} \xrightarrow[\rightarrow]{\mathscr{F}}$ induced by it we transfer it to a nondegenerate complex bilinear form $\langle\ldots\rangle$ on $\mathscr{F} \times \mathscr{F}$. For $\xi \in \mathscr{F}$ we write $H_{\xi}$ for its image in $a_{c}$; then, for $\xi, \xi^{\prime} \in \mathscr{F}$, $\left\langle H_{\xi}, H_{\xi^{\prime}}\right\rangle=\left\langle\xi, \xi^{\prime}\right\rangle=\xi\left(H_{\xi^{\prime}}\right)=\xi^{\prime}\left(H_{\xi}\right)$. The Hermitian form $\xi, \xi^{\prime} \mapsto\left\langle\xi, \xi^{\text {conj }}\right\rangle$ is then positive definite and converts $\mathscr{F}$ into a Hilbert space. We write $\|\cdot\|$ for the corresponding norm; $\|\xi\|^{2}=\left\|\xi_{R}\right\|^{2}+\left\|\xi_{I}\right\|^{2}$. We denote by $\log : A \rightarrow \mathfrak{a}$ the inverse of exp: $\mathfrak{a} \rightarrow A$. For any $\xi \in \mathscr{F}$ we denote by $\eta_{\xi}$ the quasicharacter of $A$ given by

$$
\begin{equation*}
\eta_{\xi}(a)=e^{\xi(\log a)} \quad(a \in A) \tag{3.1}
\end{equation*}
$$

$\eta_{\xi}$ is a character (i.e., unitary) if and only if $\xi \in \mathscr{F}_{I}$. One can select Haar measures $d a$ on $A$ and $d n$ on $N$ and fix $d x$ by

$$
\begin{equation*}
d x=\eta_{2 \rho} d k d a d n \quad(x=k a n) \tag{3.2}
\end{equation*}
$$

For our subsequent needs it is convenient to use a specific normalization of $d n$. To choose $d n$, let $\bar{N}=\theta(N)$ and let $d n=\theta(d \bar{n})$, with $d \bar{n}$ the Haar measure on $\bar{N}$ such that $\int_{N} \eta_{-2 \rho}(a(\bar{n})) d \bar{n}=1$ (cf. Harish-Chandra [18, Lemma 44]). The polar decomposition formulae

$$
\begin{equation*}
G=K A K ; \quad A^{\theta}=A ; \quad a^{\theta}=a^{-1} \quad(a \in A), \tag{3.3}
\end{equation*}
$$

show that we are in the framework discussed in Section 2.
3.2. The Abel Transform. The central fact of the theory for semisimple $G$ is the existence of the Abel transform that is an algebra isomorphism of $\mathscr{E}^{\prime}(G / / K)$ with $\mathscr{E}^{\prime \prime}(A)^{\mathfrak{w}}$ (cf. Harish-Chandra [18], Gel'fand and Graev [12]). To define it, write, for any element $x \in G, x=k(x) a(x) n(x)(k(x) \in K, a(x) \in A, n(x) \in N), H(x)=\log a(x)$. $\pi: x \mapsto a(x)$ is an analytic map of $G$ onto $A$. Let $\pi^{*}$ be the pullback map $\pi^{*}$ : $C^{\infty}(A) \rightarrow C^{\infty}(G)$. By duality, this gives the push-forward map $\pi_{*}: \mathscr{E}^{\prime}(G) \rightarrow \mathscr{E}^{\prime}(A)$. $\pi_{*}$ is continuous map of $C_{c}^{\infty}(G)$ into $C_{c}^{\infty}(A)$, and is given by (use (3.2))

$$
\begin{equation*}
\left(\pi_{*} f\right)(a)=\eta_{2 p}(a) \int_{K \times N} f(k a n) d k d n\left(f \in C_{c}^{\infty}(G), a \in A\right) . \tag{3.4}
\end{equation*}
$$

We introduce the spherical pull-back $\pi^{\#}=P_{K} \circ \pi^{*}: C^{\infty}(A) \rightarrow C^{\infty}(G / / K)$, so that, for any $h \in C^{\infty}(A), x \in G$,

$$
\begin{equation*}
\left(\pi^{\#} h\right)(x)=\int_{K}\left(\pi^{*} h\right)(x k) d k=\int_{K} h(a(x k)) d k, \tag{3.5}
\end{equation*}
$$

and correspondingly the spherical push-forward $\pi_{\#}: \mathscr{E}^{\prime}(G / / K) \rightarrow \mathscr{E}^{\prime}(A)$. It is then clear that $\pi_{\#}(T)=\pi_{*}(T)\left(T \in \mathscr{E}^{\prime}(G / / K)\right)$. For $f \in C_{c}^{\infty}(G / / K)$, (3.4) reduces to

$$
\begin{equation*}
\left(\pi_{\#} f\right)(a)=\eta_{2 \rho}(a) \int_{N} f(a n) d n \quad(a \in A) \tag{3.6}
\end{equation*}
$$

Following Harish-Chandra we define the Abel transform

$$
\begin{equation*}
\mathscr{A}=\eta_{-\rho} \circ \pi_{\#}: \mathscr{E}^{\mathscr{E}^{\prime}}(G / / K) \rightarrow \mathscr{E}^{\prime}(A) \tag{3.7}
\end{equation*}
$$

The basic fact is that $\mathscr{A}$ is a homomorphism of algebras. Of course, $\pi_{\#}$ is already a homomorphism; the shift by $\eta_{-\rho}$ is introduced so that $\mathscr{A}$ commutes with ${ }^{\sim}$. It is well-known (cf. Varadarajan [42, II. Proposition 8.7]) that for any $f \in C_{c}^{\infty}(G / / K), \mathscr{A} f \in C_{c}^{\infty}(A)^{\infty}$; and so $\mathscr{A} T \in \mathscr{E}^{\prime}(A)^{\infty}$ for $T \in \mathscr{E}^{\prime}(G / / K)$. We thus have

Proposition 3.1. $\mathscr{A}$ is a continuous homomorphism of $\mathscr{E}^{\prime}(G / / K)$ into $\mathscr{E}^{\prime}(A)^{\mathfrak{w}}$ that commutes with ${ }^{\sim}$ and ${ }^{\vee}$, i.e., $\widetilde{\mathscr{A} T}=\mathscr{A} \tilde{T},(\mathscr{A} T)^{\vee}=\mathscr{A} \widetilde{T}\left(T \in \mathscr{E}^{\prime}(G / / K)\right)$.

It follows easily from the definition of $\pi_{*}$ that $\operatorname{supp}\left(\pi_{*}(T)\right) \subset \pi(\operatorname{supp}(T))$, for $T \in \mathscr{E}^{\prime}(G)$. Hence $\operatorname{supp}(\mathscr{A} T) \subset \pi(\operatorname{supp}(T))\left(T \in \mathscr{E}^{\prime}(G / / K)\right)$. In particular, if $T \in \mathscr{E}_{K}^{\prime}(G / / K)(\mathrm{cf} .(2.2)), \operatorname{supp}(\mathscr{A} T) \subset\{1\}$ and hence $\mathscr{A} T$ can be identified with an element of $U\left(\mathfrak{a}_{c}\right)$ which is $\mathfrak{w}$-invariant. So, by (2.3) we obtain a homomorphism $\gamma: U\left(\mathfrak{g}_{c}\right)^{K} \rightarrow U\left(\mathfrak{a}_{c}\right)^{\mathbf{w}}$. From the definition of $\pi_{\text {击 }}$ and the various identifications used above we get, for any $h \in C^{\infty}(A), q \in U\left(g_{c}\right)^{K}$ that $h(1 ; \gamma(q))=\left(\pi^{*}\left(\eta_{-\rho} h\right)\right)(1 ; q)$. It follows from this that, if $\gamma^{\prime}(q)$ is the unique element in $U\left(a_{c}\right)$ such that

$$
\begin{equation*}
q \equiv \gamma^{\prime}(q) \bmod \left(f U\left(\mathfrak{g}_{c}\right)+U\left(\mathfrak{g}_{c}\right) n\right) \tag{3.8}
\end{equation*}
$$

then

$$
\begin{equation*}
\gamma(q)=\eta_{\rho} \circ \gamma^{\prime}(q) \circ \eta_{-\rho} \quad\left(q \in U\left(\mathfrak{g}_{c}\right)^{K}\right) . \tag{3.9}
\end{equation*}
$$

$\gamma$ is thus Harish-Chandra's homomorphism $U\left(\mathfrak{g}_{c}\right)^{K} \rightarrow U\left(\mathfrak{a}_{c}\right)^{\boldsymbol{w}}$ (cf. Varadarajan, [42, II, subsection 8.3]). We describe these results in

Proposition 3.2. The restriction of $\mathscr{A}$ to $\mathscr{E}_{K}^{\prime}(G / / K)$ is a homomorphism into $\mathscr{E}_{1}^{\prime \prime}(A)^{w}$. The homomorphism that one obtains from this via natural identifications, from
$U\left(\mathrm{~g}_{c}\right)^{K}$ to $U\left(\mathfrak{a}_{c}\right)^{\mathfrak{w}}$ is none other than the Harish-Chandra homomorphism $\gamma$ : $U\left(\mathfrak{g}_{c}\right)^{K} \rightarrow U\left(\mathfrak{a}_{c}\right)^{\boldsymbol{m}}$. In particular, the restriction of $\mathscr{A}$ to $\mathscr{E}_{\mathrm{K}}^{\prime}(G / / K)$ is an isomorphism of $\mathscr{E}_{K}^{\prime}(G / / K)$ onto $U\left(\mathfrak{a}_{c}\right)^{m}$.
3.3. The Structure of $\Lambda(S)$. Formula for the Functions $\varphi_{\chi}$. We denote by ${ }^{\wedge}$ the Fourier-Laplace transform on $\mathscr{E}^{\prime}(A)$. Thus

$$
\begin{equation*}
\hat{T}(\lambda)=\left\langle T, \eta_{\lambda}\right\rangle \quad(T \in \mathscr{E}(A), \lambda \in \mathscr{F}) . \tag{3.10}
\end{equation*}
$$

The composition of ${ }^{\wedge}$ with the Abel transform is the Harish-Chandra transform $\mathscr{H}: \mathscr{E}^{\prime}(G / / K) \rightarrow \widehat{E^{\prime}}(A)^{1 \infty} ;$

$$
\begin{equation*}
(\mathscr{H} T)(\lambda)=\mathscr{A} \hat{T}(\lambda)=\left\langle\mathscr{A} T, \eta_{\lambda}\right\rangle \quad\left(T \in \mathscr{E}^{\prime}(G / / K), \lambda \in \mathscr{F}\right) . \tag{3.11}
\end{equation*}
$$

Since $\mathscr{H}$ is a homomorphism, for each $\lambda \in \mathscr{F}$, the specialization

$$
\begin{equation*}
\chi_{\lambda}: \mathscr{E}^{\mathscr{E}^{\prime \prime}}(G / / K) \rightarrow \mathbb{C} ; \quad \chi_{2}(T)=\mathscr{H} T(\lambda) \quad\left(T \in \mathscr{E}^{\prime}(G / / K)\right) \tag{3.12}
\end{equation*}
$$

is a continuous homomorphism. Let $\varphi_{\lambda}=\varphi_{\chi_{\lambda}}$ be the corresponding eigenfunction in $C^{\infty}(G / / K)$ as in (2.8). Since $\left\langle T, \varphi_{\lambda}\right\rangle=\chi_{\lambda}(T)=\left\langle\mathscr{A} T, \eta_{\lambda}\right\rangle=\left\langle T, \pi^{\#} \eta_{\lambda \ldots \rho}\right\rangle$ for all $T \in \mathscr{E}^{\prime}(G / / K)$, we recover, using (3.5), Harish-Chandra's famous formula

$$
\begin{equation*}
\varphi_{\lambda}(x)=\int_{K} \eta_{\lambda-\rho}(a(x k)) d k \quad(\lambda \in \mathscr{F}, x \in G) . \tag{3.13}
\end{equation*}
$$

From (3.11) and the fact that $\mathscr{A}$ commutes with ${ }^{\vee}$ and $\sim$ (Proposition 3.1), we get

$$
(\mathscr{H} \check{T})(\lambda)=(\mathscr{H} T)(-\lambda) ; \quad(\mathscr{H} \tilde{T})(\lambda)=(\mathscr{H} T)\left(-\lambda^{\mathrm{conj}}\right) \quad\left(T \in \mathscr{E}^{\prime \prime}(G / / K), \lambda \in \mathscr{F}\right) .
$$

If we now use these relations in conjunction with $\mathscr{H} T(\lambda)=\left\langle T, \varphi_{\lambda}\right\rangle$, it follows that

$$
\begin{equation*}
\check{\varphi}_{\lambda}=\varphi_{-\lambda} ; \quad \tilde{\varphi}_{\lambda}=\varphi_{-\lambda}{ }^{\mathrm{conj}} \quad(\lambda \in \mathscr{F}) \tag{3.14}
\end{equation*}
$$

Proposition 3.3. The $\chi_{\lambda}(\lambda \in \mathscr{F})$ are precisely all the continuous homomorphisms of $\left.\mathscr{E}^{\prime}(G / / K)\right)$ into $\mathbb{C}$; and $\chi_{\lambda}=\chi_{\lambda^{\prime}} \Leftrightarrow \mathfrak{w} \cdot \lambda=\mathfrak{w} \cdot \lambda^{\prime}$. For any $\lambda \in \mathscr{\mathscr { F }}$ the corresponding eigenfunction $\varphi_{\lambda}=\varphi_{\chi_{\lambda}} \in C^{\infty}(G / / K)$ is given by (3.13); it satisfies the system of equations

$$
\begin{equation*}
\varphi_{\lambda} * \check{T}=\mathscr{H} T(\lambda) \varphi_{\lambda} \quad\left(T \in \mathscr{E}^{\prime \prime}(G / / K)\right) ; \quad \varphi_{\lambda}(e)=1 \tag{3.15}
\end{equation*}
$$

and is the unique solution in $C^{\infty}(G / / K)$ for this system. In particular,

$$
q \varphi_{\lambda}=\gamma(q)(\lambda) \varphi_{\lambda} \quad\left(\lambda \in \mathscr{F}, q \in U\left(\mathfrak{g}_{c}\right)^{K}\right) ; \quad \varphi_{\lambda}=\varphi_{\lambda^{\prime}} \Leftrightarrow \mathfrak{w} \cdot \lambda=\mathfrak{w} \cdot \hat{\lambda}^{\prime}
$$

We recall that as $\mathfrak{a}$ is Abelian, $U\left(\mathfrak{a}_{\mathfrak{c}}\right)$ is canonically isomorphic to the symmetric algebra $S\left(a_{c}\right)$ and hence to all polynomial functions on $\mathscr{F}$. The present Proposition follows without much difficulty from the Propositions 2.1 and 3.2 in conjunction with the well-known result that the homomorphisms of $U\left(\mathfrak{a}_{c}\right)^{\mathbf{m}}$ into $\mathbb{C}$ are canonically parametrized in the usual way by the $\mathfrak{w}$-orbits in $\mathscr{F}$ (cf. Helgason [24, X, Lemma 6.9]).

It is clear that from this point of view, the Abel transform is the fundamental object. The fact that both the integral map (3.6) and the differential map (3.9)
given by the Harish-Chandra homomorphism, are specializations of $\mathscr{A}$ is one of the major unifying features of the Abel transform.

It follows from Proposition 3.3 that, in the notation of $(2.7), \Lambda(S) \approx \mathscr{F} / \mathfrak{w}$; we shall commit a mild abuse of notation and write $\Lambda(S)=\mathscr{F}$. Similarly we write (cf. (2.12))

$$
\begin{equation*}
A=\left\{\lambda: \lambda \in \mathscr{F}, C^{\infty}\left(X: \chi_{\lambda}\right) \neq(0)\right\} . \tag{3.16}
\end{equation*}
$$

Proposition 3.4. (i) $\rho \in \Lambda$; (ii) $\Lambda=(-\Lambda)^{\mathrm{conj}}$; (iii) $\varphi_{\mu}(\mu \in \mathscr{F})$ is bounded if and only if $\mu_{R}$ lies in the convex hull of the points $s \cdot \rho(s \in \mathfrak{w})$. In particular, $\left\|\lambda_{R}\right\| \leqq\|\rho\|$ if $\lambda \in \Lambda$.

Assertion (i) follows from Proposition $2.3(\mathrm{i})$ as $\varphi_{\rho}=1$ (cf. (3.13)) whereas (ii) is obtained from (2.13) and (3.14). The first assertion in (iii) is a result of Helgason and Johnson [24, Theorem 2.1]. By (2.13) $\varphi_{\lambda}$ is bounded if $\lambda \in \Lambda$. Since $\|s \cdot \rho\|$ $=\|\rho\|$ and since the set $\left\{\mu: \mu \in \mathscr{F}_{R},\|\mu\| \leqq\|\rho\|\right\}$ is convex we see that $\varphi_{\mu}$ bounded implies $\left\|\mu_{R}\right\| \leqq\|\rho\|$.

Corollary 3.5. For every $s \in \mathfrak{w}$, let $\mathscr{F}(s)=\left\{\mu: \mu \in \mathscr{F}, s \cdot \mu_{I}=\mu_{I}, s \cdot \mu_{R}=-\mu_{R}\right\}$. Then $\mathscr{F}(1)=\mathscr{F}_{I}$ and $\Lambda \subset \bigcup_{s \in \mathrm{w}} \mathscr{F}(s)$.

Finally let us recall the Laplace-Beltrami operator $\omega_{S}$ on $S(c f .(2.6))$. The $\varphi_{\lambda}$ are eigenfunctions for $\omega_{S}$ and a simple calculation shows that

$$
\omega_{S} \varphi_{\lambda}=(\langle\lambda, \lambda\rangle-\langle\rho, \rho\rangle) \varphi_{\lambda} \quad(\lambda \in \mathscr{F}) .
$$

If $\lambda \in \Lambda$, then $\langle\lambda, \lambda\rangle-\langle\rho, \rho\rangle$ is real and $\leqq 0$ and we have

$$
\begin{equation*}
\omega_{S} \varphi_{\lambda}=\left(\left\|\lambda_{R}\right\|^{2}-\|\rho\|^{2}-\left\|\lambda_{I}\right\|^{2}\right) \varphi_{\lambda} \quad(\lambda \in A) \tag{3.17}
\end{equation*}
$$

The classical example is: $G=S L(2, \mathbb{R}), K=S O(2, \mathbb{R})$ and $S$ the Poincare upper half-plane; then with $z=x+i y \in S(y>0)$, we have $2 \omega_{S}=\mathrm{y}^{2}\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}\right)$.
3.4. The Analytic Theory of the Harish-Chandra Transform. The Theorems of Harish-Chandra and Gangolli. For any $f \in C_{c}^{\infty}(G / / K)$ the restriction to $\mathscr{F}_{I}$ of the Harish-Chandra transform $\mathscr{H} f$ lies in the Schwartz space $\mathscr{S}\left(\mathscr{F}_{I}\right)$; so we can regard $\mathscr{H}$ as a continuous map $C_{c}^{\infty}(G / / K) \rightarrow \mathscr{S}\left(\mathscr{F}_{I}\right)^{w}$. Observe that this map depends on our choice of $d x$. The Plancherel measure for the Harish-Chandra transform is then a measure $\beta d v$ where (a) $\beta \in C^{\infty}\left(\mathscr{F}_{I}\right)^{\mathbf{m}} ; \beta \geqq 0$; (b) $\beta$ and all its derivatives grow at most polynomially; (c) if $d v$ is the Lebesgue measure on $\mathscr{F}_{I}$ dual to the Haar measure $d a$ on $A$,

$$
\begin{equation*}
f(x)=|\mathfrak{w}|^{-1} \int_{\mathscr{F}_{\mathfrak{F}}}(\mathscr{H} f)(v) \varphi_{v}(x)^{\mathrm{conj}} \beta(v) d v\left(f \in C_{\mathrm{c}}^{\infty}(G / / K), x \in G\right) \tag{3.18}
\end{equation*}
$$

In particular, for $f \in C_{c}^{\infty}(G / / K)$,

$$
\begin{equation*}
f(e)=|\mathfrak{w}|^{-1} \int_{\mathscr{F}_{I}}(\mathscr{H} f)(v) \beta(v) d v=\langle\hat{\beta}, \mathscr{A} f\rangle \tag{3.19}
\end{equation*}
$$

We note that $\hat{\beta}$ makes sense as a tempered distribution on $\mathscr{F}_{I}$ in view of (b). It is known that $\beta$ exists and is uniquely determined by (a)-(c); and that $\operatorname{supp}(\beta)$
$=\mathscr{F}_{I}$. There is in addition an explicit formula for $\beta$, due to Gindikin and Karpelevič [13]; see Section 3.8 infra for a more detailed discussion of $\beta$. Moreover, the map $f \mapsto \mathscr{H} f$ extends to a unitary isomorphism (also denoted by $\mathscr{H})$ between $L^{2}(G / / K)$ and $L^{2}\left(\mathscr{F}_{I}, \beta d v\right)^{w}$ so that $\mathscr{H}\left(C_{c}^{\infty}(G / / K)\right)$ is dense in $L^{2}\left(\mathscr{F}_{I}, \beta d \nu\right)^{\text {w }}$, and, for all $f \in C_{c}^{\infty}(G / / K)$,

$$
\begin{equation*}
\int_{G}|f(x)|^{2} d x=|\mathfrak{w}|^{-1} \int_{\mathscr{F _ { I }}}|\mathscr{H} f(v)|^{2} \beta(v) d v . \tag{3.20}
\end{equation*}
$$

The Fourier-Laplace transform (cf. (3.10)) is an algebra isomorphism of $C_{c}^{\infty}(A)$ with the algebra $\mathscr{P}(\mathscr{F})$ of entire functions on $\mathscr{F}$, satisfying the wellknown conditions of the Paley-Wiener Theorem. As $\mathscr{H} f=\mathscr{A} \hat{f}$, we have a commutative diagram


It was proved by Gangolli [8] that in this diagram both $\mathscr{A}$ and $\mathscr{H}$ are isomorphisms for the structure of topological algebras. Moreover, he established the following more refined result: Let $A(b)$ denote, for any $b>0$, the subset of all $a \in A$ such that $\|\log a\| \leqq b$; then $A(b)$ is $\mathfrak{w}$-invariant and

$$
\begin{equation*}
\operatorname{supp}\left(\mathscr{A}^{-1} h\right) \subset K A(b) K \quad\left(h \in C_{c}^{\infty}(A(b))^{\mathfrak{w}}\right) \tag{3.22}
\end{equation*}
$$

For convenience of later use we note here that for any $h \in C_{c}^{\infty}(A(b))^{m}$,

$$
\begin{equation*}
|\widehat{h}(\lambda)| \leqq c_{m} \exp \left(b\left\|\lambda_{R}\right\|\right)(1+\|\lambda\|)^{-m} \quad(\lambda \in \mathscr{F}, m=0,1, \ldots), \tag{3.23}
\end{equation*}
$$

where $c_{m}=c_{m}(h)$ are constants $>0$.
The Harish-Chandra and the Abel transforms are actually defined on the (spherical) Schwartz space $\mathscr{C}(G / / K)$ of Harish-Chandra (cf. [21, p.46]); and Harish-Chandra's fundamental theorem asserts that the following diagram, where $\mathscr{F}$ denotes Fourier transform,

is commutative, all the arrows being linear topological isomorphisms.
It follows from the theorems of Gangolli and Harish-Chandra that for any $U \in \mathscr{D}^{\prime}(G / / K)\left(\right.$ resp. $U \in \mathscr{C}^{\prime}(G / / K)$ ) there is a unique $U^{\dagger} \in \mathscr{D}^{\prime}(A)^{w}$ (resp. $\left.U^{\dagger} \in \mathscr{S}^{\prime}(A)^{w}\right)$ such that

$$
\langle U, f\rangle=\left\langle U^{\dagger}, \mathscr{A} f\right\rangle \quad\left(f \in C_{c}^{\infty}(G / / K), \text { resp. } \mathscr{C}(G / / K)\right) ;
$$

and further that $U \mapsto U^{\dagger}$ is an isomorphism of $\mathscr{D}^{\prime}(G / / K)$ with $\mathscr{D}^{\prime}(A)^{\text {w }}$ (resp. $\mathscr{C}^{\prime}(G / / K)$ with $\left.\mathscr{S}^{\prime}(A)^{w}\right)$. It is not hard to show that if $T=\mathscr{E}^{\prime}(G / / K)$, the tempered distribution $T^{\dagger}$ on $A$ is given by

$$
\begin{equation*}
T^{\dagger}=|\mathfrak{w}|^{-1} \mathscr{F}^{-1} \circ \beta \circ \mathscr{\mathscr { F }} \circ \eta_{-\rho} \pi_{*} T . \tag{3.24}
\end{equation*}
$$

In fact, if $f \in C_{c}^{\infty}(G / / K)$, we find, on using the Plancherel formula (3.20) for $G$ as well as the usual Plancherel formula for $A$, that, for all $g \in C_{c}^{\infty}(G / / K)$,

$$
\left.\langle f, g\rangle=\left.\langle | \mathfrak{w}\right|^{-1} \mathscr{F}^{-1} \circ \beta \circ \mathscr{F} \circ \mathscr{A} f, \mathscr{A} g\right\rangle ;
$$

since $\mathscr{A} f=\eta_{-\rho} \pi_{*} f$ (cf. (3.6)) we get (3.24) in this case. For an arbitrary $T \in \mathscr{E}^{\prime \prime}(G / / K)$, (3.24) follows by a density argument.
3.5. The Principal and Complementary Spectra of $X$. Let $\Gamma \subset G$ be as in subsection 2.2. The requirement that $\Gamma$ acts freely on $G / K$ is not too serious a condition. Indeed, according to Borel [3], if $\Gamma \subset G$ is a discrete subgroup with compact quotient $\Gamma \backslash G$, there exists a normal subgroup $\Gamma^{\prime} \subset \Gamma$ of finite index, without elliptic elements other than the identity. If $\gamma \in \Gamma,[\gamma]_{G}$ is closed according to Lemma 2.5 (i); this implies that $\gamma$ is a semisimple element (cf. Varadarajan [42, II, Theorem 2.17]). The centralizer $G_{\gamma}$ is then reductive and so, in particular, is a unimodular group. The results of subsection 2.3 are thus available.

We define the spectrum $\Lambda$ of $X$ by (3.16), and the subsets $\Lambda_{p}$ and $\Lambda_{c}$ are now defined by

$$
\Lambda_{p}=\Lambda \cap \mathscr{F}_{I} ; \quad \Lambda_{c}=\Lambda \backslash \Lambda_{p}
$$

$\Lambda_{p}$ (resp. $\Lambda_{c}$ ) is called the principal (resp. complementary) spectrum. $\Lambda_{c}$ is nonempty by Proposition 3.4(i). Both $\Lambda_{p}$ and $\Lambda_{c}$ are $\mathfrak{w}$-stable. This separation of $\Lambda$ into principal and complementary spectrum seems to be justified in view of the fact (cf. (3.20)) that $\mathscr{F}_{I}$ is the spherical $L^{2}$-spectrum of $S$. The occurrence of the complementary spectrum is however a distinctly nonclassical phenomenon. It appears to be related to the "degeneracies" in the Plancherel density $\beta$. Indeed, $\beta$ has zeros in $\mathscr{F}_{I}$, the points of $\Lambda \backslash \mathscr{F}_{I}$ always have their imaginary parts located at these zeros, and the complementary spectrum seems to emerge as a "compensating factor".
3.6. The Trace Formulae. Since we are treating $\mathscr{F}$ and not $\mathscr{F} / \mathfrak{w}$ as the spectrum $\Lambda(S)$, we should modify our definition of the multiplicities of the points of the spectrum of $X$. We do it in the obvious manner: for $\lambda \in \Lambda$ we define its multiplicity $m(\lambda)$ in $L^{2}(X)$ by (cf. (2.15))

$$
\begin{equation*}
m(\lambda)=m(s \cdot \lambda)=|\mathfrak{w} \cdot \lambda|^{-1} n\left(\chi_{\lambda}\right) \quad(s \in \mathfrak{w}) \tag{3.25}
\end{equation*}
$$

Proposition 3.6. $A$ is a discrete subset of $\mathscr{F}$. There are a constant $c>0$ and an integer $M>0$ such that

$$
\begin{equation*}
\sum_{\lambda \in A,\|\lambda\| \leqq t} m(\lambda) \leqq c t^{M} \quad(t \geqq 1) \tag{3.26}
\end{equation*}
$$

We use the Laplace-Beltrami operator $\omega_{S}$. For any $t \geqq 1$, let $N(t)$ be the number of eigenvalues (with multiplicities) of $-\omega_{S}$ in $L^{2}(X)$ which are $\leqq t$. We then see from (3.17) and Proposition 3.4(iii) that $\sum_{\substack{\lambda \in \mathcal{A},\|\lambda\| t}} m(\lambda) \leqq N\left(t^{2}+\|\rho\|^{2}\right)$; so (3.26) follows from standard results on elliptic operators on compact manifolds (cf. Duistermaat-Guillemin [6, formula (1.13)]).

The trace formulae of Section 2.3 now take a much better form since the Theorems of Gangolli and Harish-Chandra give definitive information on the transforms of the test functions.

Proposition 3.7. For any $h \in C_{c}^{\infty}(A)^{w o}$

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) \tilde{h}(\lambda)=|\mathfrak{w}|^{-1} \operatorname{vol}(\Gamma \backslash G) \int_{\mathscr{F}_{I}} \tilde{r}(v) \beta(v) d v+\sum_{c \neq|e|]_{r}} J_{c}\left(\mathscr{A}^{-1} h\right), \tag{3.27}
\end{equation*}
$$

where the series on the left converges absolutely, the $J_{c}$ are as in (2.18) and $\mathscr{A}^{-1}$ : $C_{c}^{\infty}(A)^{m} \rightarrow C_{c}^{\infty}(G / / K)$ is the inverse Abel transform.
Proposition 3.8. (i) For any $h \in C_{c}^{\infty}(A)$ the series $\sum_{\lambda \leqslant 1} m(\lambda) h(\lambda)$ converges absolutely.
(ii) There exists an open neighborhood $V$ of 1 in $A$ satisfying: (a) $V=V^{-1}$ and $V$ is w-stable; (b) for all $h \in C_{c}^{\infty}(V)$ we have

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) \hat{h}(\lambda)=|\boldsymbol{w}|^{-1} \operatorname{vol}(\Gamma \backslash G) \int_{\hat{F}_{I}} \hat{h}(v) \beta(v) d v . \tag{3.28}
\end{equation*}
$$

Since $\left\|\lambda_{R}\right\| \leqq\|\rho\|$ for $\lambda \in \Lambda$, the Paley-Wiener estimate (3.23) gives assertion (i) when combined with (3.26). Let $V=\{a: a \in A,\|\log a\|<b\}, b>0$ being small enough so that $A(\mathrm{~b})$ is contained in the nbhd $U$ given by Proposition 2.8 . Then $K A(b) K \subset U$, and (3.22) in combination with Proposition 2.8 gives (3.28) for all $h \in C_{c}^{\infty}(V)^{\omega}$. Let now $\sigma$ be the measure on $\mathscr{F}$ which assigns the mass $m(\lambda)$ to $\lambda \in \Lambda$ and define the measure $\tau$ on $\mathscr{F}$ as $|\mathfrak{w}|^{-1} \operatorname{vol}(\Gamma \backslash G) \beta d \nu$. Then $\sigma$ and $\tau$ are both $\mathfrak{w}$-invariant; and for any $h \in C_{c}^{\infty}(A), \widehat{h}$ lies in both $L^{1}(\mathscr{F}, \sigma)$ and $L^{1}(\mathscr{F}, \tau)$. Since $\mathfrak{w}$ is finite we can average over $\mathfrak{w}$ to get, with $\bar{h}=|\mathfrak{w}|^{-1} \sum_{s \in \mathfrak{w}} h^{s},\langle\sigma, \hat{h}\rangle=\langle\sigma, \hat{\bar{h}}\rangle=\langle\tau, \hat{h}\rangle$
$=\langle\tau, \hat{h}\rangle$ $=\langle\tau, \hat{h}\rangle$.
3.7. The Volume of $X$. It has to be observed that $\operatorname{vol}(\Gamma \backslash G)$ depends on our choice of the Haar measure $d x$. We fix $d x$, or what amounts to the same, $d a$, in the following manner. We note that $(X, Y)=-\langle X, \theta Y\rangle(X, Y \in \mathfrak{g})$ provides $\mathfrak{g}$ with the structure of a Hilbert space. For any subspace $I \subset \mathfrak{g}, d_{0} I$ is the standard Lebesgue measure on I (with (...) induced by (...)); and let us write $d_{0} I$ also for the exterior differential form that gives rise to the measure $d_{0} I$ (after I has been oriented). If $L$ is a closed subgroup of $G$ corresponding to $I$, there is a unique left invariant exterior differential form on $L$ that corresponds to $d_{0} \mathrm{I}$ at $e \in L$; this form and the corresponding Haar measure on $L$ are denoted by $d_{0} l$. If $L$ is compact, we write $\operatorname{vol}_{0}(L)=v_{0}(L)=\int_{L} d_{0} l$. We now fix $d x$ by

$$
\begin{equation*}
d x=\eta_{2 \rho} d k d_{0} a d n \quad(x=k a n) . \tag{3.29}
\end{equation*}
$$

Here we recall that $d k$ and $d n$ are defined in Section 3.1. Let $\operatorname{vol}(X)$ and $\operatorname{vol}(\Gamma \backslash G)$ denote the volumes on $X$ and $\Gamma \backslash G$ defined by $d x$, then

$$
\begin{equation*}
\operatorname{vol}(X)=\operatorname{vol}(\Gamma \backslash G) \tag{3.30}
\end{equation*}
$$

On the other hand, the form (.,.) gives rise to a left invariant metric on $G$ which is right- $K$-invariant. On $G / K$ it induces the same metric as $\langle\ldots .$,$\rangle . So vol { }_{0}(X)$, the volume of $X$ defined by the Riemannian metric on $S$ coming from the Killing form, is evaluated by means of $d_{0} x$, and we have

$$
\begin{equation*}
\operatorname{vol}_{0}(X)=\operatorname{vol}_{0}(\Gamma \backslash G) v_{0}(K)^{-1} \tag{3.31}
\end{equation*}
$$

If $d(G)>0$ is the constant such that $d x=d(G) d_{0} x$, it follows that

$$
\begin{equation*}
\frac{\operatorname{vol}(X)}{\operatorname{vol}_{0}(X)}=d(G) v_{0}(K) . \tag{3.32}
\end{equation*}
$$

We wish to find a more explicit expression for $d(G)$; such a formula would enable us to compare our results on the asymptotic behaviour of the spectrum with classical ones (for instance, Minakshisundaram-Pleijel [32], DuistermaatGuillemin [6]).

According to Harish-Chandra [22, §37, Lemma 2] (see also Varadarajan [42, II, Section 17]) we have, with $n=\operatorname{dim}(G / K)$ and $r=\operatorname{dim} A$,

$$
\begin{equation*}
d_{0} x=2^{-(n-r) / 2} v_{0}(K) \eta_{2 \rho} d k d_{0} a d_{0} n \tag{3.33}
\end{equation*}
$$

and therefore $d(G)$ is known if we determine the constant $\gamma=\gamma(G)>0$ satisfying

$$
\begin{equation*}
d n=\gamma d_{0} n ; \quad d \bar{n}=\gamma d_{0} \bar{n} \tag{3.34}
\end{equation*}
$$

For the normalization of $d \bar{n}$ we have the result of Harish-Chandra [18, Lemma 44] according to which the map $\psi: \bar{N} \times M \rightarrow K$ given by $\psi(\bar{n}, m)=m k(\bar{n})(\bar{n} \in \bar{N}$, $m \in M$, the centralizer of $\mathfrak{a}$ in $K$ ) is a diffeomorphism onto an open subset of $K$ whose complement has measure zero, and has the property that $d k$ goes over to the measure $\eta_{-2 \rho}(a(\bar{n})) d \bar{n} d m$, if $\int_{M} d m=1$. Let $m$ be the Lie algebra of $M$.

Lemma 3.9. (i) If $\Psi=(d \psi)_{(1,1)}$, then $\Psi(X, Y)=X+\theta X+Y(X \in \bar{n}, Y \in \mathfrak{m})$. (ii) Let $q$ $=n-r$ and $X_{1}, \ldots, X_{q}$ be an orthonormal basis for $\bar{n}$ with respect to (.,.). Then $2^{-\frac{1}{2}}\left(X_{1}+\theta X_{1}\right), \ldots, 2^{-\frac{1}{2}}\left(X_{q}+\theta X_{q}\right)$ is an orthonormal basis for $\mathfrak{f} \ominus m$, the orthocomplement of m in f .

Obviously $\quad \Psi(X, Y)=\Psi(X, 0)+\Psi(0, Y)=\Psi(X, 0)+Y(X \in \bar{n}, Y \in \mathfrak{m})$. Now $\Psi(X, 0)$ is the projection of $X$ on $\mathfrak{f}$ according to $\mathfrak{g}=\mathfrak{f}+\mathfrak{a}+\mathfrak{n}$, which is $X+\theta X$, since $X=(X+\theta X)+0+(-\theta X)$. Regarding (ii), observe that $X \mapsto X+\theta X(X \in \overline{\mathfrak{n}})$ defines a bijection between $\bar{n}$ and $\mathfrak{t} \ominus \mathfrak{m}$. So it is a question of calculating scalar products. Now $\left(X_{i}+\theta X_{i}, X_{j}+\theta X_{j}\right)=2 \delta_{i j}+\left(X_{i}, \theta X_{j}\right)+\left(\theta X_{i}, X_{j}\right)=2 \delta_{i j}$ $-2\left\langle X_{i}, X_{j}\right\rangle=2 \delta_{i j}$, since $\langle\bar{n}, \bar{n}\rangle=0$.

Let the form $\omega_{0}$ (resp. $\omega$ ) on $\bar{N} \times M$ denote the pull-back under $\psi$ of the differential form $d_{0} k$ (resp. $d k$ ). Then $\left(\omega_{0}\right)_{(1,1)}=v_{0}(K) \omega_{(1,1)}$ and $\omega_{(1,1)}$ $=(d \bar{n} d m)_{(1,1)}$. If $Y_{1}, \ldots, Y_{t}$ is an orthonormal basis for $m$, we get, using

Lemma 3.9, that $\omega_{0}\left(\left(X_{1}, 0\right), \ldots,\left(X_{q}, 0\right),\left(0, Y_{1}\right), \ldots,\left(0, Y_{t}\right)\right)=d_{0} k\left(X_{1}+\theta X_{1}, \ldots, X_{q}\right.$ $\left.+\theta X_{q}, Y_{1}, \ldots, Y_{t}\right)= \pm 2^{(n-r) / 2}$. It follows that $\left(\omega_{0}\right)_{(1,1)}= \pm 2^{(n-r) / 2} d_{0} \bar{n} d_{0} m$, and we obtain $2^{(n-r) / 2} d_{0} \bar{n} d_{0} m=v_{0}(K) d \bar{n} d m$. But then after integration over $M$, we get

$$
2^{\frac{1}{2}(n-r)} \operatorname{vol}_{0}(M) d_{0} \bar{n}=\operatorname{vol}_{0}(K) d \bar{n},
$$

from which we obtain

$$
\begin{equation*}
d n=2^{\frac{1}{2}(n-r)} \operatorname{vol}_{0}(K / M)^{-1} d_{0} n \tag{3.35}
\end{equation*}
$$

Since $\int_{N} e^{-2 \rho(H(\bar{n}))} d \bar{n}=1$, we get

$$
\begin{equation*}
\operatorname{vol}_{0}(K / M)=2^{\frac{1}{2}(n-r)} \int_{N} e^{-2 \rho(H(\bar{n}))} d_{0} \bar{n} . \tag{3.36}
\end{equation*}
$$

Combining (3.32)-(3.36) we obtain
Lemma 3.10. With notation as above we have

$$
\frac{\operatorname{vol}(X)}{\operatorname{vol}_{0}(X)}=2^{n-r} \operatorname{vol}_{0}(K / M)^{-1} ; \quad \operatorname{vol}_{0}(K / M)=2^{\frac{1}{2}(n-r)} \int_{N} e^{-2 \rho(H(\bar{n}))} d_{0} \bar{n} .
$$

It must be remembered that $\operatorname{vol}_{0}(K / M)$ is to be calculated with respect to the exterior differential form coming from the Killing form of $\mathfrak{g}$ and not of $\mathfrak{l}$.

In $\S 3.9$ we shall compute $\operatorname{vol}_{0}(K / M)$ explicitly, following the technique of Gindikin-Karpelevič [13]. This method, which consists of reduction to rank one groups, was originally devised by them to evaluate integrals of the form

$$
\int_{N} e^{(v-\rho)(H(\bar{n}))} d \bar{n} \quad(v \in \mathscr{F})
$$

that appear in the expression for the Plancherel density $\beta$. Since the properties of $\beta$ are extremely important for us we shall now turn to a brief discussion of them.
3.8. The Plancherel Density $\beta$ and Its Behavior at Infinity. The starting point is Harish-Chandra's formula [19, p. 611, Corollary 2]

$$
\begin{equation*}
\beta(v)=|c(v)|^{-2} \quad\left(v \in \mathscr{F}_{I}\right) \tag{3.37}
\end{equation*}
$$

Here $\underset{\sim}{c}$ is the meromorphic function on $\mathscr{F}$ which is given on the domain

$$
\begin{equation*}
\left\{v: v \in \mathscr{F},\left\langle v_{R}, \alpha\right\rangle>0 \forall \alpha \in \Delta^{+}\right\} \tag{3.38a}
\end{equation*}
$$

by the convergent integral

$$
\begin{equation*}
\mathcal{C}(v)=\int_{N} e^{-(\nu+\rho)(H(\bar{n}))} d \bar{n} \tag{3.38b}
\end{equation*}
$$

while the Haar measure $d \bar{n}$ is normalized by

$$
\begin{equation*}
\int_{N} e^{-2 \rho(H(\bar{n}))} d \bar{n}=1 . \tag{3.39}
\end{equation*}
$$

The integral (3.38b) does not converge for $v \in \mathscr{F}_{I}$, and so it is necessary to interpret (3.37) using analytic continuation.

Let $\Delta^{++}$be the set of short positive roots, namely the set of roots $\alpha \in \Delta^{+}$such that $\frac{1}{2} \alpha$ is not a root. Then Gindikin and Karpelevič proved (loc. cit.) that

$$
\begin{equation*}
\underset{c}{c}(v)=\frac{I(v)}{I(\rho)}, \quad I(v)=\prod_{\alpha \in \Delta^{+}} I(\alpha: v) \quad(v \in \mathscr{F}) \tag{3.40}
\end{equation*}
$$

where the functions $I(\alpha: \cdot)$ are given by the following formulae. If

$$
\begin{equation*}
n(\alpha)=\operatorname{dim}\left(\mathfrak{g}_{\alpha}\right) ; \quad n(2 \alpha)=\operatorname{dim}\left(\mathfrak{g}_{2 \alpha}\right) ; \quad d(\alpha)=n(\alpha)+n(2 \alpha) \tag{3.41}
\end{equation*}
$$

for $\alpha \in A^{++}(n(\alpha)>0, n(2 \alpha) \geqq 0)$, and if $\Gamma$ is the classical Gamma function, then

$$
\begin{equation*}
I(\alpha: v)=\langle\alpha, \alpha\rangle^{-\frac{1}{2} d(\alpha)} \frac{\Gamma\left(\frac{\langle v, \alpha\rangle}{\langle\alpha, \alpha\rangle}\right) \Gamma\left(\frac{1}{4} n(\alpha)+\frac{1}{2} \frac{\langle v, \alpha\rangle}{\langle\alpha, \alpha\rangle}\right)}{\Gamma\left(\frac{1}{2} n(\alpha)+\frac{\langle v, \alpha\rangle}{\langle\alpha, \alpha\rangle}\right) \Gamma\left(\frac{1}{2} n(2 \alpha)+\frac{1}{4} n(\alpha)+\frac{1}{2} \frac{\langle v, \alpha\rangle}{\langle\alpha, \alpha\rangle}\right)} . \tag{3.42}
\end{equation*}
$$

To each $\alpha \in \Delta^{++}$one can canonically associate the connected semisimple group $G^{\alpha} \subset G$ whose Lie algebra $\mathfrak{g}^{\alpha} \subset \mathfrak{g}$ is the algebra generated by the root spaces $\mathfrak{g}_{ \pm \alpha}$ and $\mathfrak{g}_{ \pm 2 \alpha}$ (if the latter are nonzero). $G^{\alpha}$ and $\mathfrak{g}^{\alpha}$ are $\theta$-stable and the symmetric space associated with $G^{\alpha}$ has rank one. The function $I(\alpha: \cdot)$ is essentially the $\underset{\sim}{c}$-function of the group $G^{\alpha}$. Note that (cf. (3.41))

$$
\begin{equation*}
d(\alpha)=1 \Leftrightarrow \mathfrak{g}^{\alpha} \cong \mathfrak{s} \mathfrak{v}(2,1) \cong \mathfrak{s} l(2, \mathbb{R}) \tag{3.43}
\end{equation*}
$$

We also remark that $\underset{\sim}{c}(\rho)=1$ by (3.40); this is just the normalization condition (3.39).

The formulae (3.40)-(3.42) can be used to determine how $\underset{\sim}{c}$ behaves at infinity. We begin with the following well-known asymptotic expansion for the $\Gamma$-function:

$$
\begin{aligned}
\Gamma(z) & =e^{z \log z-z} \int_{-\infty}^{\infty} e^{z\left(\tau+1-e^{\tau}\right)} d \tau \\
& \sim e^{z \log z-z}\left(\frac{2 \pi}{z}\right)^{\frac{1}{2}}\left(1+\sum_{k=1}^{\infty} a_{k} z^{-k}\right),
\end{aligned}
$$

which is valid for $\operatorname{Re} z>0,|z| \rightarrow+\infty$. The proof is by applying the method of steepest descent to the above integral which has been written in a convenient form for this purpose. From this we obtain $|I(\alpha: v)|^{-2}=f_{\alpha}(\langle\alpha, v\rangle)\left(\alpha \in \Delta^{++}, v \in \mathscr{F}_{I}\right)$, where $f_{\alpha}$ is a function on $(-1)^{\frac{1}{2}} \mathbb{R}$ satisfying, for suitable constants $c_{\alpha, k} \in \mathbb{R}, k$ $=1,2 \ldots$,

$$
f_{a}(z) \sim 2^{-n(2 \alpha)}|z|^{d(x)}\left(1+\sum_{k=1}^{\infty} c_{\alpha, k}|z|^{-2 k}\right), \quad|z| \rightarrow+\infty .
$$

Therefore we can find constants $c^{\prime}, c>0$ such that, for $\alpha \in \Delta^{++}$and $v \in \mathscr{F}_{I}$,

$$
\begin{align*}
& |I(\alpha: v)|^{-2} \leqq c^{\prime}(1+\mid\langle\alpha, v\rangle)^{d(\alpha)} \leqq c(1+\|v\|)^{d(\alpha)} \\
& \quad \beta(v) \leqq c(1+\|v\|)^{n-r} . \tag{3.44}
\end{align*}
$$

It must be observed that, although the $f_{\alpha}(z)$ have a full asymptotic expansion in terms of decreasing negative powers of $|z|, \beta(v)$ has no corresponding asymptotic expansion in terms which are $O\left(\|v\|^{-k}\right), k \rightarrow+\infty$, except for the case that $G$ has a single conjugacy class of Cartan subgroups (cf. formula (4.18) below). However, we obtain

Lemma 3.11. We have, for $v \in \mathscr{F}_{I}$ and $\|v\| \rightarrow+\infty$,

$$
\beta(v)=I(\rho)^{2} 2^{-2} \sum_{x \alpha}{ }^{n(2 \alpha)}\left|\sigma_{s}(v)\right|+O\left(\|v\|^{n-r-1}\right) .
$$

Here

$$
\sigma_{S}(v)=\prod_{\alpha \in A^{+}}\langle\alpha, v\rangle^{\eta(\alpha)}\left(v \in \mathscr{F _ { H }}\right) .
$$

Moreover, there exists $\alpha \in \Delta^{++}$such that $\langle\alpha, v\rangle=0$ if $v \in, \mathscr{F}_{I}$ satisfies $\beta(v)=0$. We use this to prove that, given $\varepsilon>0$, there is a constant $c=c(\varepsilon)>0$ such that

$$
\begin{equation*}
v \in \mathscr{F}_{I},|\langle\alpha, v\rangle| \geqq \varepsilon \quad \forall \alpha \in \Delta^{++} \Rightarrow c \prod_{\alpha \in \Delta^{+}}(1+|\langle\alpha, v\rangle|)^{d(\alpha)} \leqq \beta(v) \tag{3.44a}
\end{equation*}
$$

Let $S(\mathscr{F})$ be the symmetric algebra over $\mathscr{F}$; for any $u \in S(\mathscr{F}), \hat{c}(u)$ is the corresponding differential operator acting in $\mathscr{F}_{I}$. We set

$$
\begin{equation*}
d=\min _{\alpha \in A^{++}} d(\alpha) . \tag{3.45}
\end{equation*}
$$

Let $m$ be an integer with $0 \leqq m \leqq d$ and $u \in S(\mathscr{F})$ an element which is homogeneous and of degree $m$. Then we can find a constant $c=c(u)>0$ such that

$$
\begin{equation*}
|\beta(v ; \partial(u))| \leqq c(1+\|v\|)^{n-r-m}\left(v \in \mathscr{F}_{1}\right) . \tag{3.44b}
\end{equation*}
$$

3.9. Computation of $\mathrm{vol}_{0}(K / M)$. To the best of our knowledge the explicit values for $\frac{d_{0} n}{d n}$ or $\operatorname{vol}_{0}(K / M)$ are not available in the literature. The determination of these numbers is the same as evaluating

$$
\begin{equation*}
J(v)=\int_{N} e^{-(v+\rho)(H(\bar{n}))} d_{0} \bar{n} \tag{3.46}
\end{equation*}
$$

for $v$ satisfying ( 3.38 a ). Of course the method to be used is that of GindikinKarpelevič (loc. cit). However, we need to keep track of the various constants that come up during the course of the evaluation; this precaution was not necessary for the calculation of $\mathcal{c}(v)$ because one can always use the normalization $\underset{\sim}{c}(\rho)=1$ at the end. This also explains the form of the expression (3.40) for $c(\cdot)$ as a ratio.

We shall briefly sketch the modifications that are needed to adapt the Gindikin-Karpelevič argument to our present need. Actually it is more convenient to follow Schiffmann [39] (pp. 10-18). For any element $w$ of the Weyl group $w$ let

$$
\begin{equation*}
J(w: v)=\int_{N \cap w^{-1} N w} e^{-(v+\rho)(H(n))} d_{0} \bar{n} \quad(v \in C(w)), \tag{3.47}
\end{equation*}
$$

where, with $\Delta^{+}(w)=\Delta^{+} \cap w^{-1}\left(-\Delta^{+}\right)$,

$$
C(w)=\left\{v: v \in \mathscr{F},\left\langle v_{R}, \alpha\right\rangle>0 \forall \alpha \in \Delta^{+}(w)\right\} .
$$

The basic result is that the integral (3.47) is absolutely convergent for $v \in C(w)$ and satisfies the following functional equation:

$$
\begin{equation*}
J(w: v)=J\left(w^{\prime}: w^{\prime \prime} \cdot v\right) J\left(w^{\prime \prime}: v\right) \tag{3.48}
\end{equation*}
$$

whenever

$$
\begin{equation*}
v \in C(w), w=w^{\prime} w^{\prime \prime}\left(w^{\prime}, w^{\prime \prime} \in \mathfrak{w}\right) \quad \text { with } l(w)=l\left(w^{\prime}\right)+l\left(w^{\prime \prime}\right) \tag{3.49}
\end{equation*}
$$

Here $l(\cdot)$ is the length function on $\mathfrak{w}$ corresponding to $\Delta^{++}$and one should remember that

$$
\begin{equation*}
\Delta^{+}(w)=\Delta^{+}\left(w^{\prime \prime}\right) \cup w^{\prime \prime-1}\left(\Delta^{+}\left(w^{\prime}\right)\right) \tag{3.50}
\end{equation*}
$$

so, for $v \in C(w)$, we get $v \in C\left(w^{\prime \prime}\right)$ and $w^{\prime \prime} \cdot v \in C\left(w^{\prime}\right)$. Formula (3.48) is the same as Schiffmann's formula (1.6.3). In deriving it we simply follow Schiffmann. However Schiffmann's calculations depend on the following fact: if we set $\bar{N}_{s}$ $=\bar{N} \cap s^{-1} N s$ for any $s \in \mathfrak{w}$, then, under the analytic isomorphism (of varieties)

$$
\begin{aligned}
& \bar{N}_{w^{\prime}} \times \bar{N}_{w^{\prime \prime}} \rightarrow \bar{N}_{w} \\
& \left(\bar{n}^{\prime}, \bar{n}^{\prime \prime}\right) \mapsto w^{\prime \prime}-1 \bar{n}^{\prime} w^{\prime \prime} \cdot \bar{n}^{\prime \prime}
\end{aligned}
$$

the Haar measures on $\bar{N}_{w^{\prime}} \times \bar{N}_{w^{\prime}}$ and $\bar{N}_{w}$ correspond to each other (see (1.4.10) of Schiffmann's article). But, a simple calculation, based on the fact that the elements of $\mathfrak{w}$ have representatives in $K$, shows that the Haar measures $d_{0} \bar{n}^{\prime} d_{0} \bar{n}^{\prime \prime}$ and $d_{0} \bar{n}$ also correspond under the above diffeomorphism. So we obtain (3.48).

The transition from (3.48) to a product formula for $J$ is carried out in the usual manner. If $\Sigma$ is the set of simple roots of $\Delta^{++}$and

$$
\begin{equation*}
w=s_{x_{m}} s_{\alpha_{m-1}} \ldots s_{\alpha_{1}} \quad\left(\alpha_{i} \in \Sigma, l(w)=m\right) \tag{3.51}
\end{equation*}
$$

is a reduced expression of $w$, then

$$
\begin{equation*}
J(w: v)=J\left(s_{\alpha_{1}}: v\right) J\left(s_{\alpha_{2}}: s_{\alpha_{1}} \cdot v\right) \ldots J\left(s_{\alpha_{m}}: s_{\alpha_{m-1}} \ldots s_{\alpha_{1}} \cdot v\right) . \tag{3.52}
\end{equation*}
$$

Moreover, if $\alpha \in \Sigma, \vec{N}_{s_{\alpha}}=\exp \left(\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2 \alpha}\right)$ so that the integrals $J\left(s_{\alpha}: \cdot\right)$ need, for their explicit evaluation, only information from the groups $G^{\alpha}$ of real rank one.

Let us now fix $\alpha \in \Sigma$ and consider $J\left(s_{\alpha}: \cdot\right)$. For simplicity, let us write $\bar{N}^{\alpha}=\bar{N}_{s_{\alpha}}$. Then

$$
\begin{equation*}
J\left(s_{\alpha}: v\right)=\int_{N^{\alpha}} e^{-\left(v+\rho_{\alpha}\right)(H(\bar{n}))} d_{0} \bar{n} \tag{3.53}
\end{equation*}
$$

where

$$
\rho_{\alpha}=\frac{1}{2}(n(\alpha)+2 n(2 \alpha)) \alpha .
$$

We are allowed to replace $\rho$ by $\rho_{\alpha}$ since $H(\bar{n})$ is a multiple of $H_{\alpha}$ when $\bar{n} \in \bar{N}^{\alpha}$. For brevity, let us put

$$
p=n(\alpha), \quad q=n(2 \alpha) .
$$

We shall now use Schiffmann's calculations for evaluating (3.53) (Schiffmann [39, Proposition 2.1]). Let $H=2\langle\alpha, \alpha\rangle^{-1} H_{\alpha}$ so that $\alpha(H)=2$; and let

$$
Q(X)=\frac{4 B_{\alpha}(X, \theta X)}{B_{\alpha}(H, \theta H)} \quad\left(X \in \overline{\mathfrak{n}}^{\alpha}=\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2 \alpha}\right)
$$

where $B_{\alpha}$ is the Killing form of $\mathfrak{g}^{\alpha}$. Then Schiffmann's formula for $H(\bar{n})$ is given by the following: if $\bar{n}=\exp (Y+Z), Y \in \mathfrak{g}_{-\alpha}, Z \in \mathfrak{g}_{-2 \alpha}$,

$$
H(\bar{n})=x H, \quad \text { with } e^{4 x}=\left(1+\frac{1}{2} Q(Y)\right)^{2}+2 Q(Z) .
$$

We now remark that the Killing form of $\mathfrak{g}$ restricts on $\mathfrak{g}^{\alpha}$ to a nonzero multiple of the Killing form of $\mathfrak{g}^{\alpha}$ so that

$$
Q(X)=\langle\alpha, \alpha\rangle\|X\|^{2} \quad\left(X \in \bar{n}^{\alpha},\|X\|^{2}=-\langle X, \theta X\rangle\right) .
$$

A simple calculation then gives the formula

$$
\begin{align*}
e^{-\left(v+\rho_{\alpha}\right)(H(\bar{n}))} & =\left[\left(1+\frac{1}{2}\langle\alpha, \alpha\rangle\|Y\|^{2}\right)^{2}+2\langle\alpha, \alpha\rangle\|Z\|^{2}\right]^{\zeta} \\
\zeta & =-\frac{1}{2} \frac{\langle v, \alpha\rangle}{\langle\alpha, \alpha\rangle}-\frac{1}{4} p-\frac{1}{2} q . \tag{3.54}
\end{align*}
$$

Since $d_{0} \bar{n}$ corresponds to $d_{0} Y d_{0} Z$ under the map $\bar{n} \mapsto(Y, Z)$, (3.53) can now be explicitly evaluated. The result is

$$
\begin{equation*}
J\left(s_{\alpha}: v\right)=2^{d(\alpha) / 2-n(2 \alpha)} \pi^{d(\alpha) / 2} I(\alpha: v), \tag{3.55}
\end{equation*}
$$

where $I(\alpha: \cdot)$ is given by (3.42). It only remains to substitute these formulae in (3.52). Let us write $\beta_{j}=s_{\alpha_{1}} s_{\alpha_{2}} \ldots s_{\alpha_{j-1}} \alpha_{j}$. Then, if we observe that

$$
\begin{aligned}
& \left\langle\alpha_{j}, s_{\alpha_{j-1}} s_{\alpha_{j-2}} \ldots s_{\alpha_{1}} \cdot v\right\rangle=\left\langle\beta_{j}, v\right\rangle \\
& \left\langle\beta_{j}, \beta_{j}\right\rangle=\left\langle\alpha_{j}, \alpha_{j}\right\rangle, \quad n\left(\alpha_{j}\right)=n\left(\beta_{j}\right), \quad n\left(2 \alpha_{j}\right)=n\left(2 \beta_{j}\right),
\end{aligned}
$$

and recall (Varadarajan, [41] Theorem 4.15.10) that

$$
\beta_{1}=\alpha_{1}, \beta_{2}, \ldots, \beta_{m}
$$

is an exact enumeration of the elements of $\Delta^{++} \cap w^{-1}\left(-\Delta^{++}\right)=\Delta^{++}(w)$, we obtain finally the following expression for $J(w: v)$ :

$$
\begin{equation*}
J(w: v)=2^{\sum_{i=1}+(w)} \pi^{\left(\frac{1}{2} d(\alpha)-n(2 x)\right)} \pi^{\frac{1}{2}} \sum_{\alpha=+\cdots(w)}{ }^{d(\alpha)} \prod_{\alpha \in \Delta^{++}(w)} I(\alpha: v) . \tag{3.56}
\end{equation*}
$$

Consider now the element $w_{0} \in \mathfrak{w}$ with the property that $w_{0} \Delta^{+}=-\Delta^{+}$. Clearly (cf. (3.46)) $J(v)=J\left(w_{0}: v\right)$ while $\Delta^{++}\left(w_{0}\right)=\Delta^{++}$. Hence we obtain from (3.56) the formula

$$
\begin{equation*}
\int_{\bar{N}} e^{-(v+\rho)(H(\bar{n}))} d_{0} \bar{n}=2^{\frac{1}{2}(n-r)-\sum_{x \in A^{+}} n(2 x)} \pi^{\frac{1}{2}(n-r)} I(v) . \tag{3.57}
\end{equation*}
$$

In view of Lemma 3.10 we get
Proposition 3.12. We have, with $I(\cdot)$ as in (3.40) and (3.42),

$$
\begin{equation*}
\operatorname{vol}_{0}(K / M)=2^{n-r-} \sum_{\Delta \in A^{-+}} n(2 \alpha) \pi^{\frac{1}{2}(n-r)} I(\rho) . \tag{3.58}
\end{equation*}
$$

For later use we derive the following corollary of this Proposition and Lemma 3.10:

Corollary 3.13. We have

$$
\begin{equation*}
\operatorname{vol}(X)=2^{\sum^{\sum N^{n(2 x)}}} \pi^{-\frac{1}{2}(n-r)} I(\rho)^{-1} \operatorname{vol}_{0}(X) . \tag{3.59}
\end{equation*}
$$

Moreover, if the constant $\gamma=\gamma(G)=\frac{d n}{d_{0} n}$ is as in (3.34)

$$
\begin{equation*}
\gamma(G)=2^{-\frac{1}{2}(n-r)+\sum_{x \in 4^{+}} n(2 x)} \pi^{-\frac{1}{2}(n-r)} I(\rho)^{-1} \tag{3.60}
\end{equation*}
$$

Formula (3.60) is obvious from (3.35) and (3.58).
Remark. One can use Harish - Chandra's results [22, §37, Lemma 4] to evaluate $\operatorname{vol}_{0}(K)$ and $\operatorname{vol}_{0}(M)$, thence $\operatorname{vol}_{0}(K / M)$, in terms of the root structures of $K$ and $M$. Such a formula for $\operatorname{vol}_{0}(K / M)$ is not adequate for our purposes since we need an expression for $\operatorname{vol}_{0}(K / M)$ that is in terms of the data supplied by the symmetric space $S=G / K$.

## 4. Orbital Integrals of Spherical Functions

4.1. For further development of the trace formula (3.27) it is necessary to study the integrals $J_{c}$ more closely. Since the closed conjugacy classes of $G$ are the semisimple ones (Varadarajan [42, II, Theorem 2.17]) we may, in view of Lemma 2.5 (i), consider the distributions

$$
\begin{equation*}
f \mapsto \int_{G / G_{h}} f\left(x h x^{-1}\right) d \bar{x} \quad\left(f \in C_{\mathrm{c}}^{\infty}(G / / K)\right) \tag{4.1}
\end{equation*}
$$

where $h \in G$ is semisimple (i.e. $\operatorname{Ad}(h)$ diagonalizable over $\mathbb{C}$ ), $G_{h}$ is the centralizer of $h$ in $G$ and $d \bar{x}$ is a $G$-invariant measure on $G / G_{h}$.

The fundamental results on these distributions (applied to not necessarily spherical $f$ ) are due to Harish-Chandra [20], [21] (see also Varadarajan [42]). Our main observations are straightforward consequences of Harish-Chandra's theory, except for some variations.
4.2. We shall begin with the normalization of the measure $d \bar{x}$ on $G / G_{h}$. For $G$ itself we have chosen the Haar measure (cf. subsection 3.7)

$$
\begin{equation*}
d x=\eta_{2 \rho} d k d_{0} a d n \quad(x=k a n) \tag{4.2}
\end{equation*}
$$

Suppose now that $Z$ is a $\theta$-stable closed subgroup of $G$ such that $\left[Z: Z^{0}\right]<\infty$. If $\mathfrak{z}$ is the Lie algebra of $Z, \theta(\mathfrak{z})=\mathfrak{z}$, and so $z$ is reductive in $\mathfrak{g}$. It is then clear that the restrictions to $3 \times 3$ of $\langle.,$.$\rangle and (.,.) inherit the same properties vis-à-vis 3$ as $\langle.,$.$\rangle and (.,.) have relative to \mathfrak{g}$. We may consequently speak of the Haar measure $d z^{0}$ on $Z^{0}$. We define the Haar measure $d z$ on $Z$ to be the Haar measure that coincides with $\left[Z: Z^{0}\right]^{-1} d z^{0}$ on $Z^{0}$. This normalization ensures that the maximal compact subgroups of $Z$ have volume 1 . If $k \in K$, the Haar measure on $Z^{k}$ corresponds to $d z$ under the conjugation induced by $k$.

Let $h \in G$ be semisimple and $G_{h}\left(\right.$ resp. $g_{h}$ ) be its centralizer in $G$ (resp.g. $)$. It is known that $\left[G_{h}: G_{h}^{0}\right]<\infty$. We say that $h$ is in standard position if $\mathfrak{g}_{h}$ is $\theta$-stable.

Lemma 4.1. The following conditions on $h$ are equivalent: (i) $h$ is in standard position; (ii) $h$ can be imbedded in a $\theta$-stable $\mathrm{CSG}^{1}$ : (iii) if $h=k \exp X(k \in K, X \in \mathfrak{s})$, then $k$ centralizes $X$; (iv) $G_{h}$ is $\theta$-stable.

The implication (iv) $\Rightarrow$ (i) is trivial. For (i) $\Rightarrow$ (ii) we note that there are $\theta$ stable CSA's of $\mathfrak{g}$ that are contained in $\mathfrak{g}_{h}$, since $\mathfrak{g}_{h}$ is $\theta$-stable and $\operatorname{rk}\left(\mathfrak{g}_{h}\right)=\operatorname{rk}(\mathfrak{g})$; $h$ is then in the corresponding CSG which is $\theta$-stable. Since (ii) $\Rightarrow$ (iii) is obvious, we are left with (iii) $\Rightarrow$ (iv). It is enough to prove that $G_{h}=G_{k} \cap G_{X}$ where $G_{k}$ (resp. $G_{X}$ ) is the centralizer of $k$ (resp. $X$ ) in $G$, since both of these are $\theta$-stable. If $y \in G_{h}$, we use the uniqueness of the decomposition of a semisimple element of $\mathrm{GL}(\mathfrak{g})$ as a product of two commuting semisimple elements in $\mathrm{GL}(\mathfrak{g})$ with eigenvalues respectively real and of absolute value 1 , to conclude that $\operatorname{Ad}(y)$ commutes with $\operatorname{Ad}(k)$ and $e^{\operatorname{ad} X}$. Hence $X^{y}=X$, giving $y \in G_{k} \cap G_{X}$. Since $G_{k} \cap G_{X} \subset G_{h}$, we are done.

Lemma 4.2. Any semisimple element of $G$ is conjugate to one in standard position. If $h_{1}, h_{2} \in G$ are semisimple and in standard position, they are conjugate under $G$ if and only if they are conjugate under $K$.

We assume for the second assertion that $h_{i}=k_{i} \exp X_{i}$ with $k_{i} \in K, X_{i} \in \mathfrak{a}, X_{i}^{k_{i}}$ $=X_{i}$. Let $y h_{1} y^{-1}=h_{2}$, so that $y k_{1} y^{-1}=k_{2}$ and $X_{1}^{y}=X_{2}$; so, using a result of Harish-Chandra ([22], $\S 5$, Lemma 1), we may assume that $X_{1}=X_{2}$ ( $=X$ say). If we write $y=v \exp Y$ where $v \in K, Y \in \mathfrak{s}$, then $\exp 2 \mathrm{Y}=\theta\left(y^{-1}\right) y$ commutes with $k_{1}$ so that $Y^{k_{1}}=Y$. Hence $k_{2}=v k_{1} v^{-1}$. Moreover, as $G_{X}$ is $\theta$-stable and $y \in G_{X}$, $\exp 2 \mathrm{Y} \in G_{X}$; this gives $[Y, X]=0$. But then $X=X^{y}=X^{v}$, giving $v h_{1} v^{-1}=h_{2}$.

It follows from the remarks made at the beginning of this subsection that for any semisimple $h$ in standard position it makes sense to speak of the measure $d x_{h}$ on $G_{h}$. We define the measure $d \bar{x}$ on $G / G_{h}$ by $d \bar{x} d x_{h}=d x$. If $h$ is semisimple but otherwise arbitrary, we choose $y \in G$ such that $h_{1}=y h y^{-1}$ is in standard position, and define $d x_{h}$ on $G_{h}$ to be the pull-back of the Haar measure $d x_{h_{1}}$ on $G_{h_{1}}$ through the conjugation by $y$; and as before, $d \bar{x}$ is defined on $G / G_{h}$ by $d \bar{x} d x_{h}$ $=d x$. Lemmas 4.1 and 4.2 show that these definitions do not depend on the choice of $y$, and that the following result is true.

Lemma 4.3. Let $h, h^{\prime} \in G$ be semisimple elements and $h^{\prime}=y h y^{-1}$ for some $y \in G$. Then conjugation by y carries $d x_{h}$ and $d \bar{x}$ to the corresponding measures $d x_{h^{\prime}}$ and $d \bar{x}^{\prime}$ on $G_{h^{\prime}}$ and $G / G_{h^{\prime}}$ respectively.

[^1]We now define, for any $f \in C_{c}^{\infty}(G)$ and any semisimple $h \in G$,

$$
\begin{equation*}
I(f: h)=I_{h}(f)=\int_{G / G_{h}} f\left(x h x^{-1}\right) d \bar{x} . \tag{4.3}
\end{equation*}
$$

It is clear from Lemma 4.3 that $I(f: h)=I\left(f: y h y^{-1}\right)(y \in G)$; we may assume therefore that $h$ is in standard position when studying $I_{h}$. The restriction of $I_{h}$ to $C_{c}^{\infty}(G / / K)$ can be interpreted as an element of $\mathscr{D}^{\prime}(G / / K)$, let us call it $I_{h}^{\#}$. We set (cf. subsection 3.4)

$$
\begin{equation*}
T_{h}=\left(I_{h}^{\#}\right)^{\dagger} \quad(h \in G \text { semisimple }) . \tag{4.4}
\end{equation*}
$$

Our interest is really in these distributions $T_{h}$. But their study depends on that of the $I_{h}$ which is essentially the Harish-Chandra theory of the invariant integral on $G$. For full details concerning this theory see Varadarajan [42].
4.3. The Invariant Integral. Let $L$ be a $\theta$-stable CSG and I the corresponding CSA. We put $L_{I}=L \cap K, \mathrm{I}_{I}=\mathrm{I} \cap \mathrm{f}, L_{R}=\operatorname{expl} \mathrm{I}_{R}$ where $\mathrm{I}_{R}=\mathrm{I} \cap \mathfrak{s}$; then $L=L_{I} L_{R} \simeq L_{I}$ $\times L_{R}$. For any root $\alpha$ of $\left(\mathfrak{g}_{c}, \mathfrak{l}_{c}\right)$ we write $\xi_{\alpha}$ for the corresponding global root, this being the homomorphism according to which $L$ acts on the root space $\mathfrak{g}_{c, \alpha}$. Let $\mathrm{m}_{1}$ be the centralizer of $\mathfrak{l}_{R}$ in $\mathfrak{g}$. The roots of $\left(\mathrm{m}_{1, c}, \mathfrak{l}_{c}\right)$ are precisely those roots of $\left(\mathfrak{g}_{c}, \mathfrak{l}_{c}\right)$ which are purely imaginary on I. We select a positive system $P_{I}$ of roots of ( $\mathrm{m}_{1, c}, l_{c}$ ) and define the functions $\Delta_{1}, \Delta_{+}$and $\Delta_{1}$ on $L$ as follows;

$$
\begin{equation*}
' \Delta_{I}=\prod_{\alpha \in P_{I}}\left(1-\xi_{-\alpha}\right) ; \quad \Delta_{+}=\left|\prod_{ \pm \alpha \notin P_{I}}\left(1-\xi_{-\alpha}\right)\right|^{\frac{1}{2}} ; \quad \Delta_{1}=\Delta^{\prime} \Delta_{I} \Delta_{+} \tag{4.5a}
\end{equation*}
$$

we have

$$
\begin{equation*}
\Delta_{+}(h)=\left|\operatorname{det}\left(\left(1-\operatorname{Ad}\left(h^{-1}\right)\right)_{\mathbf{g} / \mathbf{m}_{1}}\right)\right|^{\frac{1}{2}} . \tag{4.5b}
\end{equation*}
$$

Following Harish-Chandra we shall define, for any $f \in C_{c}^{\infty}(G)$,

$$
\begin{equation*}
{ }^{\prime} F_{f}(h)=\Delta_{1}(h) \int_{G / L_{R}} f\left(x h x^{-1}\right) d x^{*} \quad\left(h \in L^{\prime}\right) ; \tag{4.6}
\end{equation*}
$$

here $L^{\prime}$ is the set of regular points in $L$ and $d x^{*}$ is the $G$-invariant measure on $G / L_{R}$ such that $d x=d x^{*} d_{0} b, d_{0} b$ being the Haar measure defined on $L_{R}$ by $\langle.,$.$\rangle . We then have$

$$
\begin{equation*}
' F_{f}(h)=\Delta_{1}(h) I(f: h) \quad\left(f \in C_{c}^{\infty}(G), h \in L^{\prime}\right) \tag{4.7}
\end{equation*}
$$

Furthermore, for use in explicit calculation, we have the following easily proved result.

Lemma 4.4. There are positive systems $P$ of roots of $\left(\mathfrak{g}_{c}, \mathfrak{l}_{c}\right)$ such that $P_{I} \subset P$ and $P \backslash P_{I}$ is stable under complex conjugation. Fix such a $P$ and write

$$
\delta_{P}=\frac{1}{2} \sum_{\alpha \in P} \alpha ; \quad \delta_{I}=\frac{1}{2} \sum_{\alpha \in P_{I}} \alpha ; \quad \zeta_{P}=\delta_{P}-\delta_{I}
$$

Then, for $h=h_{I} \exp H_{R} \in L^{\prime}$, with $h_{I} \in L_{I}, H_{R} \in I_{R}$, and for $f \in C_{c}^{\infty}(G)$, we have

$$
\begin{align*}
& \Delta_{+}(h)=\varepsilon_{R}(h) e^{\zeta_{P}\left(H_{R}\right)} \prod_{\alpha \in P \backslash P_{T}}\left(1-\xi_{-\alpha}(h)\right) ; \\
& { }^{\prime} F_{f}(h)=\varepsilon_{R}(h) e^{\zeta_{P}\left(H_{R}\right)} \Delta_{P}(h) I(f: h), \tag{4.8}
\end{align*}
$$

where

$$
\begin{equation*}
\varepsilon_{R}(h)=\operatorname{sgn} \prod_{\substack{\alpha \in P, \alpha \text { real }}}\left(1-\xi_{-\alpha}(h)\right) \quad(h \in L) ; \quad{ }^{\prime} \Delta_{P}=\prod_{\alpha \in P}\left(1-\xi_{-\alpha}\right) . \tag{4.9}
\end{equation*}
$$

Now ${ }^{\prime} F_{f} \in C^{\infty}\left(L^{\prime}\right)$ for any $f \in C_{c}^{\infty}(G)$; moreover, if $h \in L$ is singular and $u \in U\left(\mathfrak{l}_{c}\right)$ (= subalgebra of $U\left(\mathfrak{g}_{c}\right)$ generated by 1 and I$), \lim _{h^{\prime} \rightarrow h}\left(u^{\prime} F_{f}\right)\left(h^{\prime}\right)$ will exist as long as $h^{\prime}$ approaches $h$, ultimately in one connected component of $L^{\prime}$. In general, different choices of these connected components will lead to different limiting distributions. If, however, $u={ }^{\prime} v$, with

$$
\begin{equation*}
v=e^{-\delta_{1}} \circ v \circ e^{\delta_{I}} \tag{4.10}
\end{equation*}
$$

where $v \in U\left(\mathbf{I}_{c}\right)$ and $v^{s_{\alpha}}=-v$ for all $\alpha \in P$ satisfying $\xi_{\alpha}(h)=1$, then ' $v^{\prime} F_{f}$ extends to a continuous function in a ngbd of $h$. In particular, if

$$
\begin{equation*}
P_{h}=\left\{\alpha: \alpha \in P, \xi_{\alpha}(h)=1\right\}, \quad \varpi_{h}=\prod_{\alpha \in P_{h}} H_{\alpha}, \tag{4.11}
\end{equation*}
$$

then it follows that $f \mapsto\left({ }^{\prime} w_{h}{ }^{\prime} F_{f}\right)(h)\left(f \in C_{c}^{\infty}(G)\right)$ is a well-defined invariant distribution on $G$.
4.4. Limit Formulae. Let $h \in G$ be an element in standard position and let $L$ be a $\theta$-stable CSG of $G$. We say that $L$ is aligned to $h$ if the Lie algebra I of $L$ is fundamental in $\mathfrak{g}_{h}$, i.e., $\left(\mathfrak{g}_{h, c}, \mathfrak{l}_{c}\right)$ has no real roots. Clearly we can find $\theta$-stable CSG's $L$ containing $h$ and aligned to $h$; and any two such $L$ are conjugate under $K \cap G_{h}^{0}$. It was proved by Harish-Chandra [21, Lemma 23] that for a suitable constant $c(h) \neq 0$ we have, for all $f \in C_{c}^{\infty}(G)$,

$$
\begin{equation*}
I(f: h)=c(h) \quad\left({ }^{\prime} \varpi_{h}^{\prime} F_{f}\right)(h) \quad(L \text { aligned to } h) \tag{4.12}
\end{equation*}
$$

By carefully keeping track of the constants in Harish-Chandra's proof of (4.12) we can determine the precise value of $c(h)$. To this end we proceed as follows.

Let $\mathfrak{z}$ be a reductive Lie algebra over $\mathbb{R}$ with connected adjoint group $Z$, and let $U \subset Z$ be a maximal compact subgroup with Lie algebra $u$. Let $\mathfrak{h}$ be a fundamental CSA of $\mathfrak{j}$. We define (cf. Harish-Chandra [22, §37])

$$
\begin{align*}
& \kappa(3)=(-1)^{q}(2 \pi)^{-q} 2^{-s / 2} w_{u}\left(\delta_{u}\right)^{-1}\left|\mathfrak{w}\left(Z / Z_{b}\right)\right|^{-1} \gamma(Z)^{-1},  \tag{4.13}\\
& q=\frac{1}{2}(\operatorname{dim} Z / U-\operatorname{rk} Z+\operatorname{rk} U) ; \quad s=\operatorname{dim} Z / U-\operatorname{rk} Z / U .
\end{align*}
$$

Here $\varpi_{u}\left(\delta_{u}\right)$ is defined in the obvious way using a positive system of roots for a $C S A$ of $\mathfrak{u} ; Z_{\mathfrak{b}}$ is the centralizer of $\mathfrak{b}$ in $Z$ and $\mathfrak{w}\left(Z / Z_{\mathfrak{b}}\right)$ is the normalizer of $\mathfrak{b}$ in $Z$ modulo $Z_{\mathfrak{h}}$; and $\gamma(Z)$ is the constant given by (3.60) for the group $Z$. With this notation, we get, for the constant $c(h)$ in (4.12), (cf. Lemma 4.4)

$$
\begin{equation*}
c(h)=\kappa\left(\mathfrak{g}_{h}\right) \varepsilon_{R}(h) e^{-\zeta_{P}\left(\log h_{R}\right)} \prod_{\alpha \in P \backslash P_{h}}\left(1-\xi_{-\alpha}(h)\right)^{-1} \tag{4.14}
\end{equation*}
$$

Remark 4.5. (i) The presence of the constant $\kappa\left(\mathfrak{g}_{h}\right)$ in the formula (4.14) for $c(h)$ is due to the fact that this computation is ultimately reduced to an analogous one on $\mathfrak{g}_{h}$. The constant $\kappa\left(\mathfrak{g}_{h}\right)$ is then the constant that enters the Harish-Chandra limit formula for $\mathfrak{g}_{h}$. In fact, let $P_{3}$ be a positive system of roots of $\left(\mathfrak{3}_{c}, \mathfrak{h}_{c}\right)$ and let $\pi_{3}=\prod_{\alpha \in P_{3}} \alpha, w_{3}=\prod_{\alpha \in P_{3}} H_{\alpha}$; let the invariant integral $\psi$ on $\mathfrak{z}$ be defined by

$$
\psi_{g}(H)=\pi_{\mathfrak{3}}(H) \int_{Z / Z_{\mathfrak{h}}} g\left(H^{z}\right) d z^{*} \quad\left(g \in C_{c}^{\infty}(3), H \in \mathfrak{h}^{\prime}\right)
$$

Then $\partial\left(m_{3}\right) \psi_{g}$ extends to a continuous function on all of $\mathfrak{h}$, and the constant $\kappa(\mathfrak{z})$ is defined by the limit formula

$$
\begin{equation*}
g(0)=\kappa(\mathfrak{J}) \cdot\left(\partial\left(\varpi_{\mathfrak{z}}\right) \psi_{g}\right)(0) \quad\left(g \in C_{c}^{\alpha}(\mathfrak{J})\right) . \tag{4.15}
\end{equation*}
$$

The explicit value given in (4.13) for $\kappa(3)$ is a result of Harish-Chandra [22, $\S \S 36,37]$ (cf. also Varadarajan [42, II, Section 17]).

If $Z$ is compact, the function $H \mapsto \int_{Z / Z_{6}} g\left(H^{z}\right) d z^{*}$ is itself $C^{\infty}$ and one obtains the explicit value (cf. Harish-Chandra [16, Lemma 16])

$$
\begin{equation*}
\kappa(\mathfrak{z})=\left(\partial\left(w_{z}\right) \pi_{3}\right)^{-1} . \tag{4.16}
\end{equation*}
$$

It is to be observed that $\varpi_{3}$ and $\pi_{3}$ are homogeneous of the same degree and so $\partial\left(\sigma_{3}\right) \pi_{3}$ is indeed a constant which is easily seen to be $\neq 0$.
(ii) The formula (4.12) with $h=e$ is one of the main steps in Harish-Chandra's derivation of the Plancherel formula for $G$. If $G$ has a single conjugacy class of CSG's, then the limit formula is essentially equivalent to the Plancherel formula (Harish-Chandra [15], [17]).

Let us now assume for the moment that $G$ has a single conjugacy class of CSG's and examine (4.12) for spherical $f$, i.e., $f \in C_{c}^{\infty}(G / / K)$. A straightforward calculation shows that (cf. (3.7) and (3.34)),

$$
\begin{equation*}
' F_{f}(h)==^{\prime} \Delta_{I}\left(h_{I}\right)(\mathscr{A} f)\left(h_{R}\right) \quad\left(h \in L^{\prime}, h=h_{I} h_{R}\right) . \tag{4.17}
\end{equation*}
$$

On taking $h_{I}=\exp H_{I}, h_{R}=\exp H_{R}$ with $H_{I} \in I_{I}, H_{R} \in \mathfrak{l}_{R}=a$, we get

$$
e^{\delta_{I}\left(\boldsymbol{H}_{I}\right)^{\prime}} F_{f}(h)=\sum_{s \in \mathbf{w}_{I}} \varepsilon(s)\left(e^{s \delta_{I}} \otimes \mathscr{A} f\right)\left(H_{I}, \exp H_{R}\right)
$$

where $\mathfrak{w}_{I}$ is the subgroup of $\mathfrak{w}(\mathfrak{g}, \mathrm{l})$ generated by the $s_{\alpha}\left(\alpha \in P_{I}\right)$. Now, for $s \in \mathfrak{w}_{I}$ and $m=\omega_{e}$, we have

$$
\varepsilon(s) \varpi \circ e^{s \delta_{I}}=e^{s \delta_{I}} \prod_{\alpha \in P}\left(H_{\alpha, I}^{s}+\left\langle\delta_{I}, \alpha\right\rangle+H_{\alpha, R}\right) .
$$

So, writing ' $w_{\mathfrak{a}}=\prod_{\alpha \in P}\left(H_{\alpha, R}+\left\langle\delta_{I}, \alpha\right\rangle\right)$, we find

$$
\begin{aligned}
\left({ }^{\prime} \varpi^{\prime} F_{f}\right)(e) & =\left|\mathfrak{w}_{I}\right|\left({ }^{\prime} \varpi_{\mathfrak{a}} \mathscr{A} f\right)(1) \\
& =\left|\mathfrak{w}_{I}\right| \int_{\mathscr{F}_{I}} \mathscr{H} f(v) \cdot\left({ }^{\prime} \varpi_{\mathfrak{a}}\right)(-v) d v .
\end{aligned}
$$

Using (3.19) we then obtain

$$
\beta(v)=\kappa(\mathfrak{g})|\mathfrak{w}|\left|\mathfrak{w}_{I}\right| \varpi_{I}\left(\delta_{I}\right) \prod_{\alpha \in P \backslash P_{I}}\left(-v\left(H_{\alpha, R}\right)+\left\langle\delta_{I} \alpha\right\rangle\right) \quad\left(v \in \mathscr{F}_{I}\right) .
$$

To get an explicit formula for $\beta$ we can now proceed in either of two ways. We can appeal to Lemma 3.11; or else we can use the explicit formulae for $\kappa(\mathrm{g})$ and $\gamma(G)$ together with Lemma 4 of $\S 37$ in Harish-Chandra [22]. In either case we must use the fact that $n(2 \alpha)=0$ for all $\alpha \in \Delta^{++}$(cf. Araki [1, Propositions 2.4 and 2.3]). The final formula is

$$
\begin{equation*}
\beta(v)=\mathrm{I}(\rho)^{2} \prod_{\alpha \in\left(P \backslash P_{I}\right) / \text { conj }}\left(|\langle\alpha, v\rangle|^{2}+\left\langle\alpha, \delta_{I}\right\rangle^{2}\right) \quad\left(v \in \mathscr{F}_{I}\right) . \tag{4.18}
\end{equation*}
$$

Here we must note that no root of $P \backslash P_{I}$ is real and the product on the right hand side of (4.18) is over a complete set of mutually non-conjugate roots in $P \backslash P_{I}$.
4.5. The Distributions $T_{h}$. Recalling the definition of $T_{h}$ (cf. (4.4)), we formulate

Proposition 4.6. Let $h \in G$ be a semisimple element. Then there exists a unique distribution $T_{h} \in \mathscr{D}^{\prime}(A)^{\text {w }}$ such that

$$
\begin{equation*}
I_{h}(f)=\int_{G / G_{h}} f\left(x h x^{-1}\right) d \bar{x}=\left\langle T_{h}, \mathscr{A} f\right\rangle \quad\left(f \in C_{c}^{\infty}(G / / K)\right) . \tag{4.19}
\end{equation*}
$$

$T_{h}$ is tempered and $T_{h}=T_{y h y-1}$, for all $y \in G$.
It is possible to say more about the distributions $T_{h}$, provided we exploit the relation between the invariant integral on $G$ and the invariant integral on $\bar{M}_{1}$, the Levi component of a parabolic subgroup $Q$ associated to $\mathrm{l}_{R}$, i.e., $\bar{M}_{1}$ is the centralizer of $\mathrm{I}_{R}$ in $G$. If $Q=\bar{M} L_{R} N^{+}$is the Langlands decomposition of $Q, \mathrm{n}^{+}$ $=$ Lie algebra of $N^{+}$and $d_{Q}\left(m_{1}\right)=\left|\operatorname{det}\left(\operatorname{Ad}\left(m_{1}\right)_{\mathfrak{n}^{+}}\right)\right|^{\frac{1}{2}}$, we have the following result (see Varadarajan [42, II. 10. Proposition 6]).

Proposition 4.7. For any $f \in C_{c}^{\infty}(G)$, let $\bar{f} \in C_{c}^{\infty}(G)$ and $f_{1} \in C_{c}^{\infty}\left(\bar{M}_{1}\right)$ be defined by

$$
\bar{f}(x)=\int_{K} f\left(k x k^{-1}\right) d k(x \in G) ; \quad f_{1}\left(m_{1}\right)=d_{Q}\left(m_{1}\right) \int_{N^{+}} \bar{f}\left(m_{1} n\right) d_{0} n\left(m_{1} \in \bar{M}\right) .
$$

Then, for $h=h_{I} h_{R} \in L^{\prime}$ with $h_{I} \in L_{I}, h_{R} \in L_{R}$,

$$
\begin{equation*}
' F_{f}(h)=\gamma(G) \gamma(\bar{M})^{-1} \Delta_{I}\left(h_{I}\right) \int_{\bar{M}} f_{1}\left(m h_{I} m^{-1} h_{R}\right) d m \tag{4.20}
\end{equation*}
$$

Observe that in general $\bar{M}$ is neither connected nor semi-simple; however, all the foregoing theory applies to it as it is a group of the so-called HarishChandra class $\mathfrak{5}$ (Harish-Chandra [22, §3], Varadarajan [42, II, Section 1]).

For any $f \in C_{c}^{\infty}(G / / K)$ and any $a \in L_{R}$, we define $f_{1, a} \in C_{c}^{\infty}\left(\vec{M} / / K_{M}\right)$ by $f_{1, a}(m)$ $=f_{1}(m a)(m \in \bar{M})$, where $K_{M}=K \cap \bar{M}$. Denoting by $\mathscr{A}_{M}$ the Abel transform on $\bar{M}$ and setting

$$
\begin{equation*}
{ }^{*} L=A \cap \bar{M}, \tag{4.21}
\end{equation*}
$$

we obtain, from the definitions of the Abel transforms,

$$
\begin{equation*}
\left(\mathscr{A}_{M} f_{1, a}\right)(b)=\gamma(\bar{M}) \gamma(G)^{-1}(\mathscr{A} f)(b a) \quad\left(a \in L_{R}, b \in^{*} L\right) \tag{4.22}
\end{equation*}
$$

If *I denotes the Lie algebra of * $L$, we obviously have

$$
\begin{equation*}
A=* L L_{R} \simeq * L \times L_{R} ; \quad \mathfrak{a}=* \mathrm{I} \oplus \mathrm{I}_{R} ; \quad{ }^{*} \perp \mathrm{I}_{R} \tag{4.23}
\end{equation*}
$$

Consider now the group $\bar{M}$ and its compact $\operatorname{CSG} L_{I} \subset K_{\bar{M}}$. Applying Proposition 4.6, we have for each $h^{\prime} \in L_{I}^{\prime}$, the set of elements in $L_{I}$ which are regular in $\bar{M}$, a unique tempered distribution $V_{h^{\prime}}$ on ${ }^{*} L$, invariant under the Weyl group $\mathfrak{w}_{M}$ (of the symmetric space $\left.\bar{M} / K_{\bar{M}}\right)$ such that, for $g \in C_{c}^{\infty}\left(\bar{M} / / K_{\bar{M}}\right)$,

$$
\begin{equation*}
\Delta_{I}\left(h^{\prime}\right) \int_{\bar{M}} g\left(m h^{\prime} m^{-1}\right) d m=\left\langle V_{h^{\prime}}, \mathscr{A}_{\bar{M}} g\right\rangle \tag{4.24}
\end{equation*}
$$

So, using (4.20), (4.24) and (4.22), we get

$$
\begin{equation*}
' F_{f}(h)=\left\langle V_{h_{I}} \otimes \delta_{h_{\mathbb{R}}}, \mathscr{A} f\right\rangle \tag{4.25}
\end{equation*}
$$

Here $\otimes$ is with respect to the identification $A \simeq^{*} L \times L_{R}$; and $\delta_{h R}$ is the Dirac measure at $h_{R}$. So

$$
\begin{equation*}
T_{h}=U_{1}(h)^{-1}\left(V_{h_{I}} \otimes \delta_{h_{\mathrm{R}}}\right)^{-} \quad\left(h=h_{I} h_{R} \in L^{\prime}\right), \tag{4.26}
\end{equation*}
$$

where the bar - indicates averaging with respect to $\mathfrak{w}$.
Suppose now $h$ is singular and $L$ aligned to $h$. Writing

$$
' \varpi_{h}=\sum_{1 \leqq j \leqq p_{h}} u_{j} v_{j} \quad\left(u_{j} \in U\left(\mathrm{I}_{I, c}\right), v_{j} \in U\left(\mathrm{I}_{R, \mathrm{c}}\right)\right),
$$

it follows from formula (4.12) and (4.25) that, if $h=h_{I} h_{R} \in L$,

$$
\begin{equation*}
T_{h}=c(h)\left(\sum_{1 \leqq j \leqq p_{h}} u_{j} V_{h_{I}} \otimes v_{j} \delta_{h_{R}}\right)^{-} . \tag{4.27}
\end{equation*}
$$

Here

$$
u_{j} V_{h_{I}}=\lim _{L_{I}^{+} \ni h^{\prime} \rightarrow h_{I}} u_{j} V_{h^{\prime}} \quad\left(L_{I}^{+} \text {is a connected component of } L_{I}^{\prime}\right) .
$$

In deriving (4.27) we must remember that the theory of the invariant integral guarantees that $V_{h^{\prime}}$ is $C^{\infty}$ in $h^{\prime}$ and that the derivatives of $V_{h^{\prime}}$ in $h^{\prime}$ have limits as $h^{\prime}$ approaches singular points, as long as the approach $h^{\prime} \rightarrow h_{I}$ is eventually in a connected component of $\omega \cap L_{I}^{\prime}$ for some open ngbd $\omega$ of $h_{I}$ in $L_{I}$. However, in calculating the limit, the connected component in (4.27) should be the same for all $j$. These formulae lead immediately to the following theorem.

Theorem 4.8. Let $h \in G$ be a semisimple element in standard position and let $L$ $=L_{I} L_{R}$ be a $\theta$-stable CSA containing $h$ such that $L$ is aligned to $h$ and $L_{R} \subset A$. Let ${ }^{*} L=\bar{M} \cap A$ so that ${ }^{*} L L_{R} \simeq{ }^{*} L \times L_{R}$. Then $T_{h}$ is given by (4.27) and

$$
\begin{equation*}
\operatorname{supp}\left(T_{h}\right) \subset \bigcup_{s \in \mathbf{w}} s \cdot\left({ }^{*} L h_{R}\right) \tag{4.28}
\end{equation*}
$$

Remark 4.9. Because of the differentiations $v_{j}$ in (4.27) (which are transversal to ${ }^{*} L$ ) we cannot say that $T_{h}$ lives on the union of the affine cosets $s \cdot\left({ }^{*} L h_{R}\right)$. This is the case however if $h_{R}$ is regular in $L_{R}$, i.e., if $\xi_{\alpha}\left(h_{R}\right) \neq 1$ for all $\alpha \in P \backslash P_{I}$. In this case the roots of $\left(g_{h, c}, \mathrm{l}_{c}\right)$ are in $\pm P_{I}$ so that ${ }^{\prime} \sigma_{h} \in U\left(\mathrm{l}_{I, c}\right)$. The formula (4.27) then becomes

$$
\begin{equation*}
T_{h}=c(h) \quad\left({ }^{\prime} \varpi_{h} V_{h_{R}} \otimes \delta_{h_{R}}\right)^{-} . \tag{4.29}
\end{equation*}
$$

Corollary 4.10. Suppose that $L_{R}=A$ and that $h \in L$ is such that $h_{R}$ is regular in $L_{R}$. Then

$$
\begin{equation*}
T_{h}=\Delta_{+}(h)^{-1}\left(|\mathfrak{w}|^{-1} \sum_{s \in \mathfrak{w}} \delta_{s h_{R}}\right) \tag{4.30}
\end{equation*}
$$

In this case, $\bar{M}=M, P_{h} \subset P_{I}$ : and if $M_{h_{I}}$ (resp. $\mathfrak{m}_{h_{I}}$ ) is the centralizer of $h_{I}$ in $M($ resp. m$)$,

$$
\begin{equation*}
G_{h}=M_{h_{I}} A, \quad \mathfrak{g}_{h}=\mathfrak{m}_{h_{I}}+\mathfrak{a} . \tag{4.31}
\end{equation*}
$$

We now observe that the distribution $V_{h_{I}}$ is the constant ' $\Delta_{I}\left(h_{I}\right)$. Moreover, a simple calculation shows that

$$
\begin{aligned}
\left({ }^{\prime} \varpi_{h}^{\prime} \Delta_{I}\right)\left(h_{I}\right) & =\left(\prod_{\alpha \in P_{h}}\left(H_{\alpha, I}+\left\langle\alpha, \delta_{I}\right\rangle\right) \cdot \Delta_{I}\right)\left(h_{I}\right) \\
& =\left(\partial\left(\varpi_{h}\right) \pi_{h}\right) \cdot \prod_{\alpha \in P_{I} \backslash P_{h}}\left(1-\xi_{-\alpha}(h)\right) .
\end{aligned}
$$

Hence, by (4.29), (4.14) and (4.8) we find that

$$
T_{h}=a(h)|\mathfrak{w}|^{-1} \sum_{s \in \mathfrak{w}} \delta_{s h_{R}}
$$

where $a(h)$ is the constant given by

$$
a(h)=\left(\partial\left(\varpi_{h}\right) \pi_{h}\right) \kappa\left(\mathfrak{g}_{h}\right) \Delta_{+}(h)^{-1} .
$$

On the other hand, as the group $M_{h_{I}}$ is compact, $\gamma\left(M_{h_{I}}\right)=1$ and we get

$$
\kappa\left(g_{h}\right)=\kappa\left(m_{h_{r}}\right)=\left(\partial\left(\varpi_{h}\right) \pi_{h}\right)^{-1}
$$

in view of Remark 4.5. (i). Hence we find that $a(h)=\Delta_{+}(h)^{-1}$, which gives (4.30).
Corollary 4.11. The formula (4.30) is valid if $\operatorname{rk}(G / K)=1$, for any element $h \in L$ which is not elliptic, i.e., for which $h_{R} \neq e$.
4.6. The distributions $T_{h}$ for regular $h$. We shall conclude section 4 by proving the following theorem.

Theorem 4.12. Let $h$ be a regular element of $L=L_{I} L_{R}$. Then $T_{h}$ is a $C^{\infty}$ function on $\bigcup_{s \in \boldsymbol{w}} s \cdot\left({ }^{*} L h_{R}\right)$, viz., there is a $C^{\infty}$ function $\alpha_{h}$ on ${ }^{*} L h_{R}$ such that, for all $g \in C_{c}^{\infty}(A)$,

$$
\left\langle T_{h}, g\right\rangle=|\mathfrak{w}|^{-1} \sum_{s \in \mathfrak{w} *} \int_{L} g^{s}\left(b h_{R}\right) \alpha_{h}(b) d_{0} b .
$$

It is clear from (4.26) that for proving this we may assume $L_{R}=\{1\}$. We need the following

Lemma 4.13. Let $L \subset K$ be a CSG of $G$ (so that $\operatorname{rk}(G)=\operatorname{rk}(K))$ and let $h \in L^{\prime}$. Then the projection map $\varphi: G \rightarrow A \times N$ which sends kan to ( $a, n$ ), restricts to a proper submersion of $[h]_{G}$ onto $A \times N$. In particular, the restriction to $[h]_{G}$ of the projection $\pi: G \rightarrow A$ (cf. subsection 3.2) is everywhere submersive and maps $[h]_{G}$ onto $A$.

Let $\sigma$ be the restriction to $[h]_{G}$ of $\varphi$. If $x \in G$, the tangent space to $[h]_{G}$ at $x h x^{-1}$ can be canonically identified with $\operatorname{Ad}(x)\left(\operatorname{Ad}\left(h^{-1}\right)-1\right)(\mathfrak{g})$; as $h$ is regular, $\left(\operatorname{Ad}\left(h^{-1}\right)-1\right)(\mathfrak{g})=1^{1}(\perp$ is orthogonal complement with respect to $\langle.,\rangle$.$) so that$ this becomes $\left(I^{\perp}\right)^{x}$. So to prove that $\sigma$ is submersive we should check that $\mathfrak{f}^{f}+\left(1^{\perp}\right)^{x}$ $=\mathfrak{g}$, or $\mathfrak{x}^{x^{-1}}+\mathrm{I}^{\perp}=\mathfrak{g}$. Taking orthogonal complements this reduces to proving that $\left(\mathrm{m}^{x^{-1}}\right)^{1} \cap \mathrm{I}=(0)$. But clearly $\mathrm{I} \cap \mathfrak{5}^{x^{-1}}=(0)$; for, if $H \in \mathrm{I}$, ad $H$ has only purely imaginary eigenvalues while, for any $X \in \mathfrak{s}^{x^{-1}}$, ad $X$ has real eigenvalues. The fact that $\sigma^{-1}(\omega)$ is compact for compact $\omega \subset A \times N$ is obvious. Since $\sigma$ is submersive, $\sigma\left([h]_{G}\right)$ is open in $A \times N$, while the fact that it is proper shows that $\sigma\left([h]_{G}\right)$ is closed in $A \times N$. So, by connectedness of $A \times N, \sigma$ is surjective.

We can now prove Theorem 4.12. We have $T_{h}=U^{\dagger}$ with $U=I_{h}^{\#}$ (cf. (4.4)). If $b>0$ is arbitrary and $\psi_{b}$ is an element of $C_{c}^{\infty}(G / / K)$ which is 1 on $K A(b) K$, it follows from (3.22) and the relation $\left\langle U^{\dagger}, g\right\rangle=\left\langle U, \mathscr{A}^{-1} g\right\rangle$, that $\left\langle U^{\dagger}, g\right\rangle$ $=\left\langle\left(\psi_{b} U\right)^{\dagger}, g\right\rangle$ for all $g \in C_{c}^{\infty}(A(b))^{w}$. So it is enough to prove that $(\psi U)^{\dagger} \in C^{\infty}(A)$ for any $\psi \in C_{c}^{\infty}(G / / K)$. In view of (3.24) this reduces to proving that $\pi_{*}(\psi U) \in C^{\infty}(A)$. This is obvious; for, $\psi U$ is a $C^{\infty}$ density with compact support on the manifold $[h]_{G}$, and $\pi:[h]_{G} \rightarrow A$ is a surjective map which is everywhere submersive (cf. Varadarajan [42, I.2. Lemma 1].

## 5. Periodic Geodesics in $X$ and Their Connection with the Spectrum of $X$

5.1. The Distributions $T_{c}$ and the Poisson formula for $X$. We now return to the framework of $G, \Gamma$, and $X=\Gamma \backslash G / K$. Let $\mathscr{C}(\Gamma)$ be the set of all $\Gamma$-conjugacy classes of elements of $\Gamma$. If $c \in \mathscr{C}(\Gamma)$, the tempered distribution $T_{c}$ on $A$ and the number $v_{c}>0$ are defined by

$$
\begin{equation*}
T_{c}=T_{y}, v_{c}=\operatorname{vol}\left(\Gamma_{\gamma} \backslash G_{\gamma}\right) \quad(\gamma \in \mathcal{c}) . \tag{5.1}
\end{equation*}
$$

Here $T_{\gamma}$ is defined by (4.4) or (4.19), and $\operatorname{vol}\left(\Gamma_{\gamma} \backslash G_{\gamma}\right)$ is calculated using the Haar measure $d x_{y}$ on $G_{y}$ introduced in section 4. $T_{c}$ and $v_{c}$ do not depend on the choice of $\gamma \in c$; and $T_{c}$ depends only on the $G$-conjugacy class $[\gamma]_{G}$ containing $\gamma$.

The procedure for determining $T_{c}$ is as follows. We select $h \in[c]_{G}=[\gamma]_{G}$ in standard position and a $\theta$-stable CSG $L$ containing $h$ such that $L_{R} \subset A$ and $L$ is aligned to $h$; then $T_{c}=T_{h}$ is calculated using (4.27). Let ${ }^{*} L$ be defined by (4.21) and let

$$
\begin{equation*}
{ }^{*} A(c)=\bigcup_{s \in \mathfrak{w}} s \cdot\left({ }^{*} L h_{R}\right), \quad l(c)=\left\|\log h_{R}\right\| . \tag{5.2}
\end{equation*}
$$

It is clear that * $A(c)$ and $l(c)$ depend only on $[c]_{G}$, and that $l(c)>0$ if and only if the elements of $[\mathrm{c}]_{G}$ are not elliptic. The trace formula and the results of Section 4 then lead to the following formula:

Theorem 5.1 (Poisson formula for $X$ ). The spectral multiplicities $m(\lambda)$ and the distributions $T_{c}$ are related by the following identity of distributions:

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) \eta_{\lambda}=\sum_{c \in \mathscr{\mathscr { G }}(I)} v_{c} T_{c} \tag{5.3}
\end{equation*}
$$

Here

$$
v_{[e]_{\Gamma}}=2 \sum_{\pi \in 4}^{\sum n(2 z)} \pi^{-(n-r) / 2} I(\rho)^{-1} \operatorname{vol}_{0}(X) ; \quad T_{[e]_{\Gamma}}=|\mathfrak{w}|^{-1} \hat{\beta} .
$$

## Moreover,

$\eta_{\lambda}=e^{\lambda_{0} \log }$ and $\hat{\beta}$ is the Fourier transform of $\beta$ defined by $\langle\hat{\beta}, g\rangle=\int_{\bar{F}_{7}} \hat{g} \beta d v$ $\left(g \in C_{c}^{\infty}(A)\right) . T_{c}$ is tempered and its support is contained in ${ }^{*} A(c)$. The numbers $l(c)$ are bounded away from 0 for $c \neq[e]_{\Gamma}$ and so there is an open neighborhood of 1 in A that does not meet the support of any of the $T_{c}\left(c \neq[e]_{\Gamma}\right)$.

Let $Y$ be a conjugacy class in $G$ of semisimple elements. We say that $Y$ is regular if its elements are regular in $G$. If $L$ is a $\theta$-stable CSG of $G$ with $L_{R} \subset A$, we say that $Y$ is of type $L$ if there is $h \in Y$ such that $L$ contains $h$ and is aligned to $h$. It is clear that such an $L$ always exists and is unique up to conjugacy by $K$. If $Y$ is of type $L$ with $L_{R}=A$, we say $Y$ is of I wasawa type. If we can find $h \in Y$ such that $h$ is in standard position and $h_{R}$ is a regular element of $A$ (i.e., $\log h_{R} \in \mathfrak{a}$ and $\left\langle\alpha, \log h_{R}\right\rangle \neq 0 \forall \alpha \in \Delta^{+}$), then $Y$ is certainly of Iwasawa type. We then say that $Y$ is of real regular I wasawa type. If $\operatorname{rk}(S)=1$ and $Y \neq[e]_{G}$, this is always the case. If $c \in \mathscr{C}(\Gamma)$ we say that $c$ is regular, type $L$, etc. if $[c]_{G}$ has the corresponding property. It follows from a result of Mostow [33] (see also Prasad-Raghunathan [36]) that $\mathscr{C}(\Gamma)$ contains infinitely many classes of any type, and for the Iwasawa type, infinitely many that are real regular (indeed such elements are even "projectively dense"; see Mostow [34, Lemma 8.3]). We can now restate the results of section 4 as follows.

Theorem 5.2. $T_{c}$ is a smooth density on ${ }^{*} A(c)$ if $c$ is regular. If $c$ is of real regular Iwasawa type (in particular for all $c \neq[e]_{G}$ if $\operatorname{rk}(S)=1$ ),

$$
\begin{equation*}
T_{c}=\Delta_{+}(h)^{-1}\left(|\mathfrak{w}|^{-1} \sum_{s \in \mathfrak{w}} \delta_{s h_{R}}\right), \tag{5.4}
\end{equation*}
$$

where $h$ is any element of $[c]_{G}$ in standard position with $h_{R} \in A$.
The distributions $T_{c}$ and their supports ${ }^{*} A(c)$ are intimately related to the manifolds of periodic geodesics in $X$. This circumstance makes it possible to carry out a detailed comparison with the results of Duistermat-Guillemin [6] on the singularities of the distribution $\sum_{j=1}^{\infty} e^{-i \lambda_{j}^{1 / m} t}=\hat{\sigma}(t), \lambda_{1}, \lambda_{2}, \ldots$ being the eigenvalues of a positive elliptic differential operator on $X$ of even order $m$ that comes from a $G$-invariant differential operator on $S$. We now turn to a discussion of this aspect of our problem.
5.2. $\gamma$-periodicity in the Tangent Bundle $T S$. For any $C^{\infty}$ manifold $Y$ we denote by $T_{y} Y$ the tangent space to $Y$ at $y \in Y$, and by $T Y$ the tangent bundle of $Y$. If a Lie group acts on $Y$, this action lifts to a natural action on $T Y$. Let $Y$ be a complete Riemannian manifold, and let $\left\{\Phi^{t}=\Phi_{Y}^{t}:-\infty<t<\infty\right\}$ be the geodesic flow on $T Y$. If the metric of $Y$ is invariant under a Lie group $H$ acting on $Y$, the flow $\Phi^{t}$ will commute with the action of $H$. Given $\gamma \in H$, we say that a point $z \in T Y$ is $\gamma$-periodic if $\gamma(z)=\Phi^{1}(z)$ i.e., if $z$ is a fixed point of the Poincaré map $\mathscr{P}_{\gamma}$ $=\gamma^{-1} \circ \Phi^{1}$. The tangent map

$$
\begin{equation*}
\left(d \mathscr{P}_{\gamma}\right)_{z}=P_{\gamma, z}: T_{z}(T Y) \rightarrow T_{z}(T Y) \tag{5.5}
\end{equation*}
$$

is then called the linear Poincaré map at $z$. We denote by $F(\gamma)$ the set of $\gamma$ periodic points of $T Y$. If $H_{1}$ is a discrete subgroup of $H$ that acts freely and properly discontinuously on $Y$, we can form the manifold $Y_{1}=H_{1} \backslash Y$. The map $Y \rightarrow Y_{1}$ lifts to a map $T Y \rightarrow T Y_{1}$ and $T Y_{1} \simeq H_{1} \backslash T Y$. If $\gamma \in H_{1}$, then a point $z$ $=(p, v) \in T Y$ is $\gamma$-periodic if and only if the geodesic $c$ on $Y$ defined by $z$ is the lift of a periodic geodesic of period 1 in $Y_{1}$, and satisfies $c(0)=p, c(1)=\gamma \cdot p$. Moreover, the image $z_{1}$ of $z$ in $T Y_{1}$ is a fixed point for $\Phi_{Y_{1}}^{1}$; and the tangent map $T_{z}(T Y) \rightarrow T_{z_{1}}\left(T Y_{1}\right)$ intertwines $P_{\gamma, z}$ with the tangent map of $\Phi_{Y_{1}}^{1}$ at $z_{1}$.

These definitions and remarks are applicable to the case when $Y=S=G / K$, $H=G, H_{1}=\Gamma, Y_{1}=X=\Gamma \backslash S$. The natural map $G \rightarrow S$ is denoted by $\pi$, and its tangent map at $x$ is denoted by $\dot{\pi}_{x}: \mathfrak{g} \rightarrow T_{\pi(x)} S$; in what follows, the elements of $\mathfrak{g}$ are regarded as left invariant vector fields on $G$ and the tangent spaces $T_{x} G$ are canonically identified with $\mathfrak{g}$ for all $x \in G$. The tangent map $\dot{\pi}_{e}$ is an isomorphism of $\mathfrak{s}$ with $T_{\pi(e)} S$, and the Riemannian metric of $S$ is the $G$-invariant one that restricts on $T_{\pi(e)} S$ to the image of the Killing form on $\mathfrak{s}$ by $\dot{\pi}_{e}$. The geodesics on $S$ through $\pi(e)$ are the curves

$$
\begin{equation*}
c_{X}: t \mapsto \pi(\exp t X) \quad(t \in \mathbb{R}, X \in \mathfrak{s}) \tag{5.6}
\end{equation*}
$$

We have

$$
\begin{equation*}
c_{X}(0)=\pi(e), \quad \dot{c}_{X}(0)=X \tag{5.7}
\end{equation*}
$$

while for any $x \in G$, the geodesics through $\pi(x)$ are given by

$$
\begin{equation*}
c_{X, \pi(x)}(t)=\pi(x \exp t X)=x \cdot c_{X}(t) \quad(t \in \mathbb{R}) \tag{5.8}
\end{equation*}
$$

For some computational purposes it is convenient to introduce the submanifold

$$
\begin{equation*}
\bar{S}=\exp \mathfrak{s} \tag{5.9}
\end{equation*}
$$

of $G$. It is well-known that $\bar{S}$ is a closed submanifold of $G$ and that $\exp$ is a diffeomorphism of $\mathfrak{s}$ with $\bar{S}$. We denote by $\log (\bar{S} \rightarrow \mathfrak{s})$ the inverse of $\exp (\mathfrak{s} \rightarrow \bar{S})$. We now define the transposition antiautomorphism $x \mapsto x^{\prime}$ of $G\left(\operatorname{resp} X \mapsto X^{\prime}\right.$ of g) by

$$
\begin{equation*}
x^{\prime}=\theta\left(x^{-1}\right), X^{\prime}=-\theta(X) \quad(x \in G, X \in \mathfrak{g}) \tag{5.10}
\end{equation*}
$$

We then have an action of $G$ on itself given by

$$
\begin{equation*}
(g, x) \mapsto g[x]=g x g^{\prime} \quad(g, x \in G) . \tag{5.11}
\end{equation*}
$$

For any $x \in G$ we have its Cartan decomposition

$$
x=x_{K} x_{S}=x_{K} \exp \log \left(x_{S}\right) \quad\left(x_{K} \in K, x_{S} \in \bar{S}, \log \left(x_{S}\right) \in \mathfrak{s}\right),
$$

using which it is easily seen that $\bar{S}$ is the orbit of $e$ under the action (5.11) and that

$$
\begin{equation*}
\zeta: \pi(g) \mapsto g g^{\prime} \quad(g \in G) \tag{5.12}
\end{equation*}
$$

is a diffeomorphism of $S$ with $\bar{S}$ that commutes with $G$ and takes $\pi(e)$ to $e$. We shall use this diffeomorphism to go from $S$ to $\bar{S}$ and vice versa. If $s \in \bar{S}$ we define $s^{\frac{1}{2}}$ by

$$
\begin{equation*}
s^{\frac{1}{2}}=\exp \left(\frac{1}{2} \log s\right) \quad(s \in \bar{S}) ; \tag{5.13}
\end{equation*}
$$

it is the unique element of $\bar{S}$ whose square is $s$, and $s \mapsto s^{\frac{1}{2}}$ is a diffeomorphism of $\bar{S}$ onto itself.

We have $T \bar{S} \subset T G$. The action of $G$ on $T G$ induced by (5.11) is given by

$$
\begin{equation*}
g \cdot(x, X)=\left(g[x], \operatorname{Ad}\left(g^{\prime}\right)^{-1}(X)\right) . \tag{5.14a}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
T_{e} \bar{S}=\mathfrak{s}, T_{s} \bar{S}=\operatorname{Ad}\left(s^{-\frac{1}{2}}\right)(\mathfrak{s}) . \tag{5.14b}
\end{equation*}
$$

To write down the geodesic flow in $T \bar{S}$ we note first that $\zeta$ induces the diffeomorphism

$$
\begin{equation*}
\left(\pi(x), \dot{\pi}_{x}(X)\right) \mapsto\left(x x^{\prime}, 2 X^{x^{\prime-1}}\right) \tag{5.15a}
\end{equation*}
$$

( $x \in G, X \in \mathfrak{s}$ ) of $T S$ with $T \bar{S}$. Since the geodesic flow $\Phi_{S}^{i}=\Phi^{t}$ on $T S$ is given by (with $x \in G, X \in \mathfrak{s}$ )

$$
\begin{equation*}
\Phi^{L^{\prime}}\left(\pi(x), \dot{\pi}_{x}(X)\right)=\left(\pi(x \exp t X), \dot{\pi}_{x \operatorname{expt} X}(X)\right) \tag{5.15b}
\end{equation*}
$$

it follows from these formulae that the geodesic flow $\bar{\Phi}^{\prime}=\Phi_{S}^{t}$ on $T \bar{S}$ is given by

$$
\begin{equation*}
\bar{\Phi}^{\prime}(s, Y)=(s \exp t Y, Y) \quad\left(s \in \bar{S}, Y \in T_{s} \bar{S}\right) . \tag{5.16a}
\end{equation*}
$$

This is the restriction to $T \bar{S}$ of the flow $\Psi^{t}$ on $T G$ given by

$$
\begin{equation*}
\Psi^{\prime}(y, Y)=(y \operatorname{expt} Y, Y) \quad(y \in G, Y \in \mathfrak{g}) \tag{5.16b}
\end{equation*}
$$

Comparing (5.14) and (5.16) we see that for $\gamma \in G,\left(s_{0}, Y\right) \in T \bar{S}$,

$$
\begin{equation*}
\left(s_{0}, Y\right) \text { is } \gamma \text {-periodic } \Leftrightarrow \gamma s_{0} \gamma^{\prime}=s_{0} \exp Y, Y=Y^{\gamma^{\prime}} . \tag{5.17a}
\end{equation*}
$$

Hence, for such $\bar{z}=\left(s_{0}, Y\right) \in T \bar{S}$, the Poincaré map $\mathscr{P}_{\gamma}$ is given by

$$
\begin{equation*}
\overline{\mathscr{P}}_{y}:(s, X) \mapsto\left(\gamma^{-1} s \exp X \gamma^{\prime-1}, X^{\gamma^{\prime}}\right) \quad((s, X) \in T \bar{S}) . \tag{5.17b}
\end{equation*}
$$

Proposition 5.3. (i) If $z$ is $\gamma$-periodic in $T S$ then $g \cdot z$ is $g \gamma g^{-1}$-periodic in TS. (ii) There are $\gamma$-periodic points in TS if and only if $\gamma$ is semisimple. (iii) If $Y \in \mathfrak{F}$, (resp. $Z \in \mathfrak{s})(e, Y)$ is $\gamma$-periodic in $T \bar{S}\left(\right.$ resp. $\left(\pi(e), \dot{\pi}_{e}(Z)\right)$ is $\gamma$-periodic in $\left.T S\right)$ if and only if $\gamma_{K}$ and $\gamma_{S}$ commute and

$$
\begin{equation*}
\log \gamma_{S}=\frac{1}{2} Y\left(\text { resp. } \log \gamma_{S}=Z\right) \tag{5.18}
\end{equation*}
$$

that is, if and only if $\gamma$ is semisimple, in standard position and $Y=2 \log \gamma_{s}(r e s p . Z$ $=\log \gamma_{S}$.

The assertion (i) follows from the fact that $G$ commutes with $\Phi^{i}$. From (5.17a), for $Y \in s,(e, Y)$ is $\gamma$-periodic if and only if $\gamma \gamma^{\prime}=\exp Y, Y=Y^{\gamma^{\prime}}$. Let $X$ $=\log \gamma_{S}$ so that $\gamma=\gamma_{K} \exp X$. Then $\gamma \gamma^{\prime}=\exp 2 X^{\gamma_{K}}$ so that $Y=2 X^{\gamma_{K}}$; and $Y^{\gamma^{\prime}}$ $=\left(2 X^{\gamma_{K}}\right)^{\gamma^{\prime}}=2 X^{\gamma s}=2 X=Y$. Hence $X^{\gamma_{K}}=X$, showing that $\gamma$ is semisimple, in standard position, and that $Y=2 X^{\gamma_{K}}$ (cf. Lemma 4.1). Conversely, if $\gamma$ is semisimple and in standard position, we define $Y=2 \log \gamma_{S}$ and verify (5.17a) for $(e, Y)$. This proves (iii); in view of (i), this gives (ii) also.

Let $\gamma \in G$ be semisimple. Define

$$
\begin{array}{ll}
F(\gamma)=\text { set of } \gamma \text {-periodic points in } T S & (F(\gamma) \subset T S)  \tag{5.19}\\
S(\gamma)=\text { projection of } F(\gamma) \text { in } S & (S(\gamma) \subset S)
\end{array}
$$

Proposition 5.4. Let $\gamma \in G$ be semisimple. Then $F(\gamma)$ and $S(\gamma)$ are connected smooth manifolds, stable under $G_{\gamma}$; and the projection $F(\gamma) \rightarrow S(\gamma)$ is bijective. Moreover, $G_{\gamma}$ acts transitively on $F(\gamma)$, the stabilizer in $G_{\gamma}$ of points of $F(\gamma)$ are maximal compact subgroups of $G_{\gamma}$, and $F(\gamma)$ is a symmetric space. If $\gamma$ is in standard position, $G_{\gamma}$ and $\mathbf{g}_{\gamma}$ are $\theta$-stable, $K_{\gamma}=G_{\gamma} \cap K$ is a maximal compact subgroup of $G_{\gamma}$, and $\boldsymbol{G}_{\gamma} \approx \exp \left(\mathfrak{s}_{\gamma}\right) K_{\gamma}$ where $\mathfrak{s}_{\gamma}=\mathfrak{g}_{\gamma} \cap \mathfrak{s}$.

Since $G$ is transitive over $S$, we may assume that $\gamma$ is in standard position. Formula (5.18) shows that when $\pi(e) \in S(\gamma)$, there is exactly one point in $F(\gamma)$ above $\pi(e)$. Moreover, (i) of Proposition 5.3 shows that $F(\gamma)$ (and hence $S(\gamma)$ ) are $G_{\gamma}$-stable. Suppose now that $\pi(e) \in S(\gamma)$ and $p \in S(\gamma)$. Let $Z \in \mathfrak{s}$ be the unique element such that $p=\pi(\exp Z)$ and let $y=\exp (-Z)$. Then $\pi(e)=y \cdot p \in S\left(y \gamma y^{-1}\right)$. So $\tilde{\gamma}=y \gamma y^{-1}$ is in standard position. If $\tilde{\gamma}_{K}=y \gamma_{K} y^{-1}, \tilde{\gamma}_{S}=y \gamma_{S} y^{-1}$, then $\tilde{\gamma}$ $=(\tilde{\gamma})_{K}(\tilde{\gamma})_{S}=\tilde{\gamma}_{K} \tilde{\gamma}_{S}$. Now all elements in sight are semisimple; moreover, $(\tilde{\gamma})_{K}$ and $(\tilde{\gamma})_{S}$ commute (because $\tilde{\gamma}$ is in standard position) while $\tilde{\gamma}_{K}$ and $\tilde{\gamma}_{S}$ commute also (because $\gamma_{K}$ and $\gamma_{S}$ commute). So, by the uniqueness of the decomposition of semisimple elements into a product of commuting semisimple elements one of which is elliptic and the other hyperbolic we have

$$
y \gamma_{K} y^{-1}=(\tilde{\gamma})_{K}, \quad y \gamma_{S} y^{-1}=(\tilde{\gamma})_{S} .
$$

Applying ' to the first and $\theta$ to the second of these relations we find that $y^{2}$ $=\exp (-2 Z)$ commutes with $\gamma_{K}$ and $\gamma_{S}$. Hence $Z$ commutes with $\gamma_{K}$ and $\gamma_{S}$, i.e., $Z \in \mathfrak{g}_{\gamma}$ or $p=\exp (Z) \cdot \pi(e) \in G_{\gamma}^{0} \cdot \pi(e)$. The statements of the proposition easily follow from this.

Remark 5.5. Let $\gamma$ be in standard position and write $Y=\log \gamma_{S}$. It follows from (5.18) that

$$
F(\gamma)=\left\{\left(\pi(g), \dot{\pi}_{g}(Y)\right): g \in G_{\gamma}\right\} .
$$

It is clear from this description that the manifold $F(\gamma)$ is horizontal with respect to the Levi-Civita connection.
5.3. The $\gamma$-displacement Function. For an arbitrary element $\gamma \in G$, the $\gamma$-displacement function $f^{(\gamma)}$ is defined on $S$ by

$$
\begin{equation*}
f^{(\gamma)}(p)=\operatorname{dist}(p, \gamma \cdot p) \quad(p \in S), \tag{5.20}
\end{equation*}
$$

where dist is the distance function on $S$. Actually it is more convenient to work with $g^{(y)}=\frac{1}{2}\left(f^{(v)}\right)^{2}$. Clearly, if $x=x_{K} x_{S}$, then

$$
\begin{equation*}
g^{(\gamma)}(p)=\frac{1}{2}\left\|\log \left(\gamma^{\exp (-X)}\right)_{S}\right\|^{2} \quad(p=\exp X, X \in \mathfrak{\xi}) . \tag{5.21}
\end{equation*}
$$

So $g^{(\gamma)}$ is an analytic nonnegative function. Clearly $f^{\left(x \gamma x^{-1}\right)}(x \cdot p)=f^{(\gamma)}(p)$.
Proposition 5.6. $g^{(\gamma)}$ has a nonempty set of critical points if and only if $\gamma$ is semisimple. In this case, the set of critical points of $g^{(\gamma)}$ is precisely $S(\gamma)$.

Using the transitivity of $G$ on $S$, this comes down to proving that $\pi(e)$ is a critical point of $g^{(\gamma)}$ if and only if $\gamma$ is semisimple and in standard position. Writing

$$
\begin{equation*}
\exp (-t X) \gamma \exp t X=k(t) \exp Z(t) \quad(X, Z(t) \in \mathfrak{s}) \tag{5.22}
\end{equation*}
$$

with $k(0)=\gamma_{K}, \quad Z(0)=\log \gamma_{S}=Z$ say, we find that $g^{(\gamma)}(\pi(\exp t X))=\frac{1}{2}\|Z(t)\|^{2}$. Hence $\pi(e)$ is a critical point of $g^{(\gamma)}$ if and only if $\langle\dot{Z}(0), Z\rangle=0$ for all $X \in \mathfrak{s}$. We now differentiate (5.22) at $t=0$ (in this computation, the tangent spaces to $K$ and $\mathfrak{s}$ are identified as usual with $f$ and $\mathfrak{s}$ respectively). Rewriting (5.22) as

$$
\exp \left(-t X^{\gamma^{-1}}\right) \exp t X=\gamma_{S}^{-1}\left(\gamma_{K}^{-1} k(t)\right) \gamma_{S}(\exp (-Z) \exp Z(t))
$$

and using the well-known formula for the tangent map of the exponential (cf. Helgason [24], p. 36) we find

$$
\begin{equation*}
X-\operatorname{Ad}\left(\gamma^{-1}\right) X=\operatorname{Ad}\left(\gamma_{s}^{-1}\right)(\dot{k}(0))+\frac{1-e^{-\operatorname{ad} Z}}{\operatorname{ad} Z}(\dot{Z}(0)) \tag{5.23}
\end{equation*}
$$

or,

$$
\begin{equation*}
\operatorname{Ad}\left(\gamma_{S}\right) X-\operatorname{Ad}\left(\gamma_{K}^{-1}\right) X=\dot{k}(0)+\frac{e^{\operatorname{ad} Z}-1}{\operatorname{ad} Z}(\dot{Z}(0)) \tag{5.24}
\end{equation*}
$$

We now take the component in $\mathfrak{s m o d} \mathfrak{f}$ of both sides. This gives

$$
\begin{equation*}
\cosh (\operatorname{ad} Z)(X)-\operatorname{Ad}\left(\gamma_{K}^{-1}\right) X=\left(\frac{\sinh \operatorname{ad} Z}{\operatorname{ad} Z}\right)(\dot{Z}(0)) \tag{5.25}
\end{equation*}
$$

Take scalar products with $Z$ on both sides and note that $\left\langle(\operatorname{ad} Z)^{m}\left(Z^{\prime}\right), Z\right\rangle=0$ for any $m \geqq 1$ and any $Z^{\prime}$, We then obtain

$$
\langle X, Z\rangle-\left\langle\operatorname{Ad}\left(\gamma_{K}^{-1}\right) X, Z\right\rangle=\langle\dot{Z}(0), Z\rangle
$$

or

$$
\begin{equation*}
\langle\dot{Z}(0), Z\rangle=\left\langle X, Z-Z^{\gamma_{K}}\right\rangle \tag{5.26}
\end{equation*}
$$

If this is 0 for all $X \in \mathfrak{s}$, then, as $Z-Z^{\gamma_{K}} \in \mathfrak{s}$, we find that $Z=Z^{\gamma_{K}}$. So $\gamma_{K}$ and $\gamma_{S}$ commute, and we are through.

Proposition 5.7. Let $\gamma \in G$ be semisimple. Then on the critical set $S(\gamma)$ the function $f^{(\gamma)}$ takes the same value $m(\gamma)$. Moreover, $m(\gamma)$ is the minimum value of $f^{(\gamma)}$ and $S(\gamma)$ is the set $f^{\left(\gamma \gamma^{-1}\right.}(m(\gamma))$. If $\gamma$ is in standard position, then $m(\gamma)=\left\|\log \gamma_{s}\right\|$.

We may assume that $\gamma$ is in standard position. Then, as $G_{\gamma}$ is transitive on $S(\gamma)$ and $f^{(\gamma)}$ is $G_{\gamma}$-invariant, it is a constant on $S(\gamma)$, say equal to $m(\gamma)$. Let $m$ $=\inf f^{(\gamma)}$. If $m$ is actually attained at some point, then that point must be stationary for $g^{(\gamma)}$ and so lies in $S(\gamma)$, proving $m=m(\gamma)$. To prove that $m$ is attained, it is enough to show that there is a minimizing sequence which is bounded in $S$. As $f^{(\gamma)}$ is $G_{\gamma}$-invariant, this will be done if we show that for any number $c>0$, there is a compact subset $\Omega \subset S$ such that

$$
\begin{equation*}
E=\left\{p: p \in S, f^{(\gamma)}(p) \leqq c\right\} \subset G_{\gamma} \cdot \Omega \tag{5.27}
\end{equation*}
$$

If $p=x \cdot \pi(e)$, then $f^{(\gamma)}(p) \leqq c$ if and only if $x^{-1} \gamma x=k \exp Z$ where $\|Z\| \leqq c$. Hence, there is a compact set $\Omega_{1} \subset G$ such that $x \cdot \pi(e) \in E$ if and only if $x^{-1} \gamma x \in \Omega_{1}$. Now, the conjugacy class $[\gamma]_{G}$ is closed and so the map $x \mapsto x^{-1} \gamma x$ induces a homeomorphism of $G_{\gamma} \backslash G$ onto $[\gamma]_{G}$. Hence the preimage in $G_{\gamma} \backslash G$ of the set $\Omega_{1}$ under this map is compact. So, there is a compact set $\Omega_{2} \subset G$ such that $x^{-1} \gamma x \in \Omega_{1}$ if and only if $x \in G_{\gamma} \Omega_{2}$. Then $E=G_{\gamma} \Omega_{2} \cdot \pi(e)=G_{\gamma} \Omega$ where $\Omega$ $=\Omega_{2} \cdot \pi(e)$.
Remark 5.7. The above results on the displacement function are known (see Ozols [35]). Ozols' proof makes use of a second derivative calculation to conclude that the function $t \mapsto g^{(\gamma)}(c(t))(t \in \mathbb{R})$ is convex for each geodesic $c$, i.e., $\frac{d^{2}}{d t^{2}} g^{(\gamma)}(c(t)) \geqq 0$ for all $t$. This would then imply directly that the set of critical points of $g^{(\gamma)}$ is totally geodesic and transitively acted on by $G_{\gamma}$. Our proof has the advantage of avoiding second derivative calculations and, in keeping with the theme of this article, is group theoretic in character. The convexity is itself a special case of the more general result that for any two geodesics $c_{1}, c_{2}$, the function $g: t \mapsto \operatorname{dist}\left(c_{1}(t), c_{2}(t)\right)(t \in \mathbb{R})$ is convex.

It can also be shown that the critical set $S(\gamma)$ of $g^{(\gamma)}$ is clean, i.e., that at each point $p \in S(\gamma)$, the Hessian form is nondegenerate transversal to $S(\gamma)$ (see Ozols [35]). This is however not needed for our purposes.
5.4. The Linear Poincaré Map at a $\gamma$-periodic Point in TS. Let us consider the point $\bar{z}=(e, Y) \in T \bar{S}$ where $Y=2 \log \gamma_{S}, \gamma$ being semisimple and in standard position. To determine the linear Poincare map at $\bar{z}$ we use the trivialization $T \bar{S} \approx s \times s$ given by the map (cf. ( 5.14 b ))

$$
\begin{equation*}
\tau:(X, Z) \mapsto\left(\exp X, Z^{\exp (-X / 2)}\right) \quad(X, Z \in \mathfrak{s}) \tag{5.27}
\end{equation*}
$$

So, using ( 5.17 b), we obtain for the Poincaré map in $s \times s$ the formula

$$
\begin{equation*}
\overline{\mathscr{P}}:(X, Z) \mapsto(V, W) \quad(X, Z, V, W \in \mathfrak{s}) \tag{5.28}
\end{equation*}
$$

where

$$
\begin{align*}
\exp V & =\gamma^{-1} \exp X / 2 \exp Z \exp X / 2 \gamma^{\prime-1} \\
W & =e^{\operatorname{ad} V / 2} \operatorname{Ad}\left(\gamma^{\prime}\right) e^{-\operatorname{ad} X / 2}(Z) \tag{5.29}
\end{align*}
$$

The derivative $(d \overline{\mathscr{P}})_{(0, Y)}$ of $\overline{\mathscr{P}}$ at $(0, Y)$ can then be calculated from this expression for $\overline{\mathscr{P}}$. Regarded as an endomorphism of $\mathfrak{s} \oplus \mathfrak{s}$, it is given by the following $2 \times 2$ matrix whose entries are endomorphisms of $\mathfrak{s}$ :

$$
\bar{P}_{(0, Y)}=(d \overline{\mathscr{P}})_{(0, Y)}=\operatorname{Ad}\left(\gamma_{K}^{-1}\right) \circ\left(\begin{array}{cc}
\cosh \left(\frac{1}{2} \operatorname{ad} Y\right) & \sinh \left(\frac{1}{2} \operatorname{ad} Y\right) /\left(\frac{1}{2} \operatorname{ad} Y\right)  \tag{5.30}\\
\sinh \left(\frac{1}{2} \operatorname{ad} Y\right) \circ \frac{1}{2} \operatorname{ad} Y & \cosh \left(\frac{1}{2} \operatorname{ad} Y\right)
\end{array}\right) .
$$

It is easy to determine the spectral decomposition of $\bar{P}_{(0, Y)}$ from this formula. Let us write $\mathfrak{g}_{a}$ for the eigenspace of $\frac{1}{2}$ ad $Y$ in $\mathfrak{g}$ for the eigenvalue $a$. Then $\mathfrak{g}$ is the direct sum of $g_{0}$ and the $\mathfrak{g}_{ \pm a}(a>0)$. For any $a>0, \theta\left(g_{a} \rightarrow g_{-a}\right)$ is an isomorphism and $\mathfrak{s}_{a}=\left(\mathfrak{g}_{a} \oplus \mathfrak{g}_{-a}\right) \cap \mathfrak{s}$ is the eigenspace of $\left.\left(\frac{1}{2} \text { ad } Y\right)^{2}\right|_{s}$ for the eigenvalue $a^{2} ; \mathfrak{g}_{0}$ is $\theta$-stable and we put $\mathfrak{s}_{0}=g_{0} \cap \mathfrak{s}$. Then $\mathfrak{s}$ is the direct sum of the $\mathfrak{s}_{a}(a \geqq 0)$, $\operatorname{dim}\left(\mathfrak{s}_{a}\right)$ $=\operatorname{dim}\left(\mathfrak{g}_{ \pm a}\right)(a>0)$, and the $\mathfrak{s}_{\mathfrak{a}}(a \geqq 0)$ are all $\operatorname{Ad}\left(\gamma_{K}\right)$-stable. It follows from this that

$$
\begin{align*}
& \left.\left.\bar{P}_{(0, Y)}\right|_{s_{a} \oplus \mathfrak{s}_{a}} \simeq \operatorname{Ad}\left(\gamma_{K}^{-1}\right)\left(\begin{array}{cc}
e^{a} \cdot I & 0 \\
0 & e^{-a} \cdot I
\end{array}\right) \simeq \operatorname{Ad}\left(\gamma^{-1}\right)\right|_{\mathbf{g}_{a} \oplus \mathfrak{g}-a} \quad(a>0) \\
& \left.\bar{P}_{(0, Y)}\right|_{\mathfrak{s}_{0} \oplus \mathfrak{s}_{0}}=\operatorname{Ad}\left(\gamma_{K}^{-1}\right)\left(\begin{array}{ll}
I & I \\
0 & I
\end{array}\right) . \tag{5.31}
\end{align*}
$$

Since $\mathfrak{g}_{0}=\mathfrak{g}_{\mathbf{Y}}$, the centralizer of $Y$, the first relation gives

$$
\begin{equation*}
\left.\bar{P}_{(0, Y)}\right|_{\mathfrak{s}+\oplus \mathfrak{s}^{+}} \simeq\left(\operatorname{Ad}\left(\gamma^{-1}\right)\right)_{\mathfrak{g} / \mathbf{g Y}} \quad\left(\mathfrak{s}^{+}=\underset{a>0}{\oplus} \mathfrak{s}_{\mathfrak{a}}\right) . \tag{5.32}
\end{equation*}
$$

From the second relation it is moreover clear that the eigenvalues of $\left.\bar{P}_{(0, Y)}\right|_{\mathbf{s}_{0} \oplus \mathbf{s}_{0}}$ are precisely those of $\operatorname{Ad}\left(\gamma_{\mathbf{K}}^{-1}\right)$ on $\mathfrak{s}_{0}$. We note that

$$
\begin{equation*}
\mathfrak{s}_{0}=\mathfrak{g}_{Y} \cap \mathfrak{s},\left.\quad \operatorname{Ad}\left(\gamma^{-1}\right)\right|_{\mathfrak{s}_{0}}=\left.\operatorname{Ad}\left(\gamma_{K}^{-1}\right)\right|_{s_{0}} . \tag{5.33}
\end{equation*}
$$

If $\varepsilon$ is an eigenvalue of $\left.\operatorname{Ad}\left(\gamma^{-1}\right)\right|_{s 0}, U_{\varepsilon}$ the corresponding spectral subspace (in $\left.\mathbb{C} \cdot \mathfrak{s}_{0}\right)$, then,

$$
\left.\bar{P}_{(0, Y)}\right|_{U_{t} \oplus U_{\varepsilon}} \simeq \varepsilon\left(\begin{array}{ll}
I & I  \tag{5.34}\\
0 & I
\end{array}\right)
$$

Since $Y \in \mathfrak{F}_{0}$, we see at once that $\bar{P}_{(0, Y)}$ has a nontrivial unipotent component. Indeed, if $\mathrm{g}_{\gamma}$ is the centralizer of $\gamma$ in $\mathfrak{g}$, then $\mathfrak{s}_{y}=\mathfrak{g}_{y} \cap \mathfrak{s}$ is the subspace of $\mathfrak{s}_{0}$ fixed by $\operatorname{Ad}\left(\gamma_{K}^{-1}\right)$ and so we get

$$
\begin{equation*}
\operatorname{Ker}\left(I-\bar{P}_{(0, Y)}\right)=\left\{(X, 0): X \in \mathfrak{s}_{\gamma}\right\}=\operatorname{Range}\left(\left.\left(I-\bar{P}_{(0, Y)}\right)\right|_{\mathrm{s}_{\gamma} \oplus \mathfrak{s}_{\gamma}}\right) . \tag{5.35}
\end{equation*}
$$

Proposition 5.8. For any semisimple element $\gamma \in G, \mathscr{P}_{\gamma}$, has the clean fixed point set $F(\gamma)$, i.e., $F(\gamma)$ is smooth and for any $z \in F(\gamma), T_{z}(F(\gamma))=\operatorname{Ker}\left(I-P_{\gamma, z}\right), P_{\gamma, z}$ being the tangent map of $\mathscr{P}_{\gamma, z}$ at $z$.

We may take $z$ to be above $\pi(e) . \gamma$ is then in standard position and $\operatorname{dim}\left(\operatorname{Ker}\left(I-P_{\gamma, z}\right)\right)=\operatorname{dim}\left(\mathfrak{s}_{\gamma}\right)=\operatorname{dim}(F(\gamma))$ by (5.35). Since $T_{z}(F(\gamma)) \subset \operatorname{Ker}\left(I-P_{\gamma, z}\right)$ in any case, we are done.

Remark 5.9. The formula (5.35) again shows that $F(\gamma)$ is horizontal with respect to the Levi-Civita connection.

It is useful to write down the information contained in (5.31)-(5.34) in terms of roots. We select a $\theta$-stable CSG $L$ containing $\gamma$. Let I be the corresponding CSA; we have $\mathfrak{l} \subset \mathfrak{g}_{Y}$. We now consider the roots of $\left(\mathfrak{g}_{Y, c}, l_{c}\right)$. For any such root $\alpha$, we have $\theta \alpha=-\bar{\alpha}, \xi_{x}(\gamma)=\xi_{\alpha}\left(\gamma_{K}\right), \xi_{\bar{\alpha}}(\gamma)=\xi_{\alpha}\left(\gamma^{-1}\right)$. The span of $\mathfrak{g}_{c, \pm \alpha}, \mathfrak{g}_{c, \pm \bar{\alpha}}$ is stable both under $\theta$ and complex conjugation; we put $b_{\alpha}$ for the intersection of this span with $\mathfrak{g}$. If $\beta$ is another root but $\neq \pm \alpha, \neq \pm \bar{\alpha}$, then $\mathfrak{b}_{\beta}$ and $\mathfrak{b}_{\alpha}$ are linearly independent. We consider the following special cases.
$\alpha$ neither real nor pure imaginary. The four roots $\pm \alpha, \pm \bar{\alpha}$ are distinct. So on $\mathbb{C} \cdot \mathfrak{b}_{\alpha}, \operatorname{Ad}\left(\gamma_{K}^{-1}\right)$ has the eigenvalues $\xi_{ \pm \alpha}\left(\gamma^{-1}\right)$ with multiplicity 2 . On the other hand, $\mathfrak{b}_{\alpha} \cap \mathfrak{s}$ has dimension 2 and is spanned by $X_{ \pm \alpha}-\theta X_{ \pm \alpha}$. Indeed, if $X_{ \pm \alpha}$ $-\theta X_{ \pm \alpha}=0$, we would have $\alpha=\theta \alpha$; if $X_{\alpha}-\theta X_{\alpha}$ is a multiple of $X_{-\alpha}-\theta X_{-\alpha}$, then $\left.\xi_{\alpha}^{2}\right|_{L_{I}}=1$, i.e., $\alpha$ is real. Thus $\left.\operatorname{Ad}\left(\gamma_{K}^{-1}\right)\right|_{b_{\alpha \cap 5}}$ has the eigenvalues $\xi_{ \pm \alpha}\left(\gamma^{-1}\right)$, with multiplicity 1.
$\alpha$ real. $b_{\alpha}$ is now two dimensional. $\operatorname{Ad}\left(\gamma_{K}^{-1}\right)$ has the eigenvalue $\xi_{\alpha}\left(\gamma^{-1}\right)= \pm 1$ with multiplicity 2. $\mathfrak{b}_{\alpha} \cap \mathfrak{s}\left(\right.$ resp. $\left.\mathfrak{b}_{\alpha} \cap \mathfrak{f}\right)$ is one dimensional and spanned by $X_{\alpha}-\theta X_{\alpha}$ (resp. $X_{\alpha}+\theta X_{\alpha}$ ) (as $\theta \alpha=-\alpha, X_{\alpha} \pm \theta X_{-\alpha} \neq 0$ ). So, on $b_{\alpha} \cap s, \operatorname{Ad}\left(\gamma_{\kappa}^{-1}\right)$ has the eigenvalues $\xi_{\alpha}\left(\gamma^{-1}\right)$ with multiplicity 1 .
$\alpha$ purely imaginary. $\theta \alpha$ is now $\alpha$ while $b_{\alpha}$ is two dimensional as in the preceding case. For the root vector $X_{\alpha}$ we have either $X_{\alpha} \in\left(\mathfrak{l} \cap g_{Y}\right)_{c}$ or $X_{\alpha} \in\left(\mathfrak{s} \cap g_{Y}\right)_{c} ; \alpha$ is called compact or noncompact accordingly. If $\alpha$ is compact, $\mathrm{b}_{\alpha} \cap \mathfrak{s}=0$ and there is no contribution to $P_{\gamma, z}$. If $\alpha$ is noncompact, $\mathfrak{b}_{\alpha} \subset \mathfrak{s}$, and $\left.\operatorname{Ad}\left(\gamma_{\bar{K}}^{-1}\right)\right|_{b_{\alpha}}$ has the eigenvalues $\xi_{ \pm \alpha}\left(\gamma^{-1}\right)$, each with multiplicity one.

Since the imaginary roots of $\left(\mathfrak{g}_{c}, \mathfrak{I}_{c}\right)$ are already roots of $\left(\mathfrak{g}_{Y, c}, \mathfrak{f}_{c}\right)$, we get
Proposition 5.10. Let $\gamma$ be semisimple and in standard position. Let $z \in F(\gamma)$ be above $\pi(e)$ and let $L$ be any $\theta$-stable CSG containing $\gamma$. Then, with $\lambda$ as an indeterminate, and $\alpha$ running over the roots of $\left(\mathfrak{g}_{c}, \mathfrak{l}_{c}\right)$,

$$
\begin{equation*}
\operatorname{det}\left(\lambda I-P_{\gamma, z}\right)=(\lambda-1)^{2 \operatorname{dim}\left(I_{R}\right)} \prod_{\substack{\alpha \text { not } \\ \text { imaginary }}}\left(\lambda-\xi_{-\alpha}(\gamma)\right) \prod_{\substack{\alpha \text { imaginary } \\ \text { noncompact }}}\left(\lambda-\xi_{-\alpha}(\gamma)\right)^{2} . \tag{5.36}
\end{equation*}
$$

From (5.35) we see that $I-P_{\gamma, z}$ induces an isomorphism

$$
\begin{equation*}
\left(I-P_{\gamma, 2}\right)^{\#}: R / N \rightarrow R / N \tag{5.37}
\end{equation*}
$$

where $R=$ Range $\left(I-P_{\gamma, z}\right), N=\operatorname{Ker}\left(I-P_{\gamma, z}\right)$; we refer to this as the reduced linear Poincaré map. From (5.36) we get

Corollary 5.11. We have

$$
\begin{equation*}
\operatorname{det}\left(I-P_{\gamma, 2}\right)^{\#}=\prod_{\substack{\xi_{y}(\gamma) \neq 1 \\ \text { not } \\ \text { imagnary }}}\left(1-\xi_{-\alpha}(\gamma)\right) \prod_{\substack{\xi_{1}(\gamma) \neq 1 \\ \alpha \text { magnary } \\ \text { noncompact }}}\left(1-\xi_{-\alpha}(\gamma)\right)^{2} . \tag{5.38}
\end{equation*}
$$

If $L$ is of Iwasawa type, i.e., $L_{R}=A$, then

$$
\begin{equation*}
\operatorname{det}\left(I-P_{\lambda, z}\right)^{\#}=\prod_{\substack{\xi_{s}(\gamma) \neq 1 \\ \alpha \\ \text { not } \\ \text { imagnary }}}\left(1-\xi_{-\alpha}(\gamma)\right) \tag{5.39}
\end{equation*}
$$

Corollary 5.12. Assume that $L_{R}=A$, i.e., $L$ is of Iwasawa type and that $\gamma_{S}$ is regular in $A$. Then

$$
\begin{equation*}
\left|\operatorname{det}\left(I-P_{\gamma, z}\right)^{\#}\right|^{\frac{1}{2}}=\Delta_{+}(\gamma) . \tag{5.40}
\end{equation*}
$$

For (5.39) we must remember that there are no imaginary noncompact roots for Iwasawa $L$; to get (5.40) we use (4.5b).

Remark 5.13. Formula (5.39) seems to have been first obtained by Kolk [28]; has method was more differential geometric.
5.5. Periodic Geodesics in $X$. The natural map $p: S \rightarrow X=\Gamma \backslash S$ gives rise to a map $\dot{p}: T S \rightarrow T X \approx \Gamma \backslash T S$. As $S$ is simply connected, the mapping that assigns to any $\gamma \in \Gamma$ the $p$-images of curves from $y$ to $\gamma \cdot y(y \in S)$ induces a bijection $c \mapsto p(c)$ from the set $\mathscr{C}(\Gamma)$ of $\Gamma$-conjugacy classes to the set $\pi_{1}(X)$ of free homotopy classes of closed curves in $X$. Since the elements of $\Gamma$ are all semisimple, it follows that for any $c \in \mathscr{C}(\Gamma)$ the homotopy class $p(c)$ contains periodic geodesics (of period 1), namely the $p$-images of all the $\gamma$-periodic geodesics for any $\gamma \in c$, now considered as curves in $S$ rather than as points in TS. Regarding these periodic geodesics as curves in $T X$ we can identify the set of periodic geodesics lying in the class $p(c)$ with a subset $F(c)$ of $T X$. Obviously

$$
\begin{equation*}
F(c))=\dot{p}(F(\gamma)) \quad(\gamma \in c \in \mathscr{C}(\Gamma)) \tag{5.41}
\end{equation*}
$$

Since $F(\gamma)$ has been shown to be a symmetric space (cf. Proposition 5.4) it is natural to expect that $F(c)$ is a locally symmetric space (cf. Kolk [28], VI, Theorem 6). This is indeed so, and the argument for proving it is essentially the following elementary lemma.

Lemma 5.14. Let $\gamma, \gamma^{\prime} \in \Gamma, z \in F(\gamma), z^{\prime} \in F\left(\gamma^{\prime}\right)$. Suppose $\dot{p}(z)=\dot{p}\left(z^{\prime}\right)$. Then there is a unique $\delta \in \Gamma$ such that $\delta \cdot z=z^{\prime}$; and $\gamma^{\prime}=\delta \gamma \delta^{-1}$. In particular, the $F(\gamma)$ are mutually disjoint for distinct $\gamma$; and for a given $\gamma$ in $\Gamma$, two points $z, z^{\prime}$ in $F(\gamma)$ have the same image in $T X$ under $\dot{p}$ if and only if the element $\delta \in \Gamma$ such that $\delta \cdot z=z^{\prime}$ belongs to $\Gamma_{\gamma}$.

Since $z$ can be moved to a position above $\pi(e) \in S$, we may assume that the base point of $z$ is $\pi(e)$. Then $z \in F(\gamma)$ as well as $F\left(\delta^{-1} \gamma^{\prime} \delta\right)$. Let $\gamma_{1}=\delta^{-1} \gamma^{\prime} \delta$. By Proposition 5.3, both $\gamma$ and $\gamma_{1}$ are in standard position and that $z=(\pi(e)$, $\left.\dot{\pi}_{e}\left(\log \gamma_{S}\right)\right)=\left(\pi(e), \dot{\pi}_{e}\left(\log \left(\gamma_{1}\right)_{S}\right)\right)$. So $\gamma_{S}=\left(\gamma_{1}\right)_{S}$. Using the fact that $\gamma_{K}$ and $\gamma_{S}$ (resp. $\left(\gamma_{1}\right)_{K}$ and $\left.\left(\gamma_{1}\right)_{S}=\gamma_{S}\right)$ commute, we find that

$$
\gamma_{1}^{-1} \gamma=\left(\gamma_{1}\right)_{K}^{-1}\left(\gamma_{1}\right)_{S}^{-1} \gamma_{S} \gamma_{K}=\left(\gamma_{1}\right)_{K}^{-1} \gamma_{K} \in K
$$

As $\Gamma$ contains no elliptic elements other than $e, \gamma_{1}=\gamma$, or $\gamma^{\prime}=\delta \gamma \delta^{-1}$. So the first assertion is proved. If $z \in F(\gamma) \cap F\left(\gamma^{\prime}\right)$, we take $\delta=e$ above to get $\gamma=\gamma^{\prime}$; if $z$, $z^{\prime} \in F(\gamma)$ and $z^{\prime}=\delta \cdot z$ for $\delta \in \Gamma$, then we take $\gamma^{\prime}=\gamma$ in the above result to get $\delta \in \Gamma_{\gamma}$.
Proposition 5.15. Let $c \in \mathscr{C}(\Gamma)$. Then the set $F(c)$ of periodic geodesics (lying in the class $p(c)$ ) is a smooth compact submanifold of $T X$ and is canonically isomorphic to the compact locally symmetric space $\Gamma_{\gamma} \backslash G_{\gamma} / U_{\gamma}$ where $U_{\gamma}$ is any maximal compact subgroup of $G_{\gamma}$. The elements of $F(c)$ all have the same length $l(c)$ (cf. (5.2)), considered as curves in $X$; and $l(c)$ is the minimum of the lengths of the closed curves in the homotopy class $p(c)$. For $c \in \mathscr{C}(\Gamma)$, the $F(c)$ are mutually disjoint; and the $l(c)$ form a discrete subset of the nonnegative reals with each value taken only by finitely many $c$.

It is immediate from Lemma 5.14 that $F(c) \approx \Gamma_{\gamma} \backslash F(\gamma)$ for $\gamma \in \mathcal{c}$, and that for $c$, $c^{\prime} \in \mathscr{C}(\Gamma), F(c) \cap F\left(c^{\prime}\right)=\emptyset$ whenever $c \neq c^{\prime}$. The compactness of $F(c)$ is clear since $\Gamma_{\gamma} \backslash G_{\gamma} / U_{\gamma}$ is compact (cf. Lemma 2.5). Proposition 5.7 implies that all the geodesics in $F(c)$ have the same length $l(c)$ and that this length is the minimum of the lengths of closed curves in the homotopy class $p(c)$. The last statement follows from the fact that inf $l(c)>0$ (Theorem 5.1), and the fact that a compact subset of $G$ meets only finitely many of the $c$ 's (Lemma 2.5).

The set of positive numbers $l(c)$ is of course the so-called length spectrum of $X$. The identification of the space $F(c)$ with $\Gamma_{\gamma} \backslash G_{\gamma} / U_{\gamma}$ suggests that the distributions $T_{c}$ appearing in the Poisson formula for $X$ should be intimately related to the manifolds $F(c)$. The following partial result is evidence of such a relationship.

Proposition 5.16. Let $c \in \mathscr{C}(\Gamma)$ be of real regular Iwasawa type (cf. subsection 5.1). Let $L$ be a $\theta$-stable CSG with $L_{R}=A$ and let $h=h_{I} h_{R}\left(h_{I} \in L_{I}, h_{R} \in A^{\prime}\right)$ be an element of $L$ in the $G$-conjugacy class of elements of $c$. Then

$$
\begin{equation*}
T_{c}=\left|\operatorname{det}\left(I-P_{c}\right)^{\#}\right|^{-\frac{1}{2}} \cdot\left(|\mathfrak{w}|^{-1} \sum_{s \in \mathfrak{w}} \delta_{s h_{\mathbb{R}}}\right) \tag{5.42a}
\end{equation*}
$$

where we write $\operatorname{det}\left(I-P_{c}\right)$ for the common value of $\operatorname{det}\left(I-P_{\gamma, z}\right)$ as $\gamma$ varies in $c$ and $z$ in $F(\gamma)$. Moreover, in this case, $\Gamma_{\gamma}$ is isomorphic to $\mathbb{Z}^{\mathrm{rk}(S)}$ for $\gamma \in \Gamma \backslash\{e\}: F(c)$ is a torus of dimension equal to $\operatorname{rk}(S)$ for any $c \neq[e]_{\Gamma}$; and the volume of this torus under the identifications $F(c) \approx \Gamma_{\gamma} \backslash F(\gamma) \approx \Gamma_{\gamma} \backslash G_{\gamma} / U_{\gamma}$ is precisely $v_{c}$ (cf. (5.1)):

$$
\begin{equation*}
v_{c}=\operatorname{vol}\left(\Gamma_{\gamma} \backslash G_{\gamma} / U_{\gamma}\right)=\operatorname{vol}_{0}\left(\Gamma_{\gamma} \backslash \mathrm{G}_{\gamma} / \mathrm{U}_{\gamma}\right) \tag{5.42b}
\end{equation*}
$$

The statement concerning $T_{c}$ follows from (4.30) and (5.40). For the remaining ones we may assume $\gamma \in L$ so that $\gamma=\gamma_{I} \gamma_{R}, \gamma_{R} \in A^{\prime}$. Then $G_{\gamma}=M_{\gamma_{I}} A$ where $M_{\gamma_{I}}$ is the centralizer of $\gamma_{I}$ in $M$. Since $M_{\gamma_{I}}$ is compact, $\Gamma_{\gamma} \cap M_{\gamma_{I}}=\{e\}$. So, under the projection homomorphism $M_{\gamma_{I}} A \rightarrow A, \Gamma_{\gamma}$ gets injectively mapped onto a subgroup $A^{(\gamma)}$ of $A$. The compactness of $M_{\gamma_{1}}$ also implies that $A^{(\gamma)}$ is discrete in $A$. Since $A^{(\gamma)} \backslash A \simeq \Gamma_{\gamma} \backslash M_{\gamma_{I}} A$ is compact, $A^{(y)}$ is a lattice in $A$ and so is isomorphic to $\mathbb{Z}^{\mathrm{rk}(S)} ;$ and $F(c) \simeq$ a torus of dimension equal to $\operatorname{dim}(A)$. Moreover, since $U_{\gamma}=M_{\gamma_{I}}, G_{\gamma} \simeq U_{\gamma} \times A$, so that $d x_{\gamma}=d_{0} x_{\gamma}$, and

$$
v_{c}=\operatorname{vol}\left(\Gamma_{\gamma} \backslash G_{\gamma} / U_{\gamma}\right)=\operatorname{vol}_{0}\left(\Gamma_{\gamma} \backslash G_{\gamma} / U_{\gamma}\right)=\operatorname{vol}_{0}\left(A^{(\gamma)} \backslash A\right)
$$

Suppose now that $\operatorname{rk}(S)=1$. Then, for every $c \neq[e]_{\Gamma},(5.42 \mathrm{a}, \mathrm{b})$ are valid. By the above result, the group $\Gamma_{\gamma}(\gamma \neq e)$ is infinite cyclic and so we can find $\gamma_{0} \in \Gamma_{\gamma}$ such that $\gamma_{0}$ generates $\Gamma_{\gamma}$ and $\gamma_{0}^{m}=\gamma$ for some integer $m \geqq 1$. It is obvious that $c_{0}$ $=\left[\gamma_{0}\right]_{\Gamma}$ is uniquely determined by $c ; c_{0}\left(\operatorname{resp} . l\left(c_{0}\right)=l_{0}(c)\right)$ is called the primitive class (resp. primitive length) corresponding to $c$. Clearly

$$
\begin{equation*}
l\left(c_{0}\right)=l_{0}(c)=m^{-1} l(c) \tag{5.42c}
\end{equation*}
$$

Theorem 5.17. (Poisson formula when $\operatorname{rk}(S)=1)$. We have

$$
\begin{align*}
\sum_{\lambda \in A} m(\lambda) e^{\lambda}= & 2^{n(2 \alpha)} \pi^{-(n-1) / 2} I(\rho)^{-1} \operatorname{vol}_{0}(X)\left(\frac{1}{2} \hat{\beta}\right) \\
& +\frac{1}{2} \sum_{\substack{c \in \mathscr{\mathscr { Y }}(\Gamma), c \neq[e] \Gamma}} l_{0}(c)\left|\operatorname{det}\left(I-P_{\mathrm{c}}\right)^{\#}\right|^{-\frac{1}{2}}\left(\delta_{l(c)}+\delta_{-l(\mathrm{c})}\right) \tag{5.43}
\end{align*}
$$

as an identity of distributions on $\mathfrak{a} \approx \mathbb{R}, l_{0}(c)$ being the primitive length corresponding to $c$.

Remark 5.18. For a less precise version see Kolk ([28], formula (34), p. 107). When $G=S L(2, \mathbb{R})$, formula (5.43) specializes to results of Randol [37] and LaxPhillips [29].
5.6. Comparison with the Results of Duistermaat-Guillemin. The formula (5.43) obtained above raises the question of the relation between the results of this article and those of Duistermaat-Guillemin [6] on the singularities of the Fourier transform of the spectrum of positive elliptic differential operators. In this subsection we shall indicate briefly how this comparison can be made. In what follows, we write $\Delta$ for the Laplace-Beltrami operator $\omega_{S}$ of (2.6).

We note that the Riemannian structure on $S$ induces an isomorphism $T S \rightarrow T^{*} S$. The Hamilton flow of the length function of the tangent vectors to $S\left(=\right.$ the principal symbol of $(-\Delta)^{\frac{1}{2}}$ pulled back to $T S$ ) coincides with the geodesic flow $\Phi^{t}$ on $(T S)_{1}$, the unit sphere bundle of $S$. Multiplication by $t>0$ in the fibers of $T S$ intertwines $\Phi^{1}$ on $(T S)_{t}(=$ the sphere bundle of vectors of length $t$ ) with $\Phi^{t}$ on $(T S)_{1}$. So, for any $\gamma \in G$ we have an idendification of $F(\gamma)$ with the set $F_{1}(\gamma)$ of periodic (with respect to $\gamma$ ) geodesics in $(T S)_{1}$, of period $\left\|\log \gamma_{s}\right\|$.

If we pull back the canonical symplectic form of $T^{*} S$ at a point $z$ above $\pi(e)$ to $\mathfrak{s} \oplus \mathfrak{s}$ via the identification $T^{*} S \approx T^{*} \bar{S}$ and the trivialization $\tau$ (cf. (5.27)), we find that it is the symplectic form $\sigma_{z}$ given by

$$
\begin{equation*}
\sigma_{z}((X, Y):(\tilde{X}, \tilde{Y}))=\langle Y, \tilde{X}\rangle-\langle X, \tilde{Y}\rangle \quad(X, \tilde{X}, Y, \tilde{Y} \in \mathfrak{F}) \tag{5.44}
\end{equation*}
$$

However, as $\operatorname{ker}\left(I-P_{\gamma, z}\right)=T_{z}(F(\gamma))$ is horizontal (cf. (5.35), it is isotropic for $\sigma_{z}$, so that our situation is opposite to that described in Lemma 4.4 of DuistermaatGuillemin [6] (another way of saying this is that $I-P_{\gamma, z}$ has a nonzero nilpotent component always); so, in order to calculate the canonical density on $\operatorname{ker}(I$ $-P_{\gamma, z}$ ) we must go back to its definition in loc. cit (4.1), (4.2).

Let us temporarily write $P=P_{\gamma, z}, N=T_{\pi(e)}(S(\gamma)) \times\{0\} \approx \operatorname{ker}(I-P), L=\{0\}$ $\times T_{\pi(e)}(S(\gamma)), W \approx$ Range $(I-P), V=\mathfrak{s} \oplus \mathfrak{s}$. It is then clear from (5.31)-(5.35) that $V$ $=W \oplus L$ and that $I-P$ is an isomorphism: $L \rightarrow N$. Let us also write, for any real
vector space $U$ with a scalar product, $\varepsilon_{U}$ for its corresponding Euclidean density. Then $\varepsilon_{V}$ is equal to the canonical density on $V$ defined by the symplectic form $\sigma_{z}$. The canonical density $\mu=c \varepsilon_{N}$ on $N$ defined in loc.cit (4.1), (4.2) is now determined by the equation

$$
\begin{equation*}
(I-P)_{*}\left(\varepsilon_{V} / \mu\right)=\varepsilon_{V} / c^{-1} \varepsilon_{L}, \tag{5.45}
\end{equation*}
$$

where $c^{-1} \varepsilon_{L}$ is the density on $V / W \simeq L$ defined by the pairing between $N$ and $L$ induced by $\sigma_{z}$. Now, $\varepsilon_{V} / \mu=\varepsilon_{L} \varepsilon_{W} / c \varepsilon_{N},(I-P)_{*} \varepsilon_{L}=\varepsilon_{N}$, so that (5.45) becomes $c^{-1} \varepsilon_{N}(I-P)_{*} \varepsilon_{W / N}=c \varepsilon_{W}$, or

$$
\begin{equation*}
c^{2}=\left|\operatorname{det}(I-P)^{\#}\right|^{-1} . \tag{5.46}
\end{equation*}
$$

We have thus proved:
Proposition 5.19. The density on $F(\gamma)$ obtained from the canonical density on the corresponding manifold of periodic geodesics in $(T X)_{1} \approx\left(T^{*} X\right)_{1}$ by DuistermaatGuillemin ([6], (4.1), (4.2)) is equal to $\left|\operatorname{det}\left(I-P_{\gamma}\right)^{\#}\right|^{-\frac{1}{2}}$ times the density on $F(\gamma)$ induced by using the Killing form and the identification of $F(\gamma)$ as $G_{\gamma} / U_{\gamma}$ where $U_{\gamma}$ is a maximal compact subgroup of $G_{\gamma}$.

Let us now fix a number $T>0$ in the length spectrum of $X$. By Proposition 5.15, there are finitely many $c_{j} \in \mathscr{C}(\Gamma)(1 \leqq j \leqq k)$ such that $l\left(c_{j}\right)=T$; the manifolds $Z_{j}$ in $\left(T^{*} X\right)_{1}$ that correspond to the $F\left(c_{j}\right)$ under our identifications are then disjoint, connected, compact, and their union is the set of periodic geodesics of period $T$ in $\left(T^{*} X\right)_{1}$. Now, the subprincipal symbol of $(-\Delta)^{\frac{1}{2}}$ is equal to zero, and so, by the above Proposition, the numbers $\alpha_{j, 0}$ appearing in loc. cit. Theorem 4.5 come out to be

$$
\begin{equation*}
\alpha_{j, 0}=(2 \pi)^{-1}\left|\left(I-P_{\gamma}\right)^{\#}\right|^{-\frac{1}{2}} \operatorname{vol}_{0}\left(\Gamma_{\gamma} \backslash G_{\gamma} / U_{\gamma}\right) \quad\left(\gamma \in c_{j}, 1 \leqq j \leqq k\right) \tag{5.47}
\end{equation*}
$$

Let us now return to the Poisson formula (5.3). Let $P$ be a positive elliptic differential operator of even order $m$ on $X$ arising from a $G$-invariant differential operator on $S$. Then there exists a w-invariant polynomial $p$ of degree $m$ on $\mathfrak{a}_{c}^{*}$ $=\mathscr{F}$ such that the spectrum of $P$ consists of the numbers $p(\lambda), \lambda \in \Lambda$, counted with multiplicities. Let us write $Q=P^{1 / m}, q(\mu)=p(i \mu)^{1 / m}$ for $\mu \in \mathscr{F}_{R}$. Then, for the distribution (in $t) \hat{\sigma}(t)=\operatorname{Tr}\left(e^{-i t \ell}\right)$ we find, remembering that the measure $d \mu$ on $\mathscr{F}_{R}$ is the one dual to $d_{0} \mathrm{a}$,

$$
\begin{aligned}
& \hat{\sigma}(t) \sum_{\lambda \in \Lambda} m(\lambda) e^{-i t q(\lambda / i)} \\
& =\sum_{\lambda \in \Lambda} m(\lambda) \int_{\mathscr{F}_{R}} \int_{\mathfrak{a}} e^{i\langle\lambda-\mu, \cdot\rangle} e^{-i t q(\mu)} d \mu d_{0} a \\
& =\sum_{c \in \mathscr{\mathscr { C }}(\Gamma)} v_{c}\left\langle T_{c}, \int_{\mathscr{F}_{R}} e^{i\langle\mu, \cdot\rangle} e^{-i t q(\mu)} d \mu\right\rangle,
\end{aligned}
$$

as a distribution in $t$. To make the comparison with loc. cit. Theorem 4.5, we introduce $\chi \in C_{c}^{\infty}(\mathbb{R})$ with support in a small neighborhood of $T>0$ and equal to 1 in a small neighborhood of $T$, and investigate the asymptotic behaviour, as $\tau \rightarrow \infty$, of

$$
\begin{align*}
(2 \pi)^{-1} & \int e^{i \tau(t-T)} \hat{\sigma}(t) \chi(t) d t \\
& =\sum_{1 \leqq j \leqq k}(2 \pi)^{-1} \tau^{r} v_{c_{j}}\left\langle T_{c_{j}}, \iint_{\mathbb{R}} \int_{\boldsymbol{F}_{R}} e^{i \tau[t-T-\langle\mu, \cdot\rangle-t q(\tau \mu) / \tau]} \chi(t) d \mu d t .\right. \tag{5.48}
\end{align*}
$$

If for some $j$, the class $c_{j}$ is of real regular Iwasawa type, we can use (5.42a) to give an asymptotic expansion, for $\tau \rightarrow \infty$, of the corresponding term in (5.48), provided the homogeneous part $q_{1}$ of $q$ of degree 1 is strictly convex, using the substitution of variables $\mu=\zeta \nu, q_{1}(v)=1$. For $P=-\Delta$, the Laplacian, $q_{1}(\mu)$ $=\|\mu\|$, and one obtains for such a class $c_{j}$, using (5.42a, b), the expansion

$$
\begin{aligned}
& \int e^{i(t-T)} \hat{\sigma}(t) \chi(t) d t \\
& \quad \sim(2 \pi)^{-1}\left|\left(I-P_{\gamma}\right)^{\#}\right|^{-\frac{1}{2}} \operatorname{vol}_{0}\left(\Gamma_{\gamma} \backslash G_{\gamma /} / U_{\gamma}\right)\left(\frac{\tau}{2 \pi i}\right)^{\frac{r-1}{2}}\left(1+\sum_{k=1}^{\infty} x_{j, k}^{\prime} \tau^{-k}\right) .
\end{aligned}
$$

In view of the formula (5.47) for $\alpha_{j .0}$, this result agrees with loc. cit Theorem 4.5, except for the factor $i^{-\sigma_{J}}$ occuring in (4.7) loc. cit. Now, $\sigma_{j}$ is equal to the Morse index of the periodic geodesic as a stationary point for the length function on the space of closed curves (see the end of loc. cit. Section 6). Therefore, $\sigma_{j}=0$ in our case, because, the length function restricted to the homotopy class of the periodic geodesic attains its minimum on the periodic geodesic, in view of Proposition 5.15.

Of course, the treatment of $t \mapsto \operatorname{Tr}\left(e^{-i t Q}\right)$ for only one operator $Q$ eliminates a large part of the information in the Poisson formula (5.3); only when $\mathrm{rk}(S)=1$, the coefficients of the leading terms of the singularities of $t \mapsto \operatorname{Tr}\left(e^{-i t Q}\right)$ lead to a full recovery of $\sum v_{c} T_{c}$, and therefore of the whole spectrum $A$, as is clear (with $P$ $=-4$ ) from (5.47) and (5.43).

## 6. Spectral Asymptotics: Preliminaries

6.1. From now on our aim is to use the formula (3.28) to study the spectrum $\Lambda$ of $X$. Thus we get, with a replacing $A$, and using (3.30):

Proposition 6.1. There is a w-invariant open neighborhood $U$ of 0 in a which is balanced $(U=-U)$ and starlike at $0(t U \subset U$ for $0 \leqq t \leqq 1)$ with the following property: for any $f \in C_{c}^{\infty}(U)$,

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) \hat{f}(\lambda)=\frac{\operatorname{vol}(X)}{|\mathfrak{w}|} \int_{\hat{Z}_{1}} \hat{f}(v) \beta(v) d v \tag{6.1}
\end{equation*}
$$

here, the series on the left converges absolutely, and $\hat{f}$ is the Fourier-Laplace transform of $f$ given by

$$
\begin{equation*}
\hat{f}(\xi)=\left\langle f, e^{\xi}\right\rangle=\int_{\mathrm{a}} f(H) e^{\xi(H)} d H \quad(\xi \in \mathscr{F}) . \tag{6.2}
\end{equation*}
$$

6.2. We now begin the study of the equality (6.1). The basic idea is to construct test functions with supports in $U$ whose Fourier transform are $\geqq 0$ on $\Lambda$. Let
conj be the conjugation in $\mathscr{F}$ defined by $\mathscr{F}_{R}$, so that $\left(\xi_{R}+\xi_{I}\right)^{\text {conj }}=\xi_{R}-$ $\xi_{I}\left(\xi_{R} \in \mathscr{F}_{R}, \xi_{I} \in \mathscr{F}_{I}\right)$; and define, for any $s \in \mathfrak{w}$,

$$
\begin{equation*}
\mathscr{F}(s)=\left\{\xi: \xi \in \mathscr{F}, s \cdot \xi=-\xi^{\text {conj }}\right\} \tag{6.3}
\end{equation*}
$$

Lemma 6.2. Let $U \subset \mathfrak{a}$ be as in Proposition 6.1 with the additional restriction that $\|H\| \leqq \frac{1}{3}$ for all $H \in U$. Suppose $g \in C_{c}^{\infty}(U)$ is real non-negative symmetric (i.e. $g(H)$ $=g(-H)$ for $H \in \mathfrak{a}$ ), w-invariant, and $\int_{\mathfrak{a}} g d H=2$. Then its Fourier-Laplace transform $\hat{g}$ (cf. (6.2)) has the following properties:
(i) $\hat{\mathrm{g}}(\xi)=\hat{\mathrm{g}}(-\xi)=\hat{\mathrm{g}}(s \cdot \xi)(\xi \in \mathscr{F}, s \in \mathfrak{m})$.
(ii) $\hat{\mathrm{g}}\left(\xi^{\text {conj }}\right)=\hat{\mathrm{g}}(\xi)^{\mathrm{conj}}(\xi \in \mathscr{F})$.
(iii) $\hat{g}(\xi)$ is real if $\xi \in \bigcup_{s \in \boldsymbol{w}} \mathscr{F}$ (s). In particular, $\hat{g}(\lambda)$ is real if $\lambda \in \Lambda$.
(iv) $|\hat{g}(\xi)| \geqq 1$ if $\xi \in \mathscr{F}$ and $\|\xi\| \leqq 1$.

The assertions (i) and (ii) are consequences of the assumptions that $g$ is real, symmetric and $\mathfrak{w}$-invariant. If $s \in \mathfrak{w}$ and $\xi \in \mathscr{F}$ satifies $s \cdot \xi=-\xi^{\mathrm{conj}}$, we find, using (i) and (ii), that $\hat{g}(\xi)=\hat{g}(-\xi)=\hat{g}\left(-(s \cdot \xi)^{\text {conj }}\right)=\hat{g}(-s \cdot \xi)^{\text {conj }}=\hat{g}(\xi)^{\text {conj }}$. So $\hat{g}$ is real on $\bigcup_{s \in \mathfrak{w}} \mathscr{F}(s)$; and to complete the proof of (iii) we must recall that $\Lambda \subset \bigcup_{s \in \mathfrak{w}} \mathscr{F}(s)$, according to Corollary 3.5. Suppose now $\xi \in \mathscr{F}$ with $\|\xi\| \leqq 1$. Then

$$
\hat{g}(\xi)=\int_{a}\left(1+\left(e^{\xi(H)}-1\right)\right) g(H) d H=2+\int_{U}\left(e^{\xi(H)}-1\right) g(H) d H .
$$

But as $\|H\| \leqq \frac{1}{3}$ for $H \in U,|\xi(H)| \leqq \frac{1}{3}$ for $H \in U$. Also, if $z \in \mathbb{C}$ and $|z| \leqq \frac{1}{3}$

$$
\left|e^{z}-1\right| \leqq|z|+|z|^{2}+\ldots=|z| / 1-|z| \leqq \frac{1}{2} .
$$

Therefore (iv) follows now, using the estimate

$$
\left|\int_{U}\left(\mathrm{e}^{\xi(H)}-1\right) g(H) d H\right| \leqq \frac{1}{2} \int_{\mathrm{a}} g d H=1
$$

We choose a balanced $\mathfrak{w}$-invariant open neighborhood $U_{1}$ of 0 in $\mathfrak{a}$, starlike at 0 , with $U_{1}+U_{1} \subset U$. Let $g \in C_{c}^{\infty}\left(U_{1}\right)$ and let us assume that $g$ has the properties described in Lemma 6.2. We define the functions $h, g(t: \cdot)$ and $h(t: \cdot)(t>0)$ on a by

$$
\begin{equation*}
h=g * g ; \quad g(t: H)=t^{r} g(t H)(H \in \mathfrak{a}) ; \quad h(t: \cdot)=g(t: \cdot) * g(t: \cdot) \tag{6.4}
\end{equation*}
$$

The Fourier transforms $\hat{g}(t: \cdot)$ and $\hat{h}(t: \cdot)$ then satisfy, for $t>0$ and $\xi \in \mathscr{F}$

$$
\begin{equation*}
\hat{g}(t: \xi)=\hat{g}\left(t^{-1} \xi\right) ; \quad \hat{h}(t: \xi)=\hat{g}(t: \xi)^{2}=\hat{g}\left(t^{-1} \xi\right)^{2}=\hat{h}\left(t^{-1} \xi\right) . \tag{6.5}
\end{equation*}
$$

Lemma 6.3. The functions $h(t: \cdot)$ and $\hat{h}(t: \cdot)$ satisfy for $t \geqq 1$ : (i) $\hat{h}(t: \xi) \geqq 0$, if $\xi \in \bigcup_{s \in w} \mathscr{F}(s)$, in particular $\hat{h}(t: \lambda) \geqq 0$ if $\lambda \in \Lambda$; (ii) $|\hat{h}(t: \xi)| \geqq 1$ if $\xi \in \mathscr{F}$ and $\|\xi\| \leqq t$; (iii) $h(t: \cdot) \in C_{c}^{\infty}(U)$ is real symmetric and $w$-invariant; (iv) for each integer $m \geqq 0$ and real number $a>0$ there is a constant $c=c(m: a)>0$ such that for all $t \geqq 1$ and $\xi \in \mathscr{F}$ with $\left\|\xi_{R}\right\|<a$

$$
\begin{equation*}
|\widehat{h}(t: \xi)| \leqq c t^{m}(1+\|\xi\|)^{-m} . \tag{6.6}
\end{equation*}
$$

For (i) use Lemma 6.2 (iii) and (6.5); for (ii), Lemma 6.2 (iv) and (6.5). For (iii) observe that since $t \geqq 1$ and $U_{1}$ is starlike at 0 , $\operatorname{supp} g(t: \cdot) \subset U_{1}$; hence $\operatorname{supp} h(t: \cdot) \subset U_{1}+U_{1} \subset U$, and convolutions of real symmetric $\mathfrak{w}$-invariant functions are likewise real symmetric and $\mathfrak{w}$-invariant. According to the PaleyWiener estimate given in formula (3.23), for each integer $m \geqq 0$, we can find a constant $c^{\prime}=c_{m}^{\prime}>0$ such that, for $\xi \in \mathscr{F}$,

$$
|\hat{g}(\xi)|^{2} \leqq c^{\prime}(1+\|\xi\|)^{-m} \exp \left(\left\|\xi_{R}\right\|\right)
$$

since $\|H\| \leqq \frac{1}{3}$ for $H \in \operatorname{supp}(g)$. So

$$
|\hat{h}(t: \xi)|=\left|\hat{g}\left(t^{-1} \xi\right)\right|^{2} \leqq c^{\prime}\left(1+t^{-1}\|\xi\|\right)^{-m} \exp \left(\left\|\xi_{R}\right\| t^{-1}\right) \quad(\xi \in \mathscr{\mathscr { F }}) .
$$

Now, as $t \geqq 1,1+t^{-1}\|\xi\| \geqq t^{-1}(1+\|\xi\|)$; so we find that, if $\left\|\xi_{R}\right\| \leqq a$

$$
|\widehat{h}(t: \xi)| \leqq c^{\prime} t^{m}(1+\|\xi\|)^{-m} e^{u} .
$$

We can already obtain some crude results on the asymptotic behaviour of $A$ on the basis of the above lemmas. Thus we have

Proposition 6.4. There is a constant $c>0$ such that, for all $t \geqq 1$,

$$
\sum_{\lambda \in A,\|\lambda\| \leqq t} m(\lambda) \leqq c t^{n} \quad(n=\operatorname{dim} G / K) .
$$

In view of Lemma 6.3 (iii) we can take $f=h(t: \cdot)$ in (6.1). Since $\hat{h}(t: \lambda) \geqq 0$ for $\lambda \in A$ and $\hat{h}(t: \lambda) \geqq 1$ for $\lambda \in A$ with $\|\lambda\| \leqq t$, by (i) and (ii) of Lemma 6.3, we get, with $v=\operatorname{vol}(X) /|\mathfrak{w}|$,

$$
\begin{aligned}
& \sum_{\lambda \in A,\|\lambda\| \leqq t} m(\lambda) \leqq \sum_{\lambda \in A} m(\hat{\lambda}) \hat{h}(t: \lambda) \\
& =v \int_{\mathscr{F}_{I}} \hat{h}(t: v) \beta(v) d v=v \int_{\mathscr{F}_{I}} \hat{h}\left(t^{-1} v\right) \beta(v) d v \\
& =v t^{r} \int_{\mathscr{F}_{I}} \hat{h}(v) \beta(t v) d v,
\end{aligned}
$$

the integral being convergent by the Paley-Wiener Theorem. By the growth estimates for $\beta$ given in (3.44), we have for $t \geqq 1$ and $v \in \mathscr{F}_{I}, \beta(t \nu) \leqq \operatorname{const} t^{n-r}(1$ $+\|v\|)^{n-r}$. So the last expression on the right side above is majorized by

$$
\text { const } t^{n} \int_{\mathscr{F}_{I}} \hat{h}(v)(1+\|v\|)^{n-r} d v
$$

giving the Proposition.
From this we get at once (cf. Proposition 3.6))
Corollary 6.5. $A$ is a discrete set.
Moreover we have
Proposition 6.6. For any $f \in C_{c}^{\infty}(\mathfrak{a})$, the series

$$
\sum_{\lambda \in A} m(\lambda)|\hat{f}(\lambda-\xi)|
$$

is normally convergent when $\xi$ varies over compact subsets of $\mathscr{F}$ (in particular, uniformly convergent on compacta in $\mathscr{F}$ ).

Select a constant $c_{1}>0$ such that, for all $\eta \in \mathscr{F}$,

$$
|\hat{f}(\eta)| \leqq c_{1}(1+\|\eta\|)^{-(n+2)} \exp \left(b\left\|\eta_{R}\right\|\right),
$$

where $b>0$ is a constant such that $\operatorname{supp}(f) \subset\{H: H \in \mathfrak{a},\|H\| \leqq b\}$. According to Proposition 3.4 (iii), any $\lambda \in \Lambda$ satisfies $\left\|\lambda_{R}\right\| \leqq\|\rho\|$; so if $\Omega \subset \mathscr{F}$ is a compactum, we can find an integer $p>0$ such that $\left\|(\lambda-\xi)_{R}\right\| \leqq p$ and $\|\xi\| \leqq p(\lambda \in \Lambda, \xi \in \Omega)$. Let $k \geqq p$ be any integer. Then, for all $\lambda \in \Lambda$ with $k \geqq\|\lambda\|<k+1$ and $\xi \in \Omega$,

$$
|\hat{f}(\lambda-\xi)| \leqq c_{1}(1+\|\lambda-\xi\|)^{-(n+2)} \exp \left(b\left\|(\lambda-\xi)_{R}\right\|\right) \leqq c_{2}(1+k-p)^{-(n+2)}
$$

$c_{2}>0$ a constant. According to Proposition 6.4, we have

$$
\sum_{\lambda \in A, k \leqq\|\lambda \mid\|<k+1} m(\lambda) \leqq c(k+1)^{n}
$$

hence we get

$$
\begin{aligned}
& \sum_{\lambda \in \Lambda,\|\lambda\| \geqq p} m(\lambda) \sup _{\xi \in \Omega}|\hat{f}(\lambda-\xi)| \\
& \leqq \sum_{k=p}^{\infty} \sum_{\lambda \in \Lambda, k \leqq\|\lambda\|<k+1} m(\lambda) c_{2}(1+k-p)^{-(n+2)} \\
& \leqq c c_{2} \sum_{k=p}^{\infty}(1+k-p)^{-(n+2)}(k+1)^{n}<\infty
\end{aligned}
$$

The above method of estimating a sum like $\sum_{\lambda \in A} m(\lambda) \hat{f}(\lambda-\xi)$ by partitioning it into subsums over various "shells" and estimating these individually, will be occasionally used in our subsequent work.
6.3. These results, however, are too crude. What we need are analogous results on the number of spectral points in balls around a variable point $\mu \in \mathscr{F _ { I }}$. So we introduce the test functions

$$
\begin{equation*}
h(t: \cdot: \mu)=h(t: \cdot) e^{-\mu(\cdot)} \quad\left(t>0, \mu \in \mathscr{F}_{I}\right), \tag{6.7}
\end{equation*}
$$

where the $h(t: \cdot)$ are given by (6.4). Clearly

$$
\begin{equation*}
h(t: \cdot: \mu) \in C_{c}^{\infty}(U) \quad\left(t \geqq 1, \mu \in \mathscr{F}_{I}\right) \tag{6.8}
\end{equation*}
$$

Moreover, for the Fourier-Laplace transforms $\hat{h}(t: \cdot: \mu)$ we have

$$
\begin{equation*}
\widehat{h}(t: \xi: \mu)=\hat{h}(t: \xi-\mu) \quad(t \geqq 1, \xi \in \mathscr{F}, \mu \in \mathscr{F} I) . \tag{6.9}
\end{equation*}
$$

Lemma 6.7. For each integer $m \geqq 0$ there is a constant $c=c(m)>0$ such that for all $t \geqq 1, \lambda \in \Lambda, \mu \in \mathscr{F}_{I}$,

$$
\begin{equation*}
|\hat{h}(t: \lambda-\mu)| \leqq c t^{m}(1+\|\lambda-\mu\|)^{-m} \tag{6.10}
\end{equation*}
$$

This follows from (6.6) because any $\lambda \in A$ satisfies $\left\|\lambda_{R}\right\| \leqq\|\rho\|$ and, since $\mu \in \mathscr{F}_{I}$, $(\lambda-\mu)_{R}=\lambda_{R}$.

Let us now define $\tilde{\beta} \in C^{\infty}\left(\mathscr{F}_{I}\right)^{\text {wo }}$ by

$$
\begin{equation*}
\tilde{\beta}(\mu)=\int_{\mathscr{F}_{1}}(1+\|v-\mu\|)^{-n-1} \beta(v) d v \quad\left(\mu \in \mathscr{F}_{I}\right) . \tag{6.11}
\end{equation*}
$$

The majorant (3.44) for $\beta$ shows that $\tilde{\beta}$ is well defined.
Proposition 6.8. (i) We have, for all $t \geqq 1$ and $\mu \in \mathscr{F}_{I}$,

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) \hat{h}(t: \lambda-\mu)=\operatorname{vol}(X)|\mathfrak{w}|^{-1} \int_{\mathscr{F}_{I}} \hat{h}(t: v-\mu) \beta(v) d v, \tag{6.12}
\end{equation*}
$$

where the series on the left converges absolutely. (ii) We can find a constant $c$ $=c(h)>0$ such that, for all $t \geqq 1, \mu \in \mathscr{F}_{I}$,

$$
\begin{equation*}
\left|\sum_{\lambda \in A} m(\lambda) \hat{h}(t: \lambda-\mu)\right| \leqq c t^{n+1} \tilde{\beta}(\mu) . \tag{6.13}
\end{equation*}
$$

Indeed, applying Proposition 6.1 with $f=h(t: \cdot: \mu)$, we find

$$
\sum_{\lambda \in A} m(\lambda) \widehat{h}(t: \lambda: \mu)=\operatorname{vol}(X)|\mathfrak{w}|^{-1} \int_{\mathscr{\mathscr { P }}_{I}} \hat{h}(t: v: \mu) \beta(v) d v,
$$

and assertion (i) follows from (6.9). (ii) follows from (i) and Lemma 6.7.
6.4. Estimates for $\tilde{\beta}$. Assertion (ii) of Proposition 6.8 makes it clear that the growth properties of $\tilde{\beta}$ are important for us. Let us recall the numbers $d(\alpha)\left(\alpha \in \Delta^{++}\right)$defined in (3.41); we have $\sum_{x \in \Delta^{++}} d(\alpha)=n-r$. For any subset $\Phi \subset \Delta^{++}$we define $d(\Phi) \geqq 0$ and the set $T(\Phi) \subset \mathscr{F}_{I}$ by

$$
\begin{equation*}
d(\Phi)=\sum_{\alpha \in \Phi} d(\alpha) \tag{6.14}
\end{equation*}
$$

$$
\begin{equation*}
T(\Phi)=\left\{\mu: \mu \in \mathscr{F}_{I},\langle\alpha, \mu\rangle=0 \forall \alpha \in \Phi\right\} . \tag{6.15}
\end{equation*}
$$

Proposition 6.9. There is a constant $c>0$ such that

$$
\begin{equation*}
\tilde{\beta}(\mu) \leqq c \prod_{\alpha \in \Delta^{++}}(1+\mid\langle\alpha, \mu\rangle)^{d(\alpha)} \quad\left(\mu \in \mathscr{F}_{I}\right) \tag{6.16}
\end{equation*}
$$

and, for any subset $\Phi \subset \Delta^{++}$and arbitrary $\mu \in T(\Phi)$.

$$
\begin{equation*}
\tilde{\beta}(\mu) \leqq c(1+\|\mu\|)^{n-r-d(\Phi)} . \tag{6.17}
\end{equation*}
$$

It follows from (3.37) and (3.40) that $\beta(v)=I(\rho)^{2} \prod_{\alpha \in \Delta^{+}}|I(\alpha: v)|^{-2}\left(v \in \mathscr{F}_{I}\right)$; so according to (3.44) we have, for $\mu \in T(\Phi)$ and $v \in \mathscr{F}_{I}$,

$$
\begin{aligned}
& \beta(\mu+v)=I(\rho)^{2} \prod_{\alpha \in \Phi}|I(\alpha: \mu+v)|^{-2} \prod_{\alpha \in \Delta^{+}+\mid \Phi}|I(\alpha: \mu+v)|^{-2} \\
& \quad \leqq c_{1} \prod_{\alpha \in \Phi}(1+|\langle\alpha, v\rangle|)^{d(x)} \prod_{\alpha \in \Delta^{+}+\Phi}(1+|\langle\alpha, v\rangle|+|\langle\alpha, \mu\rangle|)^{d(\alpha)} \\
& \quad \leqq c_{2}(1+\|v\|)^{n-r} \prod_{\alpha \in \Delta^{++} \mid \Phi}(1+|\langle\alpha, \mu\rangle|)^{d(\alpha)},
\end{aligned}
$$

where $c_{2}>0$ may be selected to be independent of $\Phi, \mu$ and $v$. Thus, for all $\mu \in T(\Phi)$. (cf. (6.11))

$$
\begin{aligned}
\tilde{\beta}(\mu) & =\int_{\mathscr{F}_{I}}(1+\|v\|)^{-n-1} \beta(\mu+v) d v \\
& \leqq c_{2} \prod_{\alpha \in A^{++} \mid \Phi}(1+|\langle\alpha, \mu\rangle|)^{d(\alpha)} \int_{\mathscr{F}_{I}}(1+\|v\|)^{-r-1} d v .
\end{aligned}
$$

Formula (6.16) follows now by taking as $\Phi$ the empty set, and (6.17) follows using (6.14).

The inequality

$$
\begin{equation*}
\left(1+\left\|\xi^{\prime}\right\|\right)^{-1} \leqq\left(1+\left\|\xi-\xi^{\prime}\right\|\right)(1+\|\xi\|)^{-1} \quad\left(\xi, \xi^{\prime} \in \mathscr{F}\right) \tag{6.18}
\end{equation*}
$$

leads at once to the following:

$$
\begin{equation*}
\tilde{\beta}(v) \leqq(1+\|v-\mu\|)^{n+1} \tilde{\beta}(\mu) \quad\left(v, \mu \in \mathscr{F}_{I}\right) \tag{6.19}
\end{equation*}
$$

The estimate (6.19) shows that $\tilde{\beta}$ is of regular growth; if $b>0$ is any fixed constant, we have for a suitable constant $c=c(b)>0$

$$
\begin{equation*}
\tilde{\beta}(v) \leqq c \int_{\|\mu-v\| \leqq b} \tilde{\beta}(\mu) d \mu \quad\left(v \in \mathscr{F}_{I}\right) . \tag{6.20}
\end{equation*}
$$

When making estimates it is convenient to work with functions of regular growth; however, so far as integrals over bounded sets are concerned, $\beta$ and $\tilde{\beta}$ have the same growth as we will show in the next proposition. Let $\delta$ denote the distance function in $\mathscr{F}_{I}$. We define, for any $\kappa>0$ and any subset $E \subset \mathscr{F}_{I}$,

$$
\begin{equation*}
E_{\kappa}=\left\{v: v \in \mathscr{F}_{I}, \delta(v, E) \leqq \kappa\right\} . \tag{6.21}
\end{equation*}
$$

Proposition 6.10. Given $\kappa>0$, we can find constants $a_{1}, a_{2}>0$ depending on $\kappa$, such that, for any bounded Lebesgue measurable set $E$ in $\mathscr{F}_{1}$,

$$
\begin{equation*}
a_{1} \int_{E} \beta(\mu) d \mu \leqq \int_{E_{\kappa}} \tilde{\beta}(\mu) d \mu \leqq a_{2} \int_{E_{2 \kappa}} \beta(\mu) d \mu . \tag{6.22}
\end{equation*}
$$

The proof of the first inequality in (6.22) is by observing that there is a constant $c_{1}=c_{1}(\kappa)>0$ such that, for all $E$,

$$
\begin{aligned}
& \int_{E} \beta(\mu) d \mu \leqq c_{1} \int_{E_{x}} \int_{\|-v H \leqq \kappa} \beta(v) d v d \mu \\
& \quad \leqq c_{1}(\kappa+1)^{n+1} \int_{E_{x}} \int_{\mathscr{F}_{1}}(1+\|\mu-v\|)^{-n-1} \beta(v) d v d \mu .
\end{aligned}
$$

Using formula (6.19) and the estimate (6.16), we get a constant $c_{2}=c_{2}(\kappa)>0$ such that for any $\mu \in \mathscr{F}_{I}$

$$
\begin{equation*}
\tilde{\beta}(\mu) \leqq c_{2}^{\prime} \inf _{\|\mu-v\| \leqq k} \tilde{\beta}(v) \quad \leqq c_{2} \inf _{\|\mu-v\| \leqq k} \prod_{\alpha \in d^{+}}(1+|\langle\alpha, v\rangle|)^{d(\alpha)} \tag{6.23}
\end{equation*}
$$

By choosing $\varepsilon(\kappa)>0$ small enough, we can find a constant $c_{3}=c_{3}(\kappa)>0$ with the property that for arbitrary $\mu \in \mathscr{F}_{I}$,

$$
\begin{equation*}
\operatorname{vol}\left(\left\{v: v \in \mathscr{F}_{I},\|\mu-v\| \leqq \kappa,|\langle\alpha, v\rangle| \geqq \varepsilon(\kappa) \forall \alpha \in \Delta^{++}\right\}\right) \geqq c_{3} . \tag{6.24}
\end{equation*}
$$

Now we apply (3.44a) and we obtain $c_{4}=c_{4}(\kappa)>0$ satisfying

$$
\begin{equation*}
\prod_{\alpha \in \Delta^{+}}(1+|\langle\alpha, v\rangle|)^{d(\alpha)} \leqq c_{4} \beta(v), \tag{6.25}
\end{equation*}
$$

if $v \in \mathscr{\mathscr { F }}_{I}$ and $|\langle\alpha, v\rangle| \geqq \varepsilon(\kappa)$ for all $\alpha \in \Delta^{++}$. Combining (6.23)-(6.25) we get

$$
\tilde{\beta}(\mu) \leqq c_{5} \int_{\| \mu-v \sharp \leqq \kappa} \beta(v) d v \quad\left(\mu \in \mathscr{F}_{I}\right),
$$

where $c_{5}=c_{5}(\kappa)=c_{2} c_{4} c_{3}^{-1}$ is independent of $\mu$. Therefore

$$
\begin{aligned}
& \int_{E_{\kappa}} \tilde{\beta}(\mu) d \mu \leqq c_{5} \int_{E_{\kappa}} \int_{\|\mu-v\| \leqq \kappa} \beta(v) d v d \mu \\
& \quad \leqq c_{6} \int_{E_{2 \kappa}} \beta(v) \int_{\|\mu-v\| \leqq k} d \mu d v=a_{2}(\kappa) \int_{E_{2 \kappa}} \beta(\mu) d \mu .
\end{aligned}
$$

## 7. Local Spectral Asymptotics around a Variable Point $\boldsymbol{\mu} \in \mathscr{F}_{I}$

7.1. Our aim in this section is to obtain precise estimates for sums of the form

$$
\begin{equation*}
\sum_{\lambda \in A .\|\lambda-\mu\| \leqq t} m(\lambda) \quad\left(\mu \in \mathscr{F}_{I}, t \geqq 1\right) . \tag{7.1}
\end{equation*}
$$

These estimates will of course depend on both $\mu$ and $t$. By analogy with Proposition 6.8 (ii) one would expect that the estimates will be in terms of $\tilde{\beta}(\mu)$ and powers of $t$. It turns out that high powers of $t$ do not matter; this is because $t$ is bounded whenever we apply these estimates. However, the factor $\tilde{\beta}(\mu)$ that enters our estimate for (7.1) is critical; indeed, our main effort is to show that the factor $\tilde{\beta}(\mu)$ furnished by Proposition 6.8 is in fact the one needed to majorize (7.1).

We begin with a way of partitioning the sum (7.1). Recall the definition of the $\mathbb{R}$-linear spaces $\mathscr{F}(s) \subset \mathscr{F}(s \in \mathfrak{w})$, given in (6.3):

$$
\begin{equation*}
\mathscr{\mathscr { F }}(s)=\left\{\xi: \xi \in \mathscr{F}, s \cdot \xi=-\xi^{\mathrm{conj}}\right\}, \tag{7.2}
\end{equation*}
$$

conj being the conjugation in $\mathscr{F}$ relative to $\mathscr{F}_{R}$. Since $\Lambda \subset \bigcup_{s \in w} \mathscr{F}(s)$, we can write

$$
\begin{equation*}
\left.\Lambda=\bigcup_{s \in \mathfrak{w}} A(s) \text { disjoint union }\right) ; A(s) \subset \Lambda \cap \mathscr{F}(s) . \tag{7.3}
\end{equation*}
$$

Note that for $s=1, \mathscr{\mathscr { F }}(1)=\mathscr{F}_{I}$; thus $\Lambda(1) \subset \mathscr{F}_{I}$. More generally

$$
\begin{equation*}
\mathscr{\mathscr { F }}(s) \cap \mathscr{F}_{I}=\overline{\mathscr{F}}_{I}^{s}=\left\{v: v \in \mathscr{F}_{I}, s \cdot v=v\right\} \quad(s \in \mathfrak{w}) . \tag{7.4}
\end{equation*}
$$

From (7.4) we see that if $s \neq 1$ and $\mu \in \mathscr{F}(s) \cap \mathscr{F _ { I }}, \mu$ is fixed by $s \neq 1$ so that, by a well-known theorem of Chevalley (cf. Varadarajan [41, Lemma 4.15.15]), there is $\alpha \in \Delta^{++}$with $\langle\alpha, \mu\rangle=0$ :

$$
\begin{equation*}
s \neq 1, \mu \in \mathscr{\mathscr { F }}(s) \cap \mathscr{F}_{I} \Rightarrow\langle\alpha, \mu\rangle=0 \quad \text { for some } \alpha \in \Delta^{++} . \tag{7.5}
\end{equation*}
$$

We write $l$ for the number of short positive roots:

$$
\begin{equation*}
l=\left|\Delta^{++}\right| \tag{7.6}
\end{equation*}
$$

Using the notation of (6.15), we define the (finite) classes $\mathscr{T}_{k}(0 \leqq k \leqq l)$ and $\mathscr{T}$ by

$$
\begin{equation*}
\mathscr{T}_{k}=\left\{T(\Phi): \Phi \subset \Delta^{++},|\Phi|=k\right\} ; \quad \mathscr{T}=\bigcup_{0 \leqq k \leqq l} \mathscr{T}_{k} \tag{7.7}
\end{equation*}
$$

For any $T \in \mathscr{T}$ we introduce a partition of $\mathfrak{w}$ given by

$$
\begin{equation*}
\mathfrak{w}=\mathfrak{w}_{T} \cup \mathfrak{w}_{T}^{\prime} ; \quad \mathfrak{w}_{T} \cap \mathfrak{w}_{T}^{\prime}=\emptyset ; \quad \mathfrak{w}_{T}=\{s: s \in \mathfrak{w}, s \cdot \mu=\mu \forall \mu \in T\} \tag{7.8}
\end{equation*}
$$

We now consider the sum (7.1), with $\mu \in \mathscr{F}_{I}$ and $t \geqq 1$. We have, for any $T \in \mathscr{T}$ and any $\mu \in T$,

$$
\begin{align*}
& \sum_{\lambda \in A,\|\lambda-\mu\| \leqq t} m(\lambda) \\
& =\sum_{s \in \mathbf{w}_{T}} \sum_{\lambda \in A(s),\|\lambda-\mu\| \leqq t} m(\lambda)+\sum_{s \in w_{T}^{\prime}} \sum_{\lambda \in A(s),} m(\lambda-\mu \| \leqq t \tag{7.9}
\end{align*}
$$

If $s \in \mathfrak{w}_{T}$ and $\lambda \in A(s)$, both $\lambda$ and $\mu$ lie in $\mathscr{F}(s)$; hence $\lambda-\mu \in \mathscr{F}(s)$. If we now recall (Lemma 6.3) that the Fourier transforms $\hat{h}(t: \xi)$ are $\geqq 0$ for $\xi \in \mathscr{F}(s)$ and $\geqq 1$ in absolute value for $\|\xi\| \leqq t$, we get the estimate

$$
\begin{aligned}
& \sum_{s \in w_{T}} \sum_{\lambda \in A(s),} m(\lambda) \leqq \sum_{s \in \mathfrak{w}_{T}} \sum_{\lambda \in A(s)} m(\lambda) \hat{h}(t: \lambda-\mu) \\
& \quad=\sum_{\lambda \in A} m(\lambda) \hat{h}(t: \lambda-\mu)-\sum_{s \in w_{T}} \sum_{\lambda \in A(s)} m(\lambda) \hat{h}(t: \lambda-\mu) \\
& \leqq\left|\sum_{\lambda \in A} m(\lambda) \hat{h}(t: \lambda-\mu)\right|+\sum_{s \in w_{T}} \sum_{\lambda \in \Lambda(s)} m(\lambda)|\hat{h}(t: \lambda-\mu)| .
\end{aligned}
$$

Combining this with (7.9) we get the following proposition.
Proposition 7.1. Let $0 \leqq k \leqq l$, and $T \in \mathscr{T}_{k}$. Then, for any $\mu \in T$ and any $t \geqq 1$, we have

$$
\begin{align*}
& \quad \sum_{\lambda \in A,\|\lambda-\xi\|} m(\lambda) \leqq\left|\sum_{\lambda \in A} m(\lambda) \hat{h}(t: \lambda-\mu)\right| \\
& \quad+\sum_{s \in w_{T}^{\prime}} \sum_{\lambda \in A(s)} m(\lambda)|\hat{h}(t: \lambda-\mu)|+\sum_{s \in w_{T}^{\prime}} \sum_{\lambda \in \lambda(s),\|\lambda-\mu\| \leqq t} m(\lambda) . \tag{7.10}
\end{align*}
$$

We next prove the crucial fact that the sums involving $\mathfrak{w}_{T}^{\prime}$ can be estimated in terms of $\sum_{\lambda \in A,\left\|\lambda-\mu_{1}\right\| \leqq t_{1}} m(\lambda)$, where $\mu_{1}$ varies over $T_{1} \in \mathscr{T}_{k_{1}}$ with $k_{1}$ strictly greater than $k$, and $t_{1}=c t$ for a suitable constant $c \geqq 1$. This allows downward induction on $k$ and will lead to the required estimates for (7.1).

Proposition 7.2. For $T \in \mathscr{T}_{k}(0 \leqq k \leqq l)$ and $s \in \mathfrak{w}_{T}^{\prime}$, let $T_{1}=T \cap \mathscr{F}_{I}^{s}=\{v: v \in T, s \cdot v=v\}$. Then we have the following. (i) $T_{1} \neq T$ and there is $k_{1}$ with $k<k_{1} \leqq l$ such that $T_{1} \in \mathscr{T}_{k_{1}}$. (ii) We can find a constant $c=c(k) \geqq 1$ with the following property: if $\mu \in T$ is arbitrary and $\mu_{1} \in T_{1}$ is the orthogonal projection of $\mu$ on $T_{1}$, then, for any $\mu \in \mathscr{F}(s)$, one has

$$
\begin{equation*}
\left\|\lambda-\mu_{1}\right\| \leqq c\|\lambda-\mu\| ; \quad\left\|\mu-\mu_{1}\right\| \leqq c\|\lambda-\mu\| \tag{7.11}
\end{equation*}
$$

Note first of all that $k<l$. For, if $k=l$, then $T=(0)$ and $\mathfrak{w}_{T}^{\prime}=\emptyset$. Next, $\mathscr{F}_{I}^{s} \neq T$, so that $T_{1} \neq T$. To prove that $T_{1} \in \mathscr{T}_{k_{1}}$ for some $k_{1}>k$ we must produce an $\alpha \in \Delta^{++}$such that $\alpha$ vanishes on $T_{1}$ but not on $T$, i.e., such that the reflexion $s_{\alpha}$ fixes all elements of $T_{1}$ but does not fix some elements of $T$. Now $s$ fixes each element of $T_{1}$. So, by a well-known theorem on finite reflexion groups (cf. Varadarajan [41, Lemma 4.15.15]) we can write $s=s_{\alpha_{1}} s_{\alpha_{2}} \ldots s_{\alpha_{p}}$, where $\alpha_{1}, \ldots, \alpha_{p}$ are in $\Delta^{++}$and each of the $s_{\alpha_{1}}$ fixes all elements of $T_{1}$. If each of the $s_{\alpha_{1}}$ fixes all the elements of $T$, then $s$ would fix all of $T$, which is contradictory to the assumption that $s \in \mathfrak{w}_{T}^{\prime}$. So, for some $i, s_{\alpha_{i}}$ does not fix all the elements of $T$. We are done. The proof of the second assertion is by noting that the linear map $\mathscr{F}(s)$ $\times T \rightarrow T$ given by $(\xi, v) \mapsto \xi+v$, is injective on the image of the linear map $\mathscr{F}(s)$ $\times T \rightarrow \mathscr{F}(s) \times T$ which sends $(\lambda, \mu)$ to $\left(\lambda-\mu_{1}, \mu_{1}-\mu\right)$. Indeed, if $\lambda-\mu_{1}+\mu_{1}-\mu=0$, then $\lambda=\mu \in T_{1}$; so $\mu=\mu_{1}$ and thus $\lambda-\mu_{1}=\mu_{1}-\mu=0$. But this implies at once the existence of a constant $c_{1}=c_{1}(T, s)$ such that

$$
0<c_{1} \leqq 1,\|\lambda-\mu\| \geqq c_{1}\left(\left\|\lambda-\mu_{1}\right\|+\left\|\mu_{1}-\mu\right\|\right)(\lambda \in \mathscr{F}(s) . \mu \in T),
$$

and the assertion follows by the finiteness of $\mathscr{T}_{k}$ and $\mathfrak{w}_{T}^{\prime}$.
We can now prove the main result of this section. We remark that the applications of Theorem 7.3 do not require the explicit formula (7.13) below for $n(k)$; we give this form only for having clean proofs.

Theorem 7.3. Let $l$ be as in (7.6). Then there is a constant $c>0$ with the following property. For any integer $k$ with $0 \leqq k \leqq l$, any $T \in \mathscr{T}_{k}$, and arbitrary $\mu \in T, t \geqq 1$, we have

$$
\begin{equation*}
\sum_{\lambda \in A,\|\lambda-\mu\| \leqq t} m(\lambda) \leqq c t^{n(k)} \tilde{\beta}(\mu), \tag{7.12}
\end{equation*}
$$

where $\tilde{\beta} \in C^{\infty}\left(\mathscr{F}_{I}\right)^{\text {w }}$ is defined in (6.11) and

$$
\begin{equation*}
n(k)=2(n+1)(l+1-k) \tag{7.13}
\end{equation*}
$$

We shall prove Theorem 7.3 by downward induction on $k$. We start with $k$ $=l . \mathscr{T}_{l}$ consists only of $(0)$. So $\mu=0$, and (7.12) becomes $\sum_{\lambda \in A,\|\lambda\| \leqq t} m(\lambda) \leqq c t^{2(n+1)}$, which is true, by Proposition 6.4. So consider $0 \leqq k<l$. We assume (7.12) for $k_{1}>k$. The first step is to estimate the left side of (7.12) by (7.10). Let $\Sigma_{1}, \Sigma_{2}$ and $\Sigma_{3}$ be the three terms appearing on the right side of (7.10). Then

$$
\begin{equation*}
\sum_{\lambda \in \boldsymbol{A},\|\lambda-\mu\| \leqq t} m(\lambda) \leqq \Sigma_{1}+\Sigma_{2}+\Sigma_{3} . \tag{7.14}
\end{equation*}
$$

where $\mu \in T, T \in \mathscr{T}_{k}$, and $t \geqq 1$. We shall now obtain majorants for each of $\Sigma_{i}, i$ $=1,2,3 . \Sigma_{1}$ is estimated by (6.13) of Proposition 6.8. We have

$$
\begin{equation*}
\Sigma_{1}=\left|\sum_{\lambda \in A} m(\lambda) \tilde{h}(t: \lambda-\mu)\right| \leqq c(h) t^{n+1} \tilde{\beta}(\mu) \tag{7.15}
\end{equation*}
$$

for all $t \geqq 1, \mu \in T \in \mathscr{T}_{k}, \Sigma_{2}$ and $\Sigma_{3}$ are estimated with the help of Proposition 7.2 and the induction hypothesis. First we take up $\Sigma_{3}$. Let notation be as in Pro-
position 7.2. If $s \in \mathfrak{w}_{T}^{\prime}, \lambda \in A(s)$ and $\|\lambda-\mu\| \leqq t$, then, as $\lambda \in \mathscr{F}(s)$, we have for $c$ $=c(k) \geqq 1$ as in (7.11),

$$
\left\|\lambda-\mu_{1}\right\| \leqq c\|\lambda-\mu\| \leqq c t
$$

So, we obtain

$$
\Sigma_{3}=\sum_{s \in \mathfrak{w}_{T}^{\prime}} \sum_{\lambda \in A(s),\|\lambda-\mu\| \leqq t} m(\lambda) \leqq \sum_{\lambda \in A,\left\|\lambda-\mu_{1}\right\| \leqq c t} m(\lambda) .
$$

Since $\mu_{1} \in T_{1} \in \mathscr{T}_{k_{1}}$ where $k_{1}>k$, the induction hypothesis applies, with a constant $c_{1}(k)$. Further by (6.19) and (7.11) we have

$$
\begin{equation*}
\tilde{\beta}\left(\mu_{1}\right) \leqq\left(1+\left\|\mu-\mu_{1}\right\|\right)^{n+1} \tilde{\beta}(\mu) \leqq(1+c t)^{n+1} \tilde{\beta}(\mu) \tag{7.16}
\end{equation*}
$$

while $n\left(k_{1}\right)+n+1 \leqq n(k)$. So we find a constant $c_{2}(k)>0$ such that

$$
\begin{equation*}
\Sigma_{3} \leqq c_{1}(k)(c t)^{n\left(k_{1}\right)} \tilde{\beta}\left(\mu_{1}\right) \leqq c_{2}(k) t^{n(k)} \tilde{\beta}(\mu) \tag{7.17}
\end{equation*}
$$

for all $\mu \in T, t \geqq 1$. It remains to consider $\Sigma_{2}$. We have

$$
\begin{aligned}
\Sigma_{2} & =\sum_{s \in \mathfrak{w}_{T}^{\prime}} \sum_{\lambda \in A(s)} m(\lambda)|\widehat{h}(t: \lambda-\mu)| \\
& =\sum_{s \in \mathfrak{w}_{T}^{\prime}} \sum_{j=0}^{\infty} \sum_{\lambda \in A(s), j \leqq \lambda-\mu \|<j+1} m(\lambda)|\widehat{h}(t: \lambda-\mu)|
\end{aligned}
$$

We use (6.10) to estimate $|\hat{h}(t: \lambda-\mu)|$. Select constants $b_{p}=b_{p}(h)>0(p=0,1, \ldots)$ such that, for $\lambda \in A, \mu^{\prime} \in \mathscr{F}_{1}, t \geqq 1$, we have $\left|\tilde{h}\left(t: \lambda-\mu^{\prime}\right)\right| \leqq b_{p} t^{p}\left(1+\left\|\lambda-\mu^{\prime}\right\|\right)^{-p}$.
Then

$$
\Sigma_{2} \leqq b_{p} t^{p} \sum_{s \in \boldsymbol{w}_{T}^{\prime}} \sum_{j=0}^{\infty} \sum_{\lambda \in A(s),\|\lambda-\mu\| \leqq j+1} m(\lambda)(j+1)^{-p}
$$

But, by Proposition 7.2, for a given $j \geqq 0$, using notation from the discussion of $\Sigma_{3}$,

$$
\begin{aligned}
& \sum_{s \in w_{T}} \sum_{\lambda \in \Lambda(s),\|\lambda-\mu\| \leqq j+1} m(\lambda) \leqq \sum_{\lambda \in \Lambda,\left\|\lambda-\mu_{1}\right\| \leqq c(j+1)} m(\lambda) \\
& \quad \leqq c_{3}(k)(j+1)^{n\left(k_{1}\right)} \tilde{\beta}\left(\mu_{1}\right),
\end{aligned}
$$

by the induction hypothesis, $c_{3}(k)>0$ being independent of $j$. Hence, using (7.16),

$$
\Sigma_{2} \leqq b_{p} t^{p} c_{3}(k)(1+c t)^{n+1} \tilde{\beta}(\mu) \sum_{j=0}^{\infty}(j+1)^{-p+n\left(k_{1}\right)}
$$

Take now $p=n(k+1)+2$ (cf. (7.13)), then $p+n+1 \leqq n(k)$. So we get a constant $c_{4}(k)>0$ such that

$$
\begin{equation*}
\Sigma_{2} \leqq c_{4}(k) t^{n(k)} \tilde{\beta}(\mu) \quad(\mu \in T, t \geqq 1) \tag{7.18}
\end{equation*}
$$

Combining (7.14), (7.15), (7.17) and (7.18) we get (7.12).

## 8. Estimates for the Complementary and Principal Spectra

8.1. We are now in a position to estimate both the complementary and principal spectra. It turns out, however, that the principal spectrum $\Lambda_{p}=\Lambda \cap \mathscr{F}_{I}$ cannot be treated without first proving that the complementary spectrum $\Lambda_{c}=\Lambda \backslash A_{p}$ is of lower order of magnitude. So we begin this section with the study of $\Lambda_{c}$. In Theorem 8.3 we prove the estimate $\sum_{\lambda \in \Lambda_{c},\|\lambda\| \leqq t} m(\lambda)=O\left(t^{n-d-1}\right), t \rightarrow+\infty$, which is the best possible unless we impose further conditions on $G$ and $\Gamma$ (see Example 8.4). Here, with $d(\alpha)$ as in (3.41),

$$
\begin{equation*}
d=\min _{\alpha \in \Delta^{+}} d(\alpha) . \tag{8.1}
\end{equation*}
$$

We have $d \geqq 1$, and according to (3.43) $d=1$ if and only if there is $x \in \Delta^{++}$such that $\mathfrak{g}^{\alpha} \cong \mathfrak{s l}(2, \mathbb{R})$; this is the case, for instance, if $G \cong S L(n, \mathbb{R})(n \geqq 2)$. On the other hand $d \geqq 2$, if $G$ has a single conjugacy class of CSG's (cf. Remark 4.5 (ii)).

From Proposition 3.4(iii) and the formulae (7.3) and (7.5) we obtain:
Lemma 8.1. Suppose $\lambda \in \Lambda_{c}$, then
(i) $\left\|\lambda_{R}\right\| \leqq\|\rho\|$; (ii) $\exists \alpha \in \Delta^{++}$such that $\left\langle\alpha, \lambda_{I}\right\rangle=0$.

Let $\mathscr{T}^{\times}$denote the set of all linear subspaces of $\mathscr{F}_{I}$ of the form $T(\Phi)$ (see (6.15)) for nonempty $\Phi \subset \Delta^{++}$. For any $T=T(\Phi) \in \mathscr{T}^{\times}$we now define (cf. (6.14))

$$
\begin{align*}
& \Lambda_{c}(T)=\left\{\lambda: \lambda \in \Lambda_{c}, \lambda_{I} \in T\right\}  \tag{8.2}\\
& n(T)=n(\Phi)=d(\Phi)+\operatorname{codim}_{\mathscr{F}_{I}}(T) . \tag{8.3}
\end{align*}
$$

Proposition 8.2. There is a constant $c>0$ such that for all $T=T(\Phi) \in \mathscr{T}^{\times}$and all $t \geqq 1$

$$
\begin{equation*}
\sum_{\lambda \in A_{c}(T),\|\lambda\| \leqq t} m(\lambda) \leqq c t^{n-n(T)} . \tag{8.4}
\end{equation*}
$$

We begin the proof by fixing $T \in \mathscr{T}^{\times}$. Let us choose a covering of the ball $\{v: v \in T,\|v\| \leqq t\}$ by closed balls of unit radius with centers in the original ball of radius $t$. By elementary geometry we can arrange matters so that the number of balls of unit radius needed is $\leqq c_{0} t^{\mathrm{dim}(T)}, c_{0}$ being a constant independent of $t$. So

$$
\{v: v \in T,\|v\| \leqq t\} \subset \bigcup_{1 \leqq j \leqq M(t)}\left\{v: v \in T,\left\|v-v_{j}\right\| \leqq 1\right\}
$$

with

$$
\begin{equation*}
v_{j} \in T,\left\|v_{j}\right\| \leqq t(1 \leqq j \leqq M(t)) ; \quad M(t) \leqq c_{0} t^{\operatorname{dim}(T)}(t \geqq 1) . \tag{8.5}
\end{equation*}
$$

Then

$$
\begin{aligned}
& \sum_{\lambda \in A_{c}(T),\|\lambda\| \leqq t} m(\lambda) \leqq \sum_{\lambda \in A_{c}, \lambda_{I} \in T,\left\|\lambda_{I}\right\| \leqq t} m(\lambda) \\
& \quad \leqq \sum_{1 \leqq j \leqq M(t) \lambda \in \Lambda_{c}, \lambda_{I} \in T,\left\|\lambda_{I}-v_{J}\right\| \leqq 1} m(\lambda) .
\end{aligned}
$$

But, for $\lambda \in \Lambda,\left\|\lambda_{R}\right\| \leqq\|\rho\|$ (Lemma 8.1), so that $\left\|\lambda_{I}-v_{j}\right\| \leqq 1$ implies $\left\|\lambda-v_{j}\right\| \leqq 1$ $+\|\rho\|$. Hence

$$
\begin{equation*}
\sum_{\lambda \in A_{c}(T),\|\lambda\| \leqq t} m(\lambda) \leqq \sum_{1 \leqq j \leqq M(t)} \sum_{\lambda \in A,\left\|\lambda-v_{J}\right\| \leqq 1+\|\rho\|} m(\lambda) \tag{8.6}
\end{equation*}
$$

Use now Theorem 7.3 and (6.17) to estimate the right side of (8.6). If $T=T(\Phi)$ where $\Phi \subset \Delta^{++}$has $k \geqq 1$ elements, there is a constant $c_{1}>0$ such that, for $\nu \in T, u \geqq 1$,

$$
\sum_{\lambda \in A,\|\lambda-v\| \leqq u} m(\lambda) \leqq c_{1} u^{n(k)} \quad\left(1+\|v\| \|^{n-r-d(\Phi)} .\right.
$$

So

$$
\sum_{\lambda \in A,\left\|\lambda-v_{j}\right\| \leqq 1+\|\rho\|} m(\lambda) \leqq c_{2}\left(1+\left\|v_{j}\right\|\right)^{n-r-d(\Phi)} .
$$

Summing over $j=1, \ldots M(t)$, and using the estimate (8.5) for $M(t)$, we get from (8.6) the estimate

$$
\sum_{\lambda \in A_{c}(T),\|\lambda\| \leqq t} m(\lambda) \leqq c_{0} c_{2} t^{n-r-d(\Phi)+\operatorname{dim} T},
$$

which is (8.4).
Theorem 8.3. There is a constant $c>0$ such that, for all $t \geqq 1$,

$$
\begin{equation*}
\sum_{\lambda \in A_{c},\|2\| \leqq t} m(\lambda) \leqq c t^{n-d-1} . \tag{8.7}
\end{equation*}
$$

In fact, by Lemma 8.1, $A_{c} \subset \cup\left\{\Lambda_{c}(T): T \in \mathscr{T}^{\times}\right\}$. Hence

$$
\sum_{\lambda \in A_{c},\|\lambda\| \leqq t} m(\lambda) \leqq \sum_{T \in \mathscr{F} \times} \times \sum_{\lambda \in A_{c}(T),\|\lambda\| \leqq t} m(\lambda) .
$$

For $T=T(\Phi) \in \mathscr{T}^{\times}, n(T)=d(\Phi)+\operatorname{codim}(T) \geqq d+1$. So, by (8.4), for $T \in \mathscr{T}^{\times}$

$$
\sum_{\lambda \in A_{c}(T),\|\lambda\| \leqq t} m(\lambda)=O\left(t^{n-1-d}\right) .
$$

Example 8.4. Let $X_{j}=\Gamma_{j} \backslash G_{j} / K_{j}(1 \leqq j \leqq q)$ be compact quotients with corresponding spectra $\Lambda_{j} \subset \mathscr{F}_{j}$ and multiplicities $m_{j}\left(\lambda_{j}\right)$ for $\lambda_{j} \in \Lambda_{j}$. Then the product $X$ $=\prod_{1 \leq j \leq q} X_{j}$ satisfies condition (2.9) again, while $G$, resp. $K, \Gamma, \mathscr{F}$ is equal to the product of the $G_{j}$, resp. $K_{j}, \Gamma_{j}, \mathscr{F}_{j}$. Moreover the spectrum $A$ of $X$ is equal to the product of the $\Lambda_{j}$, with multiplicities $m(\lambda)=\Pi m_{j}\left(\lambda_{j}\right)(\lambda \in \Lambda)$. It is then clear that $\Lambda_{p}=\Pi\left(\Lambda_{j}\right)_{p}$, which implies

$$
\begin{equation*}
\Lambda_{\mathrm{c}}=\bigcup_{1 \leqq j \leqq q} \Lambda_{1} \times \ldots \times \Lambda_{j-1} \times\left(\Lambda_{j}\right)_{c} \times \Lambda_{j+1} \times \ldots \times \Lambda_{q} . \tag{8.8}
\end{equation*}
$$

According to Proposition 3.4 (i) $\left(\Lambda_{j}\right)_{\mathrm{c}} \not \ddagger \emptyset$, and in Theorem 8.8 we shall prove $\sum_{\lambda_{j} \in \Lambda_{1},\left\|\lambda_{j},\right\| \leqq t} m_{j}\left(\lambda_{j}\right) \sim c_{j} t^{n_{j}}, t \rightarrow+\infty$, where $n_{j}=\operatorname{dim} X_{j}$. This implies that there is a constant $c>0$ such that, with $n_{0}=\min _{1 \leqq j \leqq q} n_{j}$,

$$
\begin{equation*}
\sum_{\lambda_{\in}=A_{c},\|\lambda\| \leqq t} m(\lambda) \geqq c t^{n-n_{0}}(t \geqq 1) . \tag{8.9}
\end{equation*}
$$

So $A_{c}$ is infinite if $q>1$. In particular, let rank $X_{j}=1(1 \leqq j \leqq q)$; then according to Theorem 8.3, $\left(\Lambda_{j}\right)_{c}$ is finite (sic), and it is easily verified that, given $T \in \mathscr{T}^{\times}$,

$$
\sum_{\lambda \in \mathcal{A}_{c}(T,\|\lambda\| \| t} m(\lambda) \sim c t^{n-n(T)}, \quad t \rightarrow+\infty .
$$

So, in this case the estimates of Proposition 8.2 are sharp.
8.2. We now deal with the principal spectrum $\Lambda_{p}=\Lambda \cap \mathscr{F}_{I}$. We consider subsets of $\mathscr{\mathscr { F }}_{I}$. For any $\Omega \subset \mathscr{F}_{I}$ we denote by $\partial \Omega$ its boundary and by $\Omega^{\prime}$ its complement:

$$
\begin{equation*}
\partial \Omega=\operatorname{Closure}(\Omega) \backslash \operatorname{Interior}(\Omega) ; \quad \Omega^{\prime}=\mathscr{F _ { I }} \backslash \Omega \tag{8.1}
\end{equation*}
$$

If $\delta$ denotes the distance function in $\mathscr{\mathscr { F }}_{I}$, we define, for any $\kappa>0$,

$$
\begin{equation*}
\partial \Omega_{\kappa}=\left\{v: v \in \mathscr{F}_{I}, \delta(v, \partial \Omega) \leqq \kappa\right\} \tag{8.11}
\end{equation*}
$$

Let $d_{0} v$ be the measure on $\mathscr{F}_{1}$ coming from the Killing form.
Theorem 8.5. Given $\kappa>0$, we can find a constant $c=c(\kappa)>0$ such that for all bounded Lebesgue measurable subsets $\Omega \subset \mathscr{F}_{I}$ we have

$$
\sum_{\lambda \in \Lambda, \lambda_{1} \in \Omega} m(\lambda)-\sigma(G) \operatorname{vol}_{0}(X)|\mathfrak{w}|^{-1} \int_{\Omega} \beta(v) d_{0} v \mid \leqq c \int_{\partial \Omega_{\kappa}} \beta(v) d_{0} v,
$$

where vol $(X)$ is as in (3.31), while (cf. (3.40) (3.42))

$$
\begin{equation*}
\sigma(G)=2^{-r+} \sum_{x, A^{\prime}}^{n(2 x)} \pi^{-\frac{1}{2}(n+n)} I(\rho)^{-1} . \tag{8.12}
\end{equation*}
$$

The starting point for the proof is the relation (i) in Proposition 6.8:

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) h(\lambda-\mu)=\operatorname{vol}(X)|\mathfrak{w}|^{-1} \int_{\mathscr{F}_{I}} h(v-\mu) \beta(v) d v . \tag{8.13}
\end{equation*}
$$

We integrate both sides of (8.13) over $\Omega$, and in view of Proposition 6.6 we are allowed to interchange integration and summation; hence

$$
\begin{equation*}
\sum_{\lambda \in A} m(\lambda) \int_{\Omega} \hat{h}(\lambda-\mu) d \mu=\operatorname{vol}(X)|\mathfrak{w}|^{-1} \int_{\Omega} \int_{\mathscr{F}_{I}} \hat{h}(v-\mu) \beta(v) d v d \mu . \tag{8.14}
\end{equation*}
$$

It is now a question of relating $\sum_{\lambda \in 1, \lambda_{1} \in \Omega} m(\lambda)$ to the left side of (8.14). To this end we rewrite the left of (8.14); we have

$$
\begin{aligned}
& \sum_{\lambda \in A} m(\lambda) \int_{\Omega} \hat{h}(\lambda-\mu) d \mu=\sum_{\lambda \in \Lambda, \lambda_{I} \in \Omega} m(\lambda) \int_{\mathscr{F}_{I}} \hat{h}(\lambda-\mu) d \mu \\
&-\sum_{\lambda \in A, \lambda_{I} \in \Omega} m(\lambda) \int_{\Omega^{\prime}} \hat{h}(\lambda-\mu) d \mu+\sum_{\lambda \in \Lambda, \lambda_{I} \in \Omega^{\prime}} m(\lambda) \int_{\Omega} \hat{h}(\lambda-\mu) d \mu .
\end{aligned}
$$

Since $\mu \mapsto \hat{h}(\xi-\mu)$ is the Fourier transform of the function $H \mapsto e^{-\xi(H)} h(-H)$ and since $d \mu$ is the measure dual to $d H$, we have $\int_{\mathscr{\mathscr { F } _ { I }}} \hat{h}(\xi-\mu) d \mu=h(0)(\xi \in \mathscr{F})$. So

$$
\begin{align*}
& \left|h(0) \sum_{\lambda \in A, \lambda_{I} \in \Omega} m(\lambda)-\sum_{\lambda \in \Lambda} m(\lambda) \int_{\Omega} \hat{h}(\lambda-\mu) d \mu\right| \leqq J_{1}(\Omega) \\
& \quad=\sum_{\lambda \in \Lambda, \lambda_{I} \in \Omega} m(\lambda) \int_{\Omega^{\prime}}|\hat{h}(\lambda-\mu)| d \mu+\sum_{\lambda \in \Lambda, \lambda_{I} \in \Omega^{\prime}} m(\lambda) \int_{\Omega}|\hat{h}(\lambda-\mu)| d \mu . \tag{8.15}
\end{align*}
$$

On the other hand,

$$
\begin{aligned}
& \int_{\Omega} \int_{\mathscr{F} \mathscr{F}_{I}} \hat{h}(v-\mu) \beta(v) d v d \mu \\
& =\int_{\Omega} \int_{\Omega} \hat{h}(v-\mu) \beta(v) d \mu d v+\int_{\Omega^{\prime}} \int_{\Omega} \hat{h}(v-\mu) \beta(v) d \mu d v \\
& =\int_{\Omega} \int_{\mathscr{F}_{I}} \hat{h}(v-\mu) \beta(v) d \mu d v-\int_{\Omega} \int_{\Omega^{\prime}} \hat{h}(v-\mu) \beta(v) d \mu d v \\
& \quad+\int_{\Omega^{\prime}} \int_{\Omega} \hat{h}(v-\mu) \beta(v) d \mu d v \\
& =h(0) \int_{\Omega} \beta(v) d v-\int_{\Omega} \int_{\Omega^{\prime}} \hat{h}(v-\mu) \beta(v) d \mu d v \\
& \quad+\int_{\Omega^{\prime}} \int_{\Omega} \hat{h}(v-\mu) \beta(v) d \mu d v .
\end{aligned}
$$

So this implies that

$$
\begin{align*}
& \left|\int_{\Omega} \int_{\mathscr{F}_{I}} \hat{h}(v-\mu) \beta(v) d v d \mu-h(0) \int_{\Omega^{\prime}} \beta(v) d v\right| \leqq J_{2}(\Omega)+J_{3}(\Omega) \\
& \quad=\int_{\Omega} \int_{\Omega^{\prime}}|\hat{h}(v-\mu)| \beta(v) d \mu d v+\int_{\Omega^{\prime}} \int_{\Omega}|\hat{h}(v-\mu)| \beta(v) d \mu d v . \tag{8.16}
\end{align*}
$$

Now $d v=(2 \pi)^{-r} d_{0} v$, while $h(0) \neq 0$. So, from (8.14)-(8.16) we get

$$
\begin{aligned}
& \left.\left|\sum_{\lambda \in \Lambda, \lambda_{I} \in \Omega} m(\lambda)-(2 \pi)^{-r} \operatorname{vol}(X)\right| \mathfrak{w}\right|^{-1} \int_{\Omega} \beta(v) d_{0} v \mid \\
& \quad \leqq h(0)^{-1}\left(J_{1}(\Omega)+J_{2}(\Omega)+J_{3}(\Omega)\right) .
\end{aligned}
$$

To complete the proof of Theorem 8.5 it is therefore sufficient to establish the following: (i) $\operatorname{vol}(X)=(2 \pi)^{r} \sigma(G) \operatorname{vol}_{0}(X)$, which is immediate from Corollary 3.13; and (ii)

$$
\sum_{i=1}^{3} J_{i}(\Omega) \leqq c \int_{\partial \Omega_{k}} \beta(v) d_{0} v
$$

where $c>0$ is a constant independent of $\Omega$. In view of Proposition 6.10 it is sufficient to obtain (ii) with $\beta$ replaced by $\tilde{\beta}$; and we shall establish this in the following technical lemma.

## Lemma 8.6. Using the above notation,

$$
J_{i}(\Omega) \leqq c(\kappa) \int_{\partial \Omega_{\kappa}} \tilde{\beta}(v) d_{0} v \quad(1 \leqq i \leqq 3)
$$

We begin with some auxiliary estimates. For any integer $p>r+1$ we have, using (6.6) constants $c=c(p)>0$ such that $|\widehat{h}(\xi-\mu)| \leqq c(1+\|\xi-\mu\|)^{-\boldsymbol{p - r}}\left(\xi \in \mathscr{F},\left\|\xi_{R}\right\|\right.$ $\leqq\|\rho\|, \mu \in \mathscr{F}_{I}$ ). As a consequence we have, for any $\Omega$ and $\zeta \in \mathscr{F}$ with $\left\|\xi_{R}\right\| \leqq\|\rho\|$ and $\xi_{I} \in \Omega^{\prime}$,

$$
\begin{aligned}
& \int_{\Omega}|h(\xi-\mu)| d \mu \leqq \int_{\left\|\xi_{I}-\mu\right\| \geqq \delta\left(\xi_{I}, \partial \Omega\right)}\left(1+\left\|\xi_{I}-\mu\right\|\right)^{-p-r} d \mu \\
& \quad \leqq c \int_{\|\nu\| \geqq \delta\left(\xi_{I}, \delta \Omega\right)}(1+\|v\|)^{-p-r} d v \leqq c^{\prime}\left(1+\delta\left(\xi_{I}, \partial \Omega\right)\right)^{-p}
\end{aligned}
$$

If we now define the functions $\varphi(p: \Omega: \cdot)=\varphi\left(p:^{\cdot}\right)$ by

$$
\begin{equation*}
\varphi(p: \xi)=\left(1+\delta\left(\xi_{I}, \partial \Omega\right)\right)^{-p}(\xi \in \mathscr{F}), \tag{8.17}
\end{equation*}
$$

we find constants $c_{1}=c_{1}(p)>0$ such that for any $\Omega$,

$$
\begin{align*}
& \int_{\Omega}|\hat{h}(\xi-\mu)| d \mu \leqq c_{1} \varphi(p: \xi) \quad\left(\xi \in \mathscr{F},\left\|\xi_{R}\right\| \leqq\|\rho\|, \xi_{I} \in \Omega^{\prime}\right),  \tag{8.18a}\\
& \int_{\Omega^{\prime}}|\widehat{h}(\xi-\mu)| d \mu \leqq c_{1} \varphi(p: \xi) \quad\left(\xi \in \mathscr{F},\left\|\xi_{R}\right\| \leqq\|\rho\|, \xi_{I} \in \Omega\right) . \tag{8.18b}
\end{align*}
$$

Further we claim that for a suitable constant $c_{2}=c_{2}(\kappa: p)>0$, we have, for all $\nu \in \mathscr{F}_{I}$ and $\Omega$,

$$
\begin{equation*}
\varphi(p: \nu) \leqq c_{2} \int_{\partial \Omega_{\kappa}}(1+\|v-\mu\|)^{-p} d \mu \tag{8.19}
\end{equation*}
$$

Indeed, given any $v \in \mathscr{F}_{I}$, we select $v^{\prime} \in \partial \Omega$ such that $\delta(v, \partial \Omega)=\left\|v-v^{\prime}\right\|$. Then (8.19) follows on considering the ball in $\mathscr{F}_{I}$ around $v^{\prime}$ with radius $\kappa$.

We are now in a position to estimate the $J_{i}(\Omega)$. In the case of $J_{2}$ we choose $p$ $=n+1$ and we have, for all $\Omega$, using (8.18b) and (8.19),

$$
\begin{aligned}
J_{2}(\Omega) & =\int_{\Omega} \int_{\Omega^{\prime}}|\hat{h}(v-\mu)| d \mu \beta(v) d v \leqq c_{1} \int_{\Omega} \varphi(n+1: v) \beta(v) d v \\
& \leqq c_{1} c_{2} \int_{\mathscr{F}_{I}} \int_{\partial \Omega_{\kappa}}(1+\|v-\mu\|)^{-n-1} d \mu \beta(v) d v \\
& =c_{1} c_{2} \int_{\partial \Omega_{\kappa}} \int_{\mathscr{F}_{I}}(1+\|v-\mu\|)^{-n-1} \beta(v) d v d \mu .
\end{aligned}
$$

We argue similarly to estimate $J_{3}(\Omega)$, using (8.18a) and (8.19). We find thus a constant $c_{3}=c_{3}(\kappa, p)>0$ such that, for all $\Omega$,

$$
\begin{equation*}
J_{i}(\Omega) \leqq c_{3} \int_{i \Omega_{\kappa}} \tilde{\beta}(\mu) d \mu \quad(i=2,3) \tag{8.20}
\end{equation*}
$$

It remains to estimate $J_{1}(\Omega)$. Using (8.18a) and (8.18b) we find (cf. (8.15)) that

$$
J_{1}(\Omega) \leqq c_{1} \sum_{\lambda \in A} m(\lambda) \varphi(p: \lambda) .
$$

We shall now estimate the sum on the right by dividing it into subsums over $\lambda \in A$ with $\lambda_{I} \in Q\left(\mu_{j}\right)$, where the $Q\left(\mu_{j}\right)(j=1,2, \ldots)$ are closed cubes in $\mathscr{F}_{I}$ with sides
of length 2 and centered at points $\mu_{j} \in \mathscr{F}$ whose coordinates are integral. According to Theorem 7.3 we have, for some constant $c>0$,

$$
\sum_{\lambda \in \Lambda, \lambda_{I} \in Q\left(\mu_{j}\right)} m(\lambda) \leqq c \tilde{\beta}\left(\mu_{j}\right) \quad(j=1,2, \ldots) .
$$

So there exists a constant $c_{4}=c_{4}(p)>0$ such that for $\Omega$

$$
\begin{equation*}
J_{1}(\Omega) \leqq c_{4} \sum_{j=1}^{\infty} \tilde{\beta}\left(\mu_{j}\right) \sup \left\{\varphi(p: \xi): \xi \in Q\left(\mu_{j}\right)\right\} \tag{8.21}
\end{equation*}
$$

If we now use (6.20) and the inequality $\varphi(p: \xi) \leqq d(p) \varphi\left(p: \xi^{\prime}\right)\left(\xi, \xi^{\prime} \in Q\left(\mu_{j}\right)\right.$ ), where $d(p)>0$ is independent of $j$, we find $c_{5}=c_{5}(p)>0$ such that, for $j=1,2, \ldots$

$$
\begin{equation*}
\tilde{\beta}\left(\mu_{j}\right) \sup \left\{\varphi(p: \xi): \xi \in Q\left(\mu_{j}\right)\right\} \leqq c_{5} \int_{Q\left(\mu_{j}\right)} \tilde{\beta}(v) \varphi(p: v) d v \tag{8.22}
\end{equation*}
$$

So using (8.21), (8.22) and (8.19) with $p=n+r+2$, we get $c_{6}=c_{6}(\kappa)>0$ such that, for all $\Omega$,

$$
\begin{align*}
\mathbf{J}_{1}(\Omega) & \leqq c_{6} \int_{\tilde{\mathscr{F}}_{1}} \tilde{\beta}(v) \int_{\partial \Omega_{\kappa}}(1+\|v-\mu\|)^{-p} d \mu d v \\
& \leqq c_{6} \int_{\partial \Omega_{\kappa}} \tilde{\beta}(\mu) \int_{\tilde{F}_{1}}(1+\|v-\mu\|)^{-p+n+1} d v d \mu \tag{8.23}
\end{align*}
$$

where we used (6.19). Hence, for a constant $c_{7}=c_{7}(\kappa)>0$, we have, for all $\Omega$,

$$
J_{1}(\Omega) \leqq c_{7} \int_{\partial \Omega_{\kappa}} \tilde{\beta}(\mu) d \mu
$$

This completes the proof of Lemma 8.6.
In order to see more clearly the fact that Theorem 8.5 gives that $\Lambda_{p}$ grows asymptotically like $\beta$, it might be illuminating to construct examples of families of sets $\Omega$ which go to infinity in such a way that $\int_{\partial \Omega_{\kappa}} \alpha(v) d_{0} v$ is of smaller order of magnitude than the main term $\int_{\Omega} \beta(v) d_{0} v$. We recall (cf. Lemma 3.11) the definition of $\varpi_{S}$ :

$$
\begin{equation*}
\varpi_{S}(v)=\prod_{\alpha \in A^{+}}\langle\alpha, v\rangle^{n(\alpha)} \quad\left(v \in \mathscr{F}_{I}\right) \tag{8.24}
\end{equation*}
$$

We are now interested in families $(\Omega(t))_{t \geqq 1}$ consisting of Lebesgue measurable sets contained in $\mathscr{F}_{I}$ which satisfy the following conditions:
(a) There are constants $c_{1}, c_{2}>0$ such that, for all $t>1$,

$$
\mu \in \Omega(t) \Rightarrow\|\mu\| \leqq c_{1} t ; \operatorname{vol}_{0}(\Omega(t)) \geqq c_{2} t^{r}
$$

(b) There is a constant $\kappa>0$ such that

$$
\begin{equation*}
\operatorname{vol}_{0}\left(\partial \Omega(t)_{k}\right)=O\left(t^{r-1}\right), t \rightarrow+\infty \tag{8.25}
\end{equation*}
$$

It is obvious that for such a family we can find constants $d_{1}, d_{2}>0$ such that

$$
d_{1} t^{n} \leqq \int_{\Omega^{(t)}}\left|\varpi_{S}(v)\right| d_{0} v \leqq d_{2} t^{n} \quad(t \geqq 1) .
$$

Lemma 8.7. Let $\Omega$ be any bounded open subset of $\mathscr{F}_{I}$ such that $\partial \Omega$ has finite $(r-1)$ dimensional Hausdorff measure, i.e. limsup $\kappa^{-1} \operatorname{vol}_{0}\left(\partial \Omega_{k}\right)<\infty$. Let $\mu \in \mathscr{F}_{I}$ be fixed and let

$$
\kappa \rightarrow 0
$$

$$
\begin{equation*}
\Omega(t)=\mu+t \Omega(t \geqq 1) \tag{8.26}
\end{equation*}
$$

Then the family $(\Omega(t))_{t \geqq 1}$ satisfies conditions (a) and (b) of (8.25). In particular this will be true if $\partial \Omega$ is smooth.

Theorem 8.8. Let $(\Omega(t))_{t \geqq 1}$ be a family of subsets of $\mathscr{F}_{I}$ satisfying the conditions (a) (b) of (8.25). Then we have, in the notation of Theorem 8.5,

$$
\begin{align*}
& \sum_{\lambda \in A_{p} \cap \Omega(t)} m(\lambda)=\sigma(G) \operatorname{vol}_{0}(X)|\mathfrak{w}|^{-1} \int_{\Omega(t)} \beta(v) d_{0} v+O\left(t^{n-1}\right), \quad t \rightarrow+\infty \\
& \quad=\left(2^{2 r} \pi^{n+r} \sigma(G)\right)^{-1} \operatorname{vol}_{0}(X)|\mathfrak{w}|^{-1} \int_{\Omega(t)}\left|\varpi_{S}(v)\right| d_{0} v+O\left(t^{n-1}\right), \quad t \rightarrow+\infty . \tag{8.27}
\end{align*}
$$

It is a consequence of Theorem 8.3 and condition (8.25a) that

$$
\begin{equation*}
\sum_{\lambda \in A, \lambda_{I} \in \Omega(t)} m(\lambda)=\sum_{\lambda \in A_{p} \cap \Omega(t)} m(\lambda)+O\left(t^{n-1}\right), \quad t \rightarrow+\infty . \tag{8.28}
\end{equation*}
$$

From Lemma 3.11 and (8.25a) again, we obtain easily

$$
\begin{align*}
& \sigma(G) \operatorname{vol}_{0}(X)|\mathfrak{w}|^{-1} \int_{\Omega(t)} \beta(v) d_{0} v \\
& \quad=\left(2^{2 r} \pi^{n+r} \sigma(G)\right)^{-1} \operatorname{vol}_{0}(X)|\mathfrak{w}|^{-1} \int_{\Omega(t)}\left|\varpi_{S}(v)\right| d_{0} v+O\left(t^{n-1}\right), \quad t \rightarrow+\infty . \tag{8.29}
\end{align*}
$$

Further formula (3.44) asserts that $\beta(\mu) \leqq c(1+\|\mu\|)^{n-r}\left(\mu \in \mathscr{F}_{I}\right)$; so by condition (8.25a) we find a constant $c>0$ such that $\beta(v) \leqq c t^{n-r}$, for all $t \geqq 1, v \in \partial \Omega(t)_{\kappa}$. By condition (8.25b) we get

$$
\begin{equation*}
\int_{\partial \Omega(t)_{k}} \beta(v) d_{0} v=O\left(t^{n-1}\right), \quad t \rightarrow+\infty . \tag{8.30}
\end{equation*}
$$

The assertion follows now from Theorem 8.5 and (8.28)-(8.30).
We shall now cast Theorem 8.8 in an alternative form. To this end we proceed as follows. Recall that $\mathfrak{g}=\mathfrak{f}+\mathfrak{s}$; we denote by $\mathfrak{s}^{*}$ the dual of $\mathfrak{s}$. For any set $\Omega$ in $\mathscr{F}_{I}$ we define the subset $\Omega_{\mathfrak{s}^{*}}$ in $\mathfrak{s}^{*}$ by

$$
\begin{equation*}
\Omega_{\mathrm{s}^{*}}=\left\{(-1)^{\frac{1}{2}} \operatorname{Ad}(k) \cdot v: v \in \Omega, k \in K\right\} . \tag{8.31}
\end{equation*}
$$

Employing as usual, standard Haar measures, we have the formula (cf. Helgason [24, p. 381]) valid for any $f \in C_{c}(\mathfrak{s})$

$$
\int_{\mathfrak{s}} f(S) d_{0} S=|\mathfrak{w}|^{-1} \int_{\alpha}\left|\prod_{\alpha \in \mathbf{A}^{+}} \alpha(H)^{n(\alpha)}\right| \int_{K / M} f(\operatorname{Ad}(k) H) d_{0}(k M) d_{0} H .
$$

Going over to $\mathfrak{s}$ via the Killing form, the set $\Omega_{\mathfrak{s}^{*}}$ goes to the set $\Omega_{\mathfrak{s}}=$ $\left\{(-1)^{\frac{1}{2}} \operatorname{Ad}(k) H_{v}: v \in \Omega, k \in K\right\}$, and so,

$$
\begin{equation*}
\left.\operatorname{vol}_{0}\left(\Omega_{\mathfrak{s}^{*}}\right)=\operatorname{vol}_{0}\left(\Omega_{\mathfrak{s}}\right)=\operatorname{vol}_{0}(K / M)|\mathfrak{w}|^{-1} \int_{\Omega} \mid \varpi_{\mathrm{S}} v\right) \mid d_{0} v, \tag{8.32}
\end{equation*}
$$

since $d_{0} H$ corresponds with $d_{0} v$. Using (8.32), we write the last term in (8.27) as

$$
\left(2^{2 r} \pi^{n+r} \sigma(G)\right)^{-1} \operatorname{vol}_{0}(K / M)^{-1} \operatorname{vol}_{0}(X) \operatorname{vol}_{0}\left(\Omega(t)_{5^{*}}\right)+O\left(t^{n-1}\right), \quad t \rightarrow+\infty .
$$

So, using Proposition 3.12, Theorem 8.8 takes the following form.
Theorem 8.9. Let the family $(\Omega(t))_{t \geqq 1}$ satisfy the conditions (a)-(b) cf. (8.25) and define $\Omega(t)_{s^{*}}$ by (8.31). Then

$$
\sum_{\lambda \in A_{p} \cap \Omega(t)} m(\lambda)=(2 \pi)^{-n} \operatorname{vol}_{0}(X) \operatorname{vol}_{0}\left(\Omega(t)_{s^{*}}\right)+O\left(t^{n-1}\right), \quad t \rightarrow+\infty .
$$

Remark 8.10. Let us consider the Laplacian $A=\omega_{S}$ on $X$ (cf. (2.6)); its eigenvalue corresponding to an element $\lambda \in \Lambda_{p}$ is $-\left(\|\lambda\|^{2}+\|\rho\|^{2}\right.$ ) (cf. (3.17)). So applying the Theorems 8.3 and 8.9 we obtain at once the Minakshisundaram-Pleijel result for the number $N(t)$ of eigenvalues of $-\Delta$ (counted with multiplicities) which are $\leqq t$, as

$$
\begin{equation*}
N(t)=(2 \sqrt{\pi})^{-n} \Gamma\left(\frac{n}{2}+1\right)^{-1} \operatorname{vol}_{0}(X) t^{n / 2}+O\left(t^{(n-1) / 2}\right), \quad t \rightarrow+\infty \tag{8.33}
\end{equation*}
$$

Indeed, $\Omega(t)_{5^{*}}$ is now the ball in $\mathfrak{s}^{*}$ of radius $t$ around the origin whose volume is $\pi^{n / 2} \Gamma\left(\frac{n}{2}+1\right)^{-1} t^{n}$, from which the above expression follows.

Remark 8.11. We observe that the deduction of the Minakshisundaram-Pleijel formula from Theorem 8.8 used the expression for $\operatorname{vol}_{0}(K / M)$ obtained in Proposition 3.12. It is clear that conversely one can start with (8.33) and find this formula for $\mathrm{vol}_{0}(K / M)$.

Remark 8.12. In fact, let $P$ be a positive elliptic differential operator of order $m$ on $X$ which comes from a $G$-invariant operator on $G / K$, i.e. $P \in \mathscr{E}_{K}^{\prime}(G / / K)$ (cf. (2.2)). The principal symbol of $P$ can be regarded as an $\operatorname{Ad}(K)$-invariant homogeneous polynomial of degree $m$ on $\mathfrak{s}^{*}$, say $p$; and $P$ being elliptic, we have $p(\eta) \neq 0$, if $\eta \in \mathfrak{s}^{*} \backslash\{0\}$. Using the Killing form, we consider $p$ as a polynomial on $\mathfrak{s}$. On the other hand, the Harish-Chandra homomorphism $\gamma$ (cf. Proposition 3.2) maps $P$ onto an element $p_{\mathfrak{a}} \in U\left(\mathfrak{a}_{c}\right)^{\mathfrak{w}}$ such that $\left.p\right|_{\mathfrak{a}}=p_{\mathrm{a}, m}$, with $p_{\mathrm{a}, m}$ as the homogeneous part of degree $m$ of $p_{a}$. The eigenvalues of $P$ are given by $\mathscr{H} P(\lambda)$ $=p_{a}(\lambda) \geqq 0(\lambda \in \Lambda)$. Observing how $\Lambda_{p}$ asymptotically fills up $\mathscr{F}_{I}$, we get $i^{m} p(\eta)>0$ if $\eta \in \mathscr{F}_{R} \backslash\{0\}$, which implies $i^{m} p(\eta)>0$ if $\eta \in \mathfrak{s}^{*} \backslash\{0\}$, as $\mathfrak{s}^{*}=\operatorname{Ad}(K) \cdot \mathscr{F}_{R}$. So, if $\lambda$ $=t \mu\left(\lambda \in A_{p}, t>0, \mu \in \mathscr{F}_{I}\right)$, then the condition that $p_{\mathfrak{a}}(\lambda) \leqq t^{m}$ is equivalent to $p(\mu) \leqq 1+O\left(t^{-1}\right)$; and using Theorem 8.3 it follows from Theorem 8.9 that the number $M(P: t)$ of eigenvalues of $P$ (counted with their multiplicities) which are $\leqq t^{m}$, is given by

$$
(2 \pi)^{-n} \operatorname{vol}_{0}(X) \operatorname{vol}_{0}\left(\Omega_{5^{*}}\right) t^{n}+O\left(t^{n-1}\right), \quad t \rightarrow+\infty
$$

where $\Omega=\left\{\mu: \mu \in \mathscr{F}_{I}, p(\mu) \leqq 1\right\}$. But $\operatorname{vol}_{0}(X) \operatorname{vol}_{0}\left(\Omega_{\S^{*}}\right)=\operatorname{vol}_{0}\left(B^{*} X\right)$ where $B^{*} X$ $=\left\{(x, \eta): x \in X, \eta \in \mathfrak{s}^{*}, i^{m} p(\eta) \leqq 1\right\}$. If we denote the principal symbol of $P$ regarded as a function on the cotangent bundle $T^{*} X$ also by $p$, then $B^{*} X$ $=\left\{(x, \eta):(x, \eta) \in T^{*} X, i^{m} p(x, \eta) \leqq 1\right\}$, and $\operatorname{vol}_{0}(X) \operatorname{vol}_{0}\left(\Omega_{s^{*}}\right)$ coincides with the volume of $B^{*} X$ with respect to the canonical density in $T^{*} X$. So we have established $M(P: t)=(2 \pi)^{-n} \operatorname{vol}\left(B^{*} X\right) t^{n}+O\left(t^{n-1}\right), t \rightarrow+\infty$ which is a result of Hörmander [26].

Remark 8.13. Even for positive non-elliptic differential operators $P$ of order $m$ on $X$ with corresponding polynomial $p$ on $s^{*}$ it is reasonable to expect that the number of eigenvalues $\leqq t^{m}$ is asymptotically given by a constant $\times \int_{\Omega(t)} \beta(v) d v$, where $\Omega(t)=\left\{\mu: \mu \in \mathscr{F}_{I}, \mathrm{p}(\mu) \leqq t^{m}\right\}$. If $p_{m}$ is the homogeneous part of degree $m$ of $p$, then the $\Omega(t)$ exhibit an unproportional stretching in the direction of the cone $\left\{\mu: \mu \in \mathscr{F}_{I}, p_{m}(\mu)=0\right\}$. How drastically the asymptotic estimates for the spectrum can change has been shown for a certain class of hypoelliptic operators on arbitrary compact manifolds by Menikoff and Sjöstrand [31].

## 9. Improved Error Estimates when $\operatorname{rk}(S)=1$

9.1. In this section we shall suppose that $r=\operatorname{rk}(S)=1$. From (5.43) and (5.40) we get, for any $f \in C_{c}^{\infty}(a)$,

$$
\begin{align*}
& \sum_{\lambda \in A} m(\hat{\lambda}) \hat{f}(\lambda)=\operatorname{vol}(X) \frac{1}{2} \int_{\mathscr{F}_{I}} \hat{f}(v) \beta(v) d v \\
& \quad+\sum_{c \neq[e]_{\Gamma}} l_{0}(c) \Delta_{+}(\gamma)^{-1} \frac{1}{2}\left(f\left(H_{\gamma}\right)+f\left(-H_{\gamma}\right)\right) \tag{9.1}
\end{align*}
$$

Here $\gamma=\gamma_{I} \exp H_{\gamma}$ is any element in $[c]_{G}$ in standard position, with $\gamma_{I} \in K, H_{\gamma}$ a regular element in $\mathfrak{a}$; we have (cf. 4.8 ) and ( 5.42 c ))

$$
\begin{equation*}
l_{0}(c) \leqq\left\|H_{\gamma}\right\| ; \quad \Delta_{+}(\gamma)^{-1}=\varepsilon_{R}(\gamma) e^{-\zeta_{P}\left(H_{R}\right)} \prod_{\alpha \in P \not P P_{I}}\left(1-\xi_{-x}(\gamma)\right)^{-1} . \tag{9.2}
\end{equation*}
$$

We will use the above formulae to obtain
Theorem 9.1. If $\operatorname{rk}(G / K)=1$, we have

$$
\begin{equation*}
\sum_{\lambda \in A_{p},\|\lambda\| \leqq t} m(\lambda)=(2 \sqrt{\pi})^{-n} \Gamma\left(\frac{n}{2}+1\right) \operatorname{vol}_{0}(X) t^{n}+O\left(t^{n-1} / \log t\right), \quad t \rightarrow+\infty . \tag{9.3}
\end{equation*}
$$

We shall obtain this estimate by proving Proposition 9.2 below; it is the equivalent of formula (60) in Berard [2] when $n>2$, and stronger than (60) loc. cit when $n=2$. Theorem 9.1 is then proved by imitating Berard's arguments as in
 estimate of Bérard in (60) loc.cit does not appear to give (61) loc.cit, and consequently, the arguments of pp.264-265 loc.cit are not applicable; it is the stronger estimate provided by Proposition 9.2 that leads to (61) loc.cit and thence to Theorem 9.1. Moreover, if $n=2$, there is a proof of $(9.3)$ using number theoretic methods by Randol [38].

We recall the test functions $h(t: \cdot) \in C_{c}^{\infty}(\mathfrak{a})^{\mathfrak{w}}(t>0)$ which are constructed in (6.4).

Proposition 9.2. We can find constants $c_{1}, c_{2}, c_{3}>0$, depending on $h$, such that, for all $\varepsilon$ with $0<\varepsilon \leqq 1$ and $\mu \in \mathscr{F}_{I}$,

$$
\begin{align*}
& \sum_{\lambda \in \Lambda_{p}} m(\lambda) \hat{h}(\varepsilon: \lambda-\mu)=h(0) \operatorname{vol}(X) \varepsilon \frac{1}{2} \beta(\mu)+E(\varepsilon: \mu) \\
& |E(\varepsilon: \mu)| \leqq c_{1} \varepsilon(1+\|\mu\|)^{n-3}+c_{2} e^{c_{3} / \varepsilon} \tag{9.4}
\end{align*}
$$

In view of (9.1) we have, for any $\mu \in \mathscr{\mathscr { F }}{ }_{I}$ and $\varepsilon>0$,

$$
\begin{aligned}
& \sum_{\lambda \in \Lambda_{p}} m(\lambda) \hat{h}(\varepsilon: \lambda-\mu)=\frac{1}{2} \operatorname{vol}(X) \int_{\mathscr{F}_{I}} \hat{h}(\varepsilon: v-\mu) \beta(v) d v \\
& \quad+\sum_{c \neq\left[e l_{l}\right.} l_{0}(c) \Delta_{+}(\gamma)^{-1} e^{-\mu\left(H_{\gamma}\right)} \varepsilon h\left(\varepsilon H_{\gamma}\right) \\
& \quad+\sum_{\lambda \in \Lambda_{c}} m(\lambda) \hat{h}(\varepsilon: \lambda-\mu)=\sum_{1 \leqq i \leqq 3} \Theta_{i}(\varepsilon: \mu)
\end{aligned}
$$

We shall now estimate the $\Theta_{i}=\Theta_{i}(\varepsilon: \mu)$ individually in terms of $\varepsilon$ and $\mu$. We treat $\Theta_{3}$ first. Using the relation

$$
\begin{equation*}
\operatorname{supp}(h(\varepsilon: \cdot)) \subset\left\{H: H \in \mathfrak{a},\|\mu\| \leqq \varepsilon^{-1}\right\} \quad(\varepsilon>0) \tag{9.6}
\end{equation*}
$$

the estimate (3.23) and Lemma 8.1, we obtain a constant $b_{1}>0$ such that $\mid \hat{h}(\varepsilon: \lambda$ $-\mu) \mid \leqq b_{1} e^{\|\rho\| / \varepsilon}\left(\varepsilon>0, \lambda \in \Lambda_{c}, \mu \in \mathscr{F}_{I}\right)$. So, as $\Lambda_{c}$ is finite (see, for instance, Theorem 8.3) there is a constant $b_{2}>0$ such that

$$
\begin{equation*}
\left|\Theta_{3}(\varepsilon: \mu)\right| \leqq \sum_{\lambda \in \mathcal{A}_{\mathrm{c}}} m(\lambda)|\hat{h}(\varepsilon: \lambda-\mu)| \leqq b_{2} e^{\|\rho\| / \varepsilon}\left(\varepsilon>0, \mu \in \mathscr{F}_{I}\right) . \tag{9.7}
\end{equation*}
$$

To estimate $\Theta_{2}$ we note that $l(c)=\left\|H_{y}\right\|$ so that, by (9.6), the summation in $\Theta_{2}$ is over those $c \neq[e]_{\Gamma}$ for which $l(c) \leqq \varepsilon^{-1}$. Using (9.2), we get, for all $\varepsilon>0$ and $\mu \in \mathscr{F}_{I}$,

$$
\begin{equation*}
\left|\Theta_{2}(\varepsilon: \mu)\right| \leqq \varepsilon b_{3} \sum_{\substack{c \neq[\tilde{c}] \\ l(c) \leqq \varepsilon, r^{\prime}}}\left\|H_{\gamma}\right\| e^{\zeta P\left(H_{\gamma}\right)} \prod_{\alpha \in P \backslash P_{I}}\left|1-\xi_{-\alpha}(\gamma)\right|^{-1} \tag{9.8}
\end{equation*}
$$

where the constant $b_{3}>0$ is independent of $\varepsilon$ and $\mu$. An elementary and wellknown measure theoretic argument gives the following estimate: there are constants $b_{4}, b_{5}>0$ such that, for $x>0$,

$$
\begin{equation*}
\sum_{c \in \mathscr{C}(I), l(c) \leqq x} l(c) \leqq b_{4} e^{b_{5} x} \tag{9.9}
\end{equation*}
$$

Further, as the numbers $l(c)\left(\mathrm{c} \neq[e]_{\Gamma}\right)$ are bounded away from 0 (Theorem 5.1) we find $b_{6}>0$ such that, for all $\gamma$,

$$
\begin{equation*}
\prod_{\alpha \in P \backslash P_{I}}\left|1-\xi_{-\alpha}(\gamma)\right|=\prod_{\alpha \in P \backslash P_{I}}\left|\xi_{x}\left(\gamma_{I}\right)-e^{-\alpha\left(H_{\gamma}\right)}\right| \geqq \prod_{\alpha \in P \backslash P_{I}}\left|1-e^{-\alpha\left(H_{\gamma}\right)}\right| \geqq b_{6} \tag{9.10}
\end{equation*}
$$

So combining (9.8)-(9.10), we find, for any $\varepsilon>0$ and $\mu \in \mathscr{F}_{I}$,

$$
\begin{equation*}
\left|\Theta_{2}(\varepsilon: \mu)\right| \leqq \varepsilon b_{3}\left(b_{4} e^{b_{5 / c}}\right) e^{\left\|/ \zeta_{p}\right\| / \varepsilon} b_{6}^{-1} \leqq b_{7} e^{b_{8} / \varepsilon} . \tag{9.11}
\end{equation*}
$$

The estimates ( 9.7 ) and ( 9.11 ) would then prove the proposition, provided we establish the following result.
Proposition 9.3. There is a constant $c>0$ such that, for $\mu \in \mathscr{F}_{I}$ and $0<\varepsilon \leqq 1$,

$$
\begin{equation*}
\int_{F_{1}} \hat{h}(\varepsilon: v-\mu) \beta(v) d \nu=h(0) \varepsilon \beta(\mu)+\varepsilon F(\mu) ; \quad|F(\mu)| \leqq c(1+\|\mu\|)^{n-3} . \tag{9.12}
\end{equation*}
$$

Let $\eta \in \mathscr{F}_{I}$ be a unit vector and let $H \in(-1)^{\frac{1}{2}}$ a be the (unique) dual basis element. By a Taylor series expansion of order 2 for $\beta(\mu+v)$ around $\mu$ and by Fourier inversion we get

$$
\begin{aligned}
& \int_{\mathscr{F}_{I}} \hat{h}(\varepsilon: v-\mu) \beta(v) d v=\int_{\mathscr{F}_{I}} \hat{h}\left(\varepsilon^{-1} v\right) \beta(\mu+v) d v \\
&= h(\varepsilon: 0) \beta(\mu)+\mathrm{h}(\varepsilon: 0 ;-\partial(H)) \beta(\mu ; \partial(\eta)) \\
&+\int_{\mathscr{F}_{I}} \int_{0}^{1}(1-\tau)\langle v, \eta\rangle^{2} \beta\left(\mu+\tau v ; \partial\left(\eta^{2}\right)\right) \hat{h}\left(\varepsilon^{-1} v\right) d v d \tau .
\end{aligned}
$$

By the symmetry of $h, h(\varepsilon: 0 ;-\partial(H))=0$, and we find

$$
\begin{aligned}
& \int_{\mathscr{F}_{I}} \hat{h}(\varepsilon: v-\mu) \beta(v) d v=\varepsilon h(0) \beta(\mu) \\
& \quad+\int_{\mathscr{F}_{I}} \int_{0}^{1}(1-\tau) \varepsilon^{2}\langle v, \eta\rangle^{2} \beta\left(\mu+\tau \varepsilon v ; \partial\left(\eta^{2}\right)\right) \hat{h}(v) \varepsilon d v d \tau .
\end{aligned}
$$

In view of the Paley-Wiener estimates for $\hat{h}$ and the estimate $1+\|\mu+\tau \varepsilon v\| \leqq(1+$ $\|\mu\|)(1+\|v\|)$, we are done as soon as we can estimate the second derivative of $\beta$. Being in the rank-one situation this comes down to the (legitimate) differentiation of the asymptotic expansion for $f_{\alpha}$, occurring supra (3.44). So (9.12) follows (even for $n=2$, when $\mathfrak{g}=\mathfrak{s l}(2, \mathbb{R})$ ).

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[^1]:    1 $\quad \mathrm{CSG}=$ Cartan subgroup, $\mathrm{CSA}=$ Cartan subalgebra

