The Daniell Closure

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1 Introduction

The following presentation of the theory of Lebesgue integration with respect to a measure μ is due to Daniell [2].

Let Ω be a set. A *ring* of subsets of Ω is a collection $\mathcal{A}$ of subsets of Ω such that

$$\emptyset \in \mathcal{A} \quad \text{and} \quad A,\, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A},\ A \cap B \in \mathcal{A},\ A \setminus B \in \mathcal{A}. \tag{1.1}$$

A *positive measure* on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to \mathbf{R}$ which is *positive* in the sense that

$$\mu(A) \geq 0 \quad \text{for every} \quad A \in \mathcal{A}, \tag{1.2}$$

additive in the sense that

$$\mu(A \cup B) = \mu(A) + \mu(B) \quad \text{whenever} \quad A,\, B \in \mathcal{A} \quad \text{and} \quad A \cap B = \emptyset, \tag{1.3}$$

and has the *monotone convergence property* in the sense that

$$\text{If } A_n \text{ is a decreasing sequence in } \mathcal{A} \text{ with } \bigcap_{n=1}^{\infty} A_n = \emptyset, \text{ then } \lim_{n\to\infty} \mu(A_n) = 0. \tag{1.4}$$

Given the ring $\mathcal{A}$ of subsets of Ω, a function $f : \Omega \to \mathbf{R}$ is called $\mathcal{A}$-*elementary*, if $f(\Omega)$ is a finite subset of $\mathbf{R}$ and $f^{-1}(\{c\} \in \mathcal{A}$ for every $c \in \mathbf{R}$ such that $c \neq 0$. let $E = E_{\mathcal{A}}$ denote the set of all $\mathcal{A}$-elementary functions on E.

Provided with the pointwise addition and scalar multiplication of functions, the space F of all functions $f : \Omega \to \mathbf{R}$ is a vector space over $\mathbf{R}$, and E is a linear subspace of F. Also, in F we have the partial ordering defined by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \Omega$, and the functions $\max(f,\, g)$ and $\min(f,\, g)$, defined by $(\max(f,\, g))(x) := \max(f(x),\, g(x))$ and $\min(f(x),\, g(x)) := \min(f(x),\, g(x))$ for every $x \in \Omega$, are the smallest and largest elements of F which are $\geq f,\, g$ and $\leq f,\, g$, respectively. The ring property of $\mathcal{A}$ implies that E has the *Riesz property*

$$f,\, g \in E \quad \Rightarrow \quad \max(f,\, g) \in E \quad \text{and} \quad \min(f,\, g) \in E. \tag{1.5}$$

For any $f \in E$, the *integral* of f with respect to the positive measure μ on $\mathcal{A}$ is defined as

$$\int f(x)\,\mu(\mathrm{d}x) := \sum_{c\in f(\Omega)\setminus\{0\}} c\,\mu(f^{-1}(\{c\}). \tag{1.6}$$

Then

$$u = \mathrm{I}_\mu : f \mapsto \int f(x)\,\mu(\mathrm{d}x) \tag{1.7}$$

defines a linear form $u : E \to \mathbf{R}$, which is *positive* in the sense that

$$u(f) \geq 0 \quad \text{if} \quad f \in E \quad \text{and} \quad f \geq 0, \tag{1.8}$$

and has the *monotone convergence property* in the sense that, for every sequence f_n in E,

$$\text{If} \quad f_n \downarrow 0 \quad \text{then} \quad u(f_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{1.9}$$

We provide a proof, which is not contained in Daniell [2], perhaps because (1.9) is a very simple case of the usual limit theorems in the theory of Lebesgue integration with respect to a measure.

Proof The linearity and positivity of $u = \mathrm{I}_\mu$ are obvious. For the proof of (1.9), we define, for each $\delta \in \mathbf{R}_{>0}$, $A_{\delta,\,n} := \{x \in \Omega \mid f_n(x) > \delta\}$. Then $f_n \in E$ and (1.1) imply that $A_{\delta,\,n} \in \mathcal{A}$, and $f_n(x) \downarrow 0$ for every $x \in \Omega$ implies that $n \mapsto A_{\delta,\,n}$ is a decreasing sequence of subsets of Ω with empty intersection. Therefore (1.4) implies that $\lim_{n\to\infty} \mu(A_{\delta,\,n}) = 0$.

Let $A_1 = \{x \in \Omega \mid f_1(x) > 0\}$. Then $f_1 \in E$ and (1.1) imply that $A_1 \in \mathcal{A}$, and $f_n \leq f_1$ implies that ${f_n}^{-1}(\{c\} \subset A_1$ for every $c \in \mathbf{R}_{>0}$. Also $\max f_n \leq \max f_1$. Because for different c's the sets ${f_n}^{-1}(\{c\}) \in \mathcal{A}$ are disjoint, we have

$$\begin{aligned} u(f_n) &= \sum_{c\in f_n(\Omega),\,0<c\leq\delta} c\,\mu({f_n}^{-1}(\{c\}) + \sum_{c\in f_n(\Omega),\,c>\delta} c\,\mu({f_n}^{-1}(\{c\}) \\ &\leq \delta\,\mu(A_1) + (\max f_1)\,\mu(A_{\delta,\,n}). \end{aligned}$$

Let $\epsilon > 0$. Choose $\delta > 0$ such that $\delta\,\mu(A_1) < \epsilon$, and then, for this δ, n such that $\delta\,\mu(A_1) + (\max f_1)\,\mu(A_{\delta,\,n}) \leq \epsilon$. Then $u(f_n) \leq \epsilon$, hence $0 \leq u(f_m) \leq u(f_n) \leq \epsilon$ for every $m \geq n$. $\square$

One of the points of Daniell [2] is that, *assuming only that E is a vector space of real valued functions on a set Ω which has the Riesz property (1.5), and $u : E \to \mathbf{R}$ is a linear form on E which satisfies (1.8) and (1.9), then there is a space $\overline{E}$ of functions on Ω with values in $\overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty\} \cup \{\infty\}$ and an extension of u to a real valued function on* E *which is denoted by the same letter u, such that the space $\overline{E}$ and the function $u : \overline{E} \to \mathbf{R}$ has all the properties of the space of Lebesgue integrable functions and the Lebesgue integral.*

If $E = E_{\mathcal{A}}$ and $u = \mathrm{I}_\mu$ for a ring $\mathcal{A}$ of subsets of Ω and a measure μ on $\mathcal{A}$, respectively, then the extension procedure in Daniell [2] is *equal* to the one in the theory of Lebesgue integration, and therefore $\overline{E}$ and $u : \overline{E} \to \mathbf{R}$ is equal to the space of Lebesgue integrable functions and Lebesgue integral, respectively. That is, the theory of Daniell [2] is a generalization of the

the theory of Lebesgue integration to arbitrary linear forms $u : E \to \mathbf{R}$ which statisfy (1.5), (1.8), and (1.9). As no additional arguments are needed, this generalization is at no extra cost. Moreover, the presentation actually becomes somewhat simpler, because no ring of subsets $\mathcal{A}$ and measure μ appear in the notations, only the linear form $u : E \to \mathbf{R}$ satisfying (1.5), (1.8), and (1.9).

We now describe $\overline{E}$, $u : \overline{E} \to \mathbf{R}$, and their main properties. The first step in the extension procedure is the definition of the set $E^{\uparrow}$ of all functions $f : \Omega \to \mathbf{R} \cup \{\infty\}$ which are the pointwise limits of a pointwise non-decreasing sequence f_n in E, notation $E \ni f_n \uparrow f$. Then the numbers $u(f_n)$ form a non-decreasing sequence, of which the limit is independent of the sequence $E \ni f_n \uparrow f$. If $f \in E$, then $f - f_n \downarrow 0$, hence $u(f) - u(f_n) = u(f - f_n) \to 0$ if $f \in E$. It follows that there is a unique $u^{\uparrow} : E^{\uparrow} \to \mathbf{R} \cup \{\infty\}$, such that $u^{\uparrow}(f) = \lim_{n\to\infty} u(f_n)$ whenever $E \ni f_n \uparrow f$, and such that $u^{\uparrow}|_E = u$. We write $u^{\uparrow} = u$ in the sequel.

The next step is the definition, for every $f : \Omega \to \overline{\mathbf{R}}$ of

$$\overline{u}(f) := \inf\{u(\varphi) \mid \varphi \in E^{\uparrow} \text{ and } f \leq \varphi\} \in \overline{\mathbf{R}}, \tag{1.10}$$

where $\overline{u}(f) := \infty$ if there is no $\varphi \in E^{\uparrow}$ such that $f \leq \varphi$. with this definition, $\overline{E} = \overline{E}^{u}$ is defined as the set of all $f : \Omega \to \overline{\mathbf{R}}$ with the property that, for every $\epsilon > 0$, there exists $\varphi \in E$ such that

$$\overline{u}(|f - \varphi|) \leq \epsilon. \tag{1.11}$$

In other words, $\overline{E}$ is the closure of E in the space of all functions $f : \Omega \to \overline{\mathbf{R}}$ if $\overline{u}(|f - \varphi|)$ is viewed as the distance between f and $\varphi \in E$. It is proved that, for any φ, $\psi \in E$, $|u(\varphi) - u(\psi)| \leq \overline{u}(|f - \varphi|) + \overline{u}(|f - \psi|)$. It follows that there is a unique $v : \overline{E} \to \mathbf{R}$ such that $|v(f) - u(\varphi)| \leq \overline{u}(|f - \varphi|)$ whenever $f \in \overline{E}$ and $\varphi \in E$, which implies that $v|_E = u$, and therefore allows to write $v = u$.

With the standard arguments of the theory of Lebesgue integration with respect to a measure, Daniell [2] subsequently proved that the extended functional u is linear in the sense that $u(f+g) = u(f) + u(g)$ and $u(c\,f) = c\,u(f)$ whenever f, $g \in \overline{E}$, $c \in \mathbf{R}$, and the functions $f+g$ and $c\,f$ are pointwise defined. Furthermore, u is positive in the sense that $u(f) \leq u(g)$ if f, $g \in \overline{E}$ and $f \leq g$. More importantly, $\overline{E}$ and $u : \overline{E} \to \mathbf{R}$ enjoy very strong limit properties. For instance, one has the *Levi property*[1], that if $\overline{E} \ni f_n \uparrow f : \Omega \to \overline{\mathbf{R}}$, and the sequence $u(f_n)$ is bounded from above by a finite number, then $f \in \overline{E}$ and $u(f) = \lim_{n\to\infty} u(f_n)$. We a similar statement for decreasing sequences in $\overline{E}$, which in the case that $f = 0$ implies that $u : \overline{E} \to \mathbf{R}$ satisfies (1.9). However, the initial space E and linear form $u : E \to \mathbf{R}$ usually is not closed under taking limits of arbitrary monotone sequences f_n for which the $u(f_n)$ are bounded.

For any subset A of Ω the *characteristic function* 1_A of A is defined by $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \in \Omega \setminus A$. Let $\mathcal{A} = \mathcal{A}^u$ denote the collection of all $A \subset \Omega$ such that $1_A \in \overline{E}$. $\mathcal{A}$ is a ring of subsets of Ω, because $1_{\emptyset} = 0$, and $1_{A\cup B} = \max(1_A,\, 1_B)$, $1_{A\cap B} = \min(1_A,\, 1_B)$, and $1_{A\setminus B} = 1_A - 1_{A\cap B}$ all belong to $\overline{E}$ if A, $B \in \mathcal{A}$.

Define $\mu = \mu^u : \mathcal{A} \to \mathbf{R}$ by $\mu(A) = u(1_A)$ for every $A \in \mathcal{A}$. We have $\mu(A) = u(1_A) \geq u(0) = 0$ because $1_A \geq 0$, and $\mu(A \cup B) = u(1_{A\cup B}) = u(1_A + 1_B) = u(1_A) + u(1_B) = \mu(A) +$

[1]There were several famous mathematicians with the name Levi. This is Beppo Levi.

$\mu(B)$ if $A,\, b \in \mathcal{A}$ and $A \cap B = \emptyset$. Let A_n be an increasing sequence in $\mathcal{A}$ for which the sequence $\mu(A_n)$ is bounded from above, and let A be the union of all the A_n. Then $\overline{E} \ni 1_{A_n} \uparrow 1_A$ and the sequence $u(1_{A_n})$ is bounded from above. Therefore the Levi property implies that $1_A \in \overline{E}$ and $u(1_A) = \lim_{n\to\infty} u(1_{A_n})$. Therefore $A \in \mathcal{A}$ and $\mu(A) = \lim_{n\to\infty} \mu(A_n)$. There is an analogous statement for decreasing sequences in $\mathcal{A}$, which implies (1.4).

In particular $\mathcal{A}$ is a ring of subsets of Ω and μ is a measure on $\mathcal{A}$. We have $E_{\mathcal{A}} \subset \overline{E}$ and I_μ is equal to the restriction of u to E_μ. Applying the extension procedure with $u : E \to \mathbf{R}$ replaced by $\mathrm{I}_\mu : E_\mu \to \mathbf{R}$, it is natural to ask whether $E \subset \overline{E_\mu}^{\mathrm{I}_\mu}$, and u is equal to the restriction to E of the extension of I_μ to $\overline{E_\mu}^{\mathrm{I}_\mu}$. However, this is not always the case. That is, for $f \in E$, $u(f)$ cannot always be recovered as the Lebesgue integral of f with respect to the measure μ_u. The following, admittedly a bit artificial, example shows this convincingly. In other words, the Daniell closure is a true generalization of the theory of Lebesgue integration with respect to measures.

Example 1.1 Let Ω be a subset of $\mathbf{R}_{\geq 0}$ with at least two strictly positive elements. Let E be the set of all restrictions to Ω of linear functions on $\mathbf{R}$. That is, $f \in \Omega$ if and only if there exists $c \in \mathbf{R}$ such that $f(x) = c\,x$ for every $x \in \Omega$. Define $u(f) = c$ if $f(x) = c\,x$ for every $x \in \Omega$, that is, if we choose $x_0 \in \Omega \setminus \{0\}$, then $u(f) = f(x_0)/x_0$ for every $f \in E$. It follows that E is a vector space of functions on Ω with the Riesz property (1.5), and that u is a linear form on E such that (1.8) and (1.9). However, $E^{\uparrow} = E$, $\overline{E} = E$, $\mathcal{A} = \{\emptyset\}$, $E_{\mu_u} = \{0\}$, and $\mathrm{I}_\mu = 0$. $\oslash$

However, the other direction one has the following result, not contained in Daniell [2]. I learned the theorem and the idea of the proof from Constantinescu, Weber and Sontag [1, Prop. 5.1.2 and Th. 5.1.6].

Theorem 1.2 *Suppose that E has the* Stone property*, that is,* $\min(f,\, 1_\Omega) \in E$ *for every* $f \in E$*. Then* $\overline{E} = \overline{E_\mu}^{\mathrm{I}_\mu}$*, and* $u(f) = \int f(x)\,\mu(\mathrm{d}x)$ *for every* $f \in \overline{E}$*.*

Proof We first prove that if $f \in E$, then $\overline{E_\mu}^{\mathrm{I}_\mu}$ and $u(f) = \int f(x)\mu(\mathrm{d}x)$. In view of $f = \max(f,\, 0) - \max(-f,\, 0)$ where $\max(f,\, 0)$ and $\max(-f,\, 0)$ are nowhere negative elements of E, we may assume that $f \geq 0$. For any $c \in \mathbf{R}_{>0}$, we have $\min(f,\, c\,1_\Omega) = c\,\min((1/c)\,f,\, 1_\Omega) \in E$. Therefore, if $0 < c' < c$, also

$$g_{c,\,c'} = \frac{1}{c - c'}\,(\min(f,\, c\,1_\Omega) - \min(f,\, c'\,1_\Omega)) \in E.$$

We have $g_{c,\,c'}(x) = 1$ when $f(x) \geq c$, $g_{c,\,c'}(x) = (f(x) - c')/(c - c') \in [0,\, 1]$ when $c' \leq f(x) \leq c$, and $g_{c,\,c'}(x) = 0$ when $f(x) \leq c'$. Let c_n be a non-decreasing sequence of positive real numbers such that $0 < c_n < c$ for every n and $c_n \uparrow c$ as $n \to \infty$. Then $h_n := \min_{1 \leq m \leq n} g_{c,\,c_n} \in E$, and $h_n \downarrow 1_{(f \geq c)}$, if $(f \geq c) := \{x \in \Omega \mid f(x) \geq c\}$. It follows from the Levi property in $\overline{E}$ that $1_{(f\geq c)} \in \overline{E}$, that is, $(f \geq c) \in \mathcal{A}^u$, and $1_{(f \geq c)} \in \overline{E}^{u}_{\mathcal{A}}$.

Let F be a finite non-empty subset of $\mathbf{R}_{>0}$. For every $c \in F$, let $c'_F = \min\{c' \in F \mid c' > c\}$, where $c'_F = \infty$ if $c = \max F$. Define

$$f_F = \sum_{c \in F} c\, 1_{(c \leq f < c'_F)},$$

where $(c \leq f < c'_F) = (f \geq c) \setminus (f \geq c') \in \mathcal{A}$, hence $f_F \in \overline{E}^u_{\mathcal{A}}$. We have $f_F \leq f$ and

$$\begin{aligned} u(f) &\geq u(f_F) = \sum_{c \in F} c\, u(1_{(c \leq f < c'_F)}) \\ &= \sum_{c \in F} c\, \mu((c \leq f < c'_F)) = \int f_F(x)\, \mu(\mathrm{d}x). \end{aligned}$$

If F_n is an increasing sequence of finite subsets of $\mathbf{R}_{>0}$ such that the union of all the F_n is dense in $\mathbf{R}_{>0}$, then $f_{F_n} \uparrow f$ as $n \to \infty$. Because the integrals of the f_{F_n} are bounded from above by $u(f)$, it follows from the Levi property for Lebesgue integration with respect to μ that f is Lebesgue integrable with respect to μ and

$$\int f(x)\, \mu(\mathrm{d}x) = \lim_{n \to \infty} \int f_{F_n}(x)\, \mu(\mathrm{d}x).$$

This is actually Lebesgue's original definition of the Lebesgue integral. On the other hand the Levi property for $u : \overline{E} \to \mathbf{R}$ implies that $u(f) = \lim_{n \to \infty} u(f_{F_n})$ where

$$u(f_{F_n}) = \int f_{F_n}(x)\, \mu(\mathrm{d}x)$$

for every n. It follows that $u(f) = \int f(x)\, \mu(\mathrm{d}x)$.

We have therefore proved that $E \subset \overline{E_\mu}^{\mathrm{I}_\mu}$ and $u|_E = \mathrm{I}_\mu\,|_E$. Because the Daniell closure of I_μ is I_μ, the Daniell cloure of $u|_E = \mathrm{I}_\mu\,|_E$ is contained in the Daniell closure of I_μ, that is, $\overline{E} \subset \overline{E_\mu}^{\mathrm{I}_\mu}$ and $u|_{\overline{E}} = \mathrm{I}_\mu\,|_{\overline{E}}$. Because on the other hand $E_\mu \subset \overline{E}$ and $u|_{\overline{E}}$ is Daniell closed, we also have $\overline{E_\mu}^{\mathrm{I}_\mu} \subset \overline{E_\mu}^{u} \subset \overline{E}$, and therefore $\overline{E} = \overline{E_\mu}^{\mathrm{I}_\mu}$ and $u = \mathrm{I}_\mu$ on $\overline{E}$. □

This manuscript grew out of an attempt to prove the following theorem without using too much measure theory.

Theorem 1.3 *Let Ω be an open subset of $\mathbf{R}^n$, and let u be a distribution on Ω. Then u is positive in the sense that $u(\varphi) \geq 0$ whenever $\varphi \in \mathrm{C}_0^\infty(\Omega)$ and $\varphi \geq 0$, if and only if there is a positive measure μ on Ω, such that for every $\varphi \in \mathrm{C}_0^\infty(\mathbf{R}^n)$ we have that φ is Lebesgue integrable with respect to μ and $u(\varphi) = \int \varphi(x)\, \mu(\mathrm{d}x)$.*

Proof The "if" part is obvious.

For the "only if" part, let u be a positive distribution on Ω. Let K be a compact subset of Ω. Choose $\chi \in \mathrm{C}_0^\infty(\Omega)$ such that $\chi \geq 1_K$. If $\varphi \in \mathrm{C}_0^\infty(\Omega)$ and $\mathrm{supp}(\varphi) \subset K$, then $\pm\varphi \leq (\sup|\varphi|)\,\chi$, hence $(\sup|\varphi|)\, u(\chi) - (\pm u(\varphi) = u((\sup|\varphi|)\,\chi - (\pm\varphi)) \geq 0$, that is,

$|u(\varphi)| \leq u(\chi) \sup |\varphi|$. It follows that u extends to a continuous linear form u on the space $\mathrm{C}_0(\Omega)$ of all continuous functions with compact support on Ω, where $u(\varphi) \geq 0$ if $\varphi \in \mathrm{C}_0(\Omega)$, $\varphi \geq 0$.

If $C_0(\Omega) \ni \varphi_n \downarrow 0$, then we have for evey n that $\operatorname{supp}\varphi_n \subset K := \operatorname{supp}\varphi_1$, where K is a compact subset of Ω. The theorem of Dini now implies that the sequence φ_n converges uniformly to zero, and the continuity of the linear form u on $\mathrm{C}_0(\Omega)$ implies that $u(\varphi_n) \to 0$ as $n \to \infty$.

We conclude that the space $E = \mathrm{C}_0(\Omega)$ and the linear form $u : E \to \mathbf{R}$ satisfy (1.5), the Stone property, (1.8), and (1.9). Theorem 1.2 therefore implies that there is a positive measure μ such that, for every $\varphi \in \mathrm{C}_0(\Omega)$, φ is Lebesgue integrable with respect to μ and $u(\varphi) = \int \varphi(x)\,\mu(\mathrm{d}x)$. □

The condition for the measure μ, that every $\varphi \in \mathrm{C}_0^\infty(\Omega)$ is Lebesgue integrable with respect tot μ, is not very direct in terms of the measure μ itself. This is remedied by the following proposition. Note that 1_A is μ-integrable if and only if A is measurable with respect to μ and $\mu(A) < \infty$.

Proposition 1.4 *Every $\varphi \in \mathrm{C}_0^\infty(\Omega)$ is μ-integrable $\Rightarrow$ 1_A is μ-integrable for every relatively compact open subset A of Ω $\Rightarrow$ every $\varphi \in \mathrm{C}_0(\Omega)$ is μ-integrable.*

Proof We have $f \in \overline{E}$ if and only if f is Lebesgue integrable with respect to μ.

Suppose that $\mathrm{C}_0^\infty(\Omega) \subset \overline{E}$ and that A is a relatively compact open subset of Ω. Choose $\chi \in \mathrm{C}_0^\infty(\Omega)$ and an increasing sequence χ_n in C_0^∞ such that $\chi_n \uparrow 1_A \leq \chi$. Because $u(\chi_n) \leq u(\chi)$ for every n, it follows from the Levi property that $1_A \in \overline{E}$.

Conversely, suppose that $1_A \in \overline{E}$ for every relatively compact open subset A of Ω. Let $\varphi \in \mathrm{C}_0(\Omega)$ and $\varphi \geq 0$. Then $(\varphi > c)$ is a relatively compact open subset of Ω for every $c \in \mathbf{R}_{>0}$, hence $1_{(\varphi > c)} \in \overline{E}$. If $0 < c_n < c$ and $c_n \uparrow c$, then then $1_{(\varphi > c_n)} \downarrow 1_{(\varphi \geq c)}$, and the Levi property implies that $1_{(\varphi \geq c)} \in \overline{E}$. Because the $A \subset \Omega$ such that $1_A \in \overline{E}$ form a ring, we have fore every $a, b \in \mathbf{R}$ such that $0 < a < b$ that $1_{(a \leq \varphi < b)} \in \overline{E}$. Approximating φ in a monotone way by a sequence of linear combinations of functions $1_{(a \leq \varphi < b)}$ as in the proof of Theorem 1.2, we conclude that $\varphi \in \overline{E}$. For any $\varphi \in \mathrm{C}_0(\Omega)$ we have $\varphi = \max(\varphi, 0) - \max(-\varphi, 0)$, where $\max(\varphi, 0)$ and $\max(-\varphi, 0)$ are ≥ 0 elements of $\mathrm{C}_0(\Omega)$. Therefore $\mathrm{C}_0(\Omega) \subset \overline{E}$. □

Daniell [2, Sec. 3, 4] also proved the following theorem.

Theorem 1.5 *Let E be a vector space of real valued functions on Ω with the Riesz property (1.5), and let $v : E \to \mathbf{R}$ be a linear form on E which has the monotone convergence property (1.9) with u replaced by v. Assume furthermore that if $f \in E$, $f \geq 0$, then*

$$v_+(f) := \sup\{v(\varphi) \mid \varphi \in E,\ 0 \leq \varphi \leq f\} < \infty. \tag{1.12}$$

For any $f \in E$, define $v_+(f) = v_+(\max(f, 0)) - v_+(\max(-f, 0))$ and $v_-(f) = v_+(f) - v(f)$. Then $v_+, v_- : E \to \mathbf{R}$ are linear forms on E such that $v_\pm$ satisfy (1.8) and (1.9) with u replaced by $v_\pm$. We have $v = v_+ - v_-$.

For a proof, see Section 6.

Corollary 1.6 *Let Ω be an open subset of $\mathbf{R}^n$, and let v be a distribution on Ω. Then v is of order zero if and only if there exist positive measures μ_+ and μ_- on Ω, such that for every $\varphi \in \mathrm{C}_0^\infty(\mathbf{R}^n)$ we have that φ is Lebesgue integrable both with respect to μ_+ and to μ_-, and $v(\varphi) = \int \varphi(x)\,\mu_+(\mathrm{d}x) - \int \varphi(x)\,\mu_-(\mathrm{d}x)$.*

Proof The condition that v is of order zero means that for every compact subset K of Ω there exists a positive constant $C = C(K)$ such that $|v(\varphi)| \leq C \max|\varphi|$ for every $\varphi \in \mathrm{C}_0^\infty(\Omega)$ with support of φ contained in K. It follows that v has an extension to a continuous linear from on $\mathrm{C}_0(K)$, denoted by the same letter, for which we have the same estimate.

The "if" part now follows from

$$\begin{aligned} & |\int \varphi(x)\,\mu_+(\mathrm{d}x) - \int \varphi(x)\,\mu_-(\mathrm{d}x)| \leq |\int \varphi(x)\,\mu_+(\mathrm{d}x)| + |\int \varphi(x)\,\mu_-(\mathrm{d}x)| \\ \leq\ & (\sup|\varphi|)\,\mu_+(K) + (\sup|\varphi|)\,\mu_-(K) = (\mu_+(K) + \mu_-(K))\,\sup|\varphi|, \end{aligned}$$

if $\varphi \in \mathrm{C}_0^\infty(\Omega)$ has support contained in the compact subset K of Ω.

For the "only if" part we observe that if v is of order zero, then the theorem of Dini implies that (1.9) holds with u replaced by v. Furthermore, if $f,\, g \in \mathrm{C}_0(\Omega)$, $0 \leq g \leq f$, then $\operatorname{supp} g \subset \operatorname{supp} f$, and therefore $|v(g)| \leq C(\operatorname{supp} f)\,\sup|g| \leq C(\operatorname{supp} f)\,\sup|f|$, and it follows that $v_+(f) \leq C(\operatorname{supp} f)\,\sup|f| < \infty$. Theorem 1.3 with u replaced by $v_\pm$ implies that there exist positive measures $\mu_\pm$ such that every $\varphi \in \mathrm{C}_0(\Omega)$ is Lebesgue integrable with respect to $\mu_\pm$ and $v_\pm(\varphi) = \int \varphi(x)\,\mu_\pm(\mathrm{d}x)$. This implies the desired expression for v in terms of μ_+ and μ_-, because $v(\varphi) = v_+(\varphi) - v_-(\varphi)$. □

Let K be a compact subset of $\mathbf{R}^n$. A *Radon measure* on K is a measure on a σ-algebra of subsets which contains all open subsets of K, or equivalently all compact susbets of K. As the *Borel σ-algebra* of K is defined as the smallest σ-algebra of subsets of K which contains every open subset of K, a radon measure on K is equivalently described as a measure on a σ-algebra of subsets of K which contains the Borel σ-algebra of K. Let $\mathrm{C}(K)$ denote the space of all continuous functions on K, provided with the topology of uniform convergence. Radon [7, Sec. III] proved that $v : \mathrm{C}(K) \to \mathbf{R}$ is a continuous linear form on $\mathrm{C}(K)$, if and only if there exists a Radon measure ν on K such that, for every $f \in \mathrm{C}(K)$, we have $v(f) = \int f(x)\,\nu(\mathrm{d}x)$. [2] Sometimes one sees in the literature that a *Radon measure* is defined as a continuous linear form on a space $\mathrm{C}(K)$, but this definition would reduce the theorem to a tautology. Corollary 1.6 is equivalent to the theorem of Radon [7, Sec. III].

[2] Radon [7] denoted the integral by $\int f\,\mathrm{d}\nu$. I prefer the notation $\int f(x)\,\nu(\mathrm{d}x)$ which I learned from Stroock [12, bottom of p. 42], because it reminds of the definition of a measure ν as a function which assigns a real number to a subset of K, where $\mathrm{d}x$ stands for an "infinitesimally small subset of K containing the point x". If one wants to delete the variable x from the notation, then the natural notation would be $\int f\,\nu$, which is a minimal notation, as one needs a symbol for the function f which is integrated, the measure ν with respect to which f is integrated, and for the operation of integrating. Here Stroock [12, bootom of p. 42] used Radon's notation $\int f\,\mathrm{d}\nu$, probably in order to conform to the majority, which is wise.

When $K = [a, b]$ is a bounded and closed interval in $\mathbf{R}$, the theorem of Radon is equivalent to the *Riesz representation theorem* [9], [10], which states that $v : \mathrm{C}([a, b]) \to \mathbf{R}$ is a continuous linear form, if and only if there exists a function $\alpha : [a, b] \to \mathbf{R}$ of bounded variation such that, for each $f \in \mathrm{C}([a, b])$, $v(f) = \int_a^b f(x)\, \mathrm{d}\alpha(x)$, the Stieltjes integral of f with respect to the function α. In the literature one sometimes sees the Riesz representation theorem described as this result only for positive linear forms, when the function α is non-decreasing. This is a bit unfair to Riesz, because [9], [10] definitely treated arbitrary continuous linear formas on $\mathrm{C}([a, b])$. Radon [7, Sec. III] referred to Riesz [9]. In [10], which is slightly later than Radon [7], Riesz did not mention Radon's generalization. Perhaps because the theorem of Radon is "only" a generalization of the Riesz representation theorem, one sometimes sees the Riesz representation theorem described as the theorem that, for any open subset Ω of $\mathbf{R}^n$, every continuous linear form on $\mathrm{C}_0(\Omega)$ is equal to integration with respect to a Radon measure on Ω. Daniell [2, Introduction] referred to Radon [7], Young [13], Riesz [10], Moore [6], and Hildebrandt [4], but did not mention the term Lebesgue integral or refer to Lebesgue.

In order to convince myself of the statements of Daniell [2], I have written down all the proofs in the remainder of this manuscript. I also have added some results in order to make a more complete fit with the theory of Lebesgue integration with respect to a measure. A very thorough exposition of the Daniell closure and its relation to Lebesgue integration with respect to a measure can be found in the book of Constantinescu, Weber and Sontag [1]. It also contains a nice discussion [1, Appendix] of the history of the theory of integration.

A bit oddly, I could not find Theorem 1.5 in [1], which has a prominent position in Daniell's paper [2, Sec. 3, 4, 8]. Nor could I find in [1] the theorem that the space L of equivalence classes of elements in $\overline{E}$ is complete with respect to the norm $f \mapsto u(|f|)$. This theorem is not discussed in Daniell [2] either. When $u = \mathrm{I}_\mu$, then $L = \mathrm{L}^1(\mu)$ is the space of all equivalence classes of Lebesgue integrable functions. The completeness of $\mathrm{L}^p(\mu)$, the space of all equivalence classes of measurable functions f such that $|f|^p$ is integrable with respect to μ, provided with the norm $f \mapsto (\int |f|^2\, \mu)^{1/p}$, is known as the *Riesz-Fischer theorem*, after Fischer [3], where it has been proved for $\Omega = [a, b]$ and $p = 2$, and Riesz [8, p. 468] for $\Omega = [a, b]$ and $1 < p < \infty$, with a footnote that it also holds for $0 < p \leq 1$. Because of the importance of the Banach space $\mathrm{L}^1(\mu)$ in analysis, I have included a discussion of L.

2 The extension to $E^{\uparrow}$

Throughout this paper, Ω is a set, E a vector space of real valued functions on Ω with the Riesz property (1.5), and $u : E \to \mathbf{R}$ a linear form on E which is positive in the sense of (1.8) and has the monotone convergence property (1.9).

Let $E^{\uparrow}$ denote the set of all $f : \Omega \to (\mathbf{R} \cup \{\infty\})$ for which there exists a sequence f_n in E such that $f_n \uparrow f$ as $n \to \infty$.

Lemma 2.1 *Let f_n be a sequence in E and $f_n \uparrow f \in E^{\uparrow}$ as $n \to \infty$. Then $u(f_n)$ is a non-decreasing sequence in $\mathbf{R}$ and therefore converges to an element in $\mathbf{R} \cup \{\infty\}$ as $n \to \infty$. If $h \in E$ and $f \geq h$, then $\lim_{n \to \infty} u(f_n) \geq u(h)$.*

Proof The monotonicity of the sequence $u(f_n)$ has been discussed after (1.9). For the second statement, let $g_n := \min(f_n, h)$. Then (1.5) implies that $g_n \in E$, and we have $g_n \uparrow h$, and therefore (1.9) implies that $u(g_n) \uparrow u(h)$. Because $f_n \geq g_n$, we have $u(f_n) \geq u(g_n)$ for every n, and the conclusion follows. □

Lemma 2.2 *Let f_n and g_n be sequences in E and $f_n \uparrow f \in E^{\uparrow}$, $g_n \uparrow g \in R^{+}$ as $n \to \infty$. If $f \geq g$ then $\lim_{n\to\infty} u(f_n) \geq \lim_{n\to\infty} u(g_n)$. If $f = g$ then $\lim_{n\to\infty} u(f_n) = \lim_{n\to\infty} u(g_n)$.*

Proof Lemma 2.1 with h replaced by g_m implies that $\lim_{n\to\infty} u(f_n) \geq u(g_m)$ for every m. Taking the limit for $m \to \infty$ yields the first inequality. if $f = g$ we can interchange the sequences f_n and g_n, which yields the inequality in the other direction. □

It follows from Lemma 2.2 that there is a unique function $u^{\uparrow} : E^{\uparrow} \to \mathbf{R} \cup \{\infty\}$ such that $\lim_{n\infty} u(f_n) = u^{\uparrow}(f)$ whenever $f \in E^{\uparrow}$ and f_n is a sequence in E such that $f_n \uparrow f$ as $n \to \infty$. If $f \in E$, then we have $f_n \uparrow f$ for the constant sequence $f_n = f$, hence $u^{\uparrow}(f) = u(f)$. That is, the function $u^{\uparrow} : E^{\uparrow} \to \mathbf{R} \cup \{\infty\}$ is an extension of the function $u : E \to \mathbf{R}$. We write $u^{\uparrow} = u$ in the sequel.

Lemma 2.3 i) *If $f \in E^{\uparrow}$ and $c \in \mathbf{R}_{>0}$, then $c\, f \in E^{\uparrow}$ and $u(c\, f) = c\, u(f)$.*
ii) *If $f,\, g \in E^{\uparrow}$, then $f + g \in E^{\uparrow}$ and $u(f + g) = u(f) + u(g)$.*
iii) *If $f,\, g \in E^{\uparrow}$, then max$(f,\, g) \in E^{\uparrow}$ and $\min(f,\, g) \in E^{\uparrow}$.*

Proof If f_n is a sequence in E then $f_n \uparrow f$ if and only if $c\, f_n \uparrow c\, f$, as $n \to \infty$, whereas $u(c\, f_n) = c\, u(f_n)$. This proves i).

For ii) we observe that $f,\, g \in E$ implies that there are sequences f_n and g_n in E such that $f_n \uparrow f$ and $g_n \uparrow g$, hence $f_n + g_n \uparrow f + g$ as $n \to \infty$. Therefore $f + g \in E^{\uparrow}$ and $u(f + g) = \lim_{n\to\infty} u(f_n + g_n) = \lim_{n\to\infty} (u(f_n) + u(g_n)) = \lim_{n\to\infty} u(f_n) + \lim_{n\to\infty} u(g_n) = u(f) + u(g)$. □

Lemma 2.4 i) *If $f,\, g \in E^{\uparrow}$ and $f \leq g$, then $u(f) \leq u(g)$.*
ii) *If f_n is a sequence in $E^{\uparrow}$, $f \in F$ and $f_n \uparrow f$, then $f \in E^{\uparrow}$ and $\lim_{n\to\infty} u(f_n) = u(f)$.*

Proof

i) This follows from the first statement in Lemma 2.2.

ii) It follows from i) that the sequence $u(f_n)$ is non-decreasing, and therefore convergers to an elment of $\mathbf{R} \cup \{\infty\}$ as $n \to \infty$. By definition, $f_n \in E^{\uparrow}$ means that there exists a sequence $m \mapsto f_{n,\, m}$ in E such that $f_{n,\, m} \uparrow f_n$ and $u(f_n) = \lim_{m\to\infty} u(f_{n,\, m})$, as $m \to \infty$. Define

$$g_l = \max_{1 \leq n \leq l,\; 1 \leq m \leq l} f_{n,\, m}.$$

Then $g_l \in E$, and $g_l \leq g_{l+1}$, hence $g_l \uparrow g \in E^{\uparrow}$ and $\lim_{l\to\infty} u(g_l) = u(g)$. For every $1 \leq n \leq l$ and $1 \leq m \leq l$ we have $f_{n,\, m} \leq f_n \leq f_l$, hence $g_l \leq f_l$, and therefore $g =$

$\lim_{l\to\infty} g_l \leq \lim_{l\to\infty} f_l = f$. On the other hand, for every $l \geq n$ we have $f_{n,l} \leq g_l$, hence $f_n = \lim_{l\to\infty} f_{n,l} \leq \lim_{l\to\infty} g_l = g$, hence $f = \lim_{n\to\infty} f_n \leq g$. Combining $g \leq f$ and $f \leq g$, we conclude that $f = g \in E^{\uparrow}$.

Because $g_l \leq f_l$, we have $u(g_l) \leq u(f_l)$, hence $u(f) = u(g) = \lim_{l\to\infty} u(g_l) \leq \lim_{l\to\infty} u(f_l)$. Furthermore, if $l \geq n$ then $f_{n,l} \leq g_l$, hence $u(f_{n,l}) \leq u(g_l)$. Therefore $u(f_n) = \lim_{l\to\infty} u(f_{n,l}) \leq \lim_{l\to\infty} u(g_l) = u(g) = u(f)$ for every n, hence $\lim_{n\to\infty} u(f_n) \leq u(f)$. □

3 The upper and the lower form

For every $f : \Omega \to \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty\} \cup \{\infty\}$, define $\overline{u}(f)$ by (1.10).

Lemma 3.1 i) *If $f : \Omega \to \overline{\mathbf{R}}$ and $c \in \mathbf{R}_{>0}$, then $\overline{u}(c\, f) = c\, \overline{u}(f)$.*
ii) *Let $f,\, g : \Omega \to \overline{\mathbf{R}}$, $\overline{u}(f) < \infty$, and $\overline{u}(g) < \infty$. Then $\overline{u}(\min(f,\, g)) \leq \overline{u}(\max(f,\, g)) < \infty$, and $\overline{u}(\max(f,\, g)) + \overline{u}(\min(f,\, g)) \leq \overline{u}(f) + \overline{u}(g)$.*
iii) *If $f,\, g : \Omega \to \overline{\mathbf{R}}$ and $f \leq g$, then $\overline{u}(f) \leq \overline{u}(g)$.*
iv) *If $0 \leq f_n \uparrow f$, then $\overline{u}(f_n) \uparrow \overline{u}(f)$ as $n \to \infty$.*
v) *If $f \in E^{\uparrow}$ then $\overline{u}(f) = u(f)$.*

Proof

i) If $\varphi \in E^{\uparrow}$ then $f \leq \varphi$ if and only if $c\, f \leq c\, \varphi$, whereas $u(c\, \varphi) = c\, u(\varphi)$ in view of i) in Lemma 2.3.

ii) Let $\varphi,\, \psi \in E^{\uparrow}$, $f \leq \varphi$, $g \leq \varphi$, where we can arrange that $u(\varphi)$ and $u(psi)$ are finite. Then it follows from iii) in Lemma 2.3 that $\max(f,\, g) \leq \max(\varphi,\, \psi) \in E^{\uparrow}$, $\min(f,\, g) \leq \min(\varphi,\, \psi) \in E^{\uparrow}$, whereas (1.5), which also holds in $E^{\uparrow}$, and ii) in Lemma 2.3 imply that

$$u(\max(\varphi,\, \psi)) + u(\min(\varphi,\, \psi)) = u(\max(\varphi,\, \psi) + \min(\varphi,\, \psi)) = u(\varphi + \psi) = u(\varphi) + u(\psi). \quad (3.1)$$

The infimum over all these φ, ψ of the right hand side of (3.1) is equal to $\overline{u}(f) + \overline{u}(g) < \infty$, and the desired inequalities now follow from the observation that $\overline{u}(\max(f,\, g)) \leq u(\max(\varphi,\, \psi))$ and $\overline{u}(\min(f,\, g)) \leq u(\min(\varphi,\, \psi))$.

iii) and v) If $\psi \in E^{\uparrow}$, $g \leq \psi$, then $f \leq \psi$, hence $\overline{u}(f) \leq u(\psi)$. Taking the infimum over all these ψ we obtain $\overline{u}(f) \leq \overline{u}(g)$. If $f \in E^{\uparrow}$, then $u(f) \leq u(\psi)$ for all $E^{\uparrow} \ni \psi \geq f$, hence $u(f) \leq \overline{u}(f)$, whereas taking $\psi = f$ we obtain that $\overline{u}(f) \leq u(f)$.

iv) If $\overline{u}(f_n) = \infty$ for some n, then it follows from iii) and $f_n \leq f$ that $\overline{u}(f) = \infty$, and we are done. Therefore we assume in the sequel that $\overline{u}(f_n) \in \mathbf{R}$ for every n. It follows from iii) that the sequence $\overline{u}(f_n)$ is non-decreasing, hence converges in $\mathbf{R} \cup \{\infty\}$, and because $f_n \leq f$ hence $\overline{u}(f_n) \leq \overline{u}(f)$ for every n, we have $\lim_{n\to\infty} \overline{u}(f_n) \leq \overline{u}(f)$.

Let $\epsilon \in \mathbf{R}_{>0}$ and choose a sequence $\epsilon_n \in \mathbf{R}_{>0}$ of which the sum is $\leq \epsilon$. Then there exists $\varphi_n \in E^{\uparrow}$ such that $f_n \leq \varphi_n$ and $u(\varphi_n) \leq \overline{u}(f_n) + \epsilon_n$. Let $\psi_n = \max_{1 \leq m \leq n} \varphi_m$. Then it folows from Lemma 2.3 by induction on n that $f_n \leq \psi_n = \max(\psi_{n-1}, \varphi_n) \in E^{\uparrow}$, where the sequence ψ_n is non-decreasing. We have $\psi_n + \chi_n = \psi_{n-1} + \varphi_n$ if $\chi_n := \min(\psi_{n-1},\, \varphi_n)$, $f_{n-1} \leq \chi_n \in E^{\uparrow}$,

$$\begin{aligned} u(\psi_n) + \overline{u}(f_{n-1}) &\leq u(\psi_n) + u(\chi_n) = u(\psi_n + \chi_n) \\ &= u(\psi_{n-1} + \varphi_n) = u(\psi_{n-1}) + u(\varphi_n) \leq \overline{u}(f_{n-1}) + \eta_{n-1} + \overline{u}(f_n) + \epsilon_n, \end{aligned}$$

hence $u(\psi_n) \leq \overline{u}(f_n) + \eta_n$ where $\eta_n := \eta_{n-1} + \epsilon_n$, and it follows by induction on n that

$$u(\psi_n) \leq \overline{u}(f_n) + \sum_{m=1}^{n} \epsilon_n \leq \overline{u}(f_n) + \epsilon.$$

for every n. Therefore $\psi_n \uparrow \psi \in E^\uparrow$, $f_n \leq \psi_n$ for all n implies that $f \leq \psi$, and therefore

$$\overline{u}(f) \leq u(\psi) = \lim_{n\to\infty} u(\psi_n) \leq \lim_{n\to\infty} \overline{u}(f_n) + \epsilon.$$

Because this holds for every $\epsilon > 0$, we conclude that $\overline{u}(f) \leq \lim_{n\to\infty} \overline{u}(f_n)$. □

The counterpart of the set $E^\uparrow$ with the reversed order is the set $E^\downarrow$ of all $f : \Omega \to \mathbf{R} \cup \{-\infty\}$ for which there exists a sequence f_n in E such that $f_n \downarrow f$. The function $u : E \to \mathbf{R}$ extends to a function $u : E^\downarrow \to \mathbf{R} \cup \{-\infty\}$ such that $u(f) = \lim_{n\to\infty} u(f_n)$ whenever f_n is a sequence in E such that $f_n \downarrow f$. We have the analogous properties for $u : E^\downarrow \to \mathbf{R} \cup \{-\infty\}$ as we had for $u : E^\uparrow \to \mathbf{R} \cup \{\infty\}$, where a short proof is by observing that $f \in E^\downarrow$ if and only if $-f \in E^\uparrow$ and, if this is the case, $u(f) = -u(-f)$.

The counterpart of $\overline{u}$ with the reversed order is

$$\underline{u}(f) := \sup\{u(\varphi) \mid \varphi \in E^\downarrow \text{ and } \varphi \leq f\} \in \overline{\mathbf{R}} \tag{3.2}$$

for every $f : \Omega \to \overline{\mathbf{R}}$, where $\underline{u}(f) = -\infty$ if there is no $\varphi \in E^\downarrow$ such that $\varphi \leq f$. We have the analogous properties for $\underline{u}$ as we had for $\overline{u}$, where the shortest proof is by observing that a short proof is by observing that $u(f) = -\overline{u}(-f)$ for every $f : \Omega \to \overline{\mathbf{R}}$.

Lemma 3.2 *Let $f,\, g : \Omega \to \overline{\mathbf{R}}$. Then:*
i) $\underline{u}(f) \leq \overline{u}(f)$.
ii) If $\overline{u}(f)$, $\underline{u}(f)$, $\overline{u}(g)$, and $\underline{u}(g)$ all are finite real numbers, then $\overline{u}(\max(f,\, g))$, $\underline{u}(\max(f,\, g))$, $\overline{u}(\min(f,\, g))$, and $\underline{u}(\min(f,\, g))$ are finite, and the sum of the non-negative numbers $\overline{u}(\max(f,\, g)) - \underline{u}(\max(f,\, g))$ and $\overline{u}(\min(f,\, g)) - \underline{u}(\min(f,\, g))$ is majorated by the sum of the non-negative numbers $\overline{u}(f) - \underline{u}(f)$ and $\overline{u}(g) - \underline{u}(g)$.
iii) If $\overline{u}(f)$ and $\underline{u}(f)$ are finite, then $\overline{u}(|f|)$ is finite and $\overline{u}(|f|) - \underline{u}(|f|) \leq \overline{u}(f) - \underline{u}(f)$.

Proof

i) Let $\varphi, \psi \in E^\uparrow$ such that $f \leq \varphi$ and $-f \leq \psi$. We have sequences φ_n and ψ_n in E such that $\varphi_n \uparrow \varphi$ and $\psi_n \uparrow \psi$, which implies $\varphi_n(x) \uparrow \varphi(x) \geq f(x)$ and $\psi_n(x) \uparrow \psi(x) \geq -f(x)$ for every $x \in \Omega$. If $f(x) \in \mathbf{R}$, then we can add the inequalities and obtain $\varphi(x) + \psi(x) \geq 0$. If $f(x) = \infty$, then $\varphi(x) = \infty$, and if $f(x) = -\infty$, then $\psi(x) = \infty$, and because $\varphi(x),\, \psi(x) \in \mathbf{R} \cup \{\infty\}$ it follows that $\varphi(x) + \psi(x) \geq 0$ for all $x \in \Omega$, that is $\varphi + \psi \geq 0$. We gave this somewhat roundabout argument because if $f(x) = \infty$, then $-f(x) = -\infty$, and $f(x) + (-f(x))$ is not defined. ii) in Lemma 2.3 now leads to $\varphi + \psi \in E^\uparrow$ and $0 \leq u(0) \leq u(\varphi + \psi) = u(\varphi) + u(\psi)$, which implies that $-u(\psi) \leq u(\varphi)$. Taking the infimum over all $E^\uparrow \ni \varphi \geq f$ of the right hand side we obtain $-u(\psi) \leq \overline{u}(f)$, and then taking the supremum over all $E^\uparrow \ni \psi \geq -f$ of the left hand side we obtain $\underline{u}(f) = -\overline{u}(-f) \leq \overline{u}(f)$.

ii) Let $\varphi,\,\psi \in E^{\uparrow}$, $\varphi',\,\psi' \in E^{\uparrow}$, $\varphi' \leq f \leq \varphi$, and $\psi' \leq g \leq \psi$, where, due the finiteness assumptions, we can arrange that the u-values of φ, ψ, φ', and ψ' all are finite. It follows from (3.1) and the analogous equation with φ and ψ repalced by φ' and ψ', respectively, that the numbers $u(\max(\varphi,\,\psi))$, $u(\min(\varphi,\,\psi))$, $u(\max(\varphi',\,\psi'))$, and $u(\min(\varphi',\,\psi'))$ all are finite, and that the sum of the non-negative real numbers $u(\max(\varphi,\,\psi)) - u(\max(\varphi',\psi'))$ and $u(\min(\varphi,\,\psi)) - u(\min(\varphi',\psi'))$ is equal to the sum of the non-negative real numbers $u(\varphi) - u(\varphi')$ and $u(\psi) - u(\psi')$. On the other hand $\max(\varphi',\psi') \leq \max(f,\,g) \leq \max(\varphi,\,\psi)$ and $\min(\varphi',\psi') \leq \min(f,\,g) \leq \min(\varphi,\,\psi)$ imply that all the numbers $\overline{u}(\max(f,\,g))$, $\underline{u}(\max(f,\,g))$, $\overline{u}(\min(f,\,g))$, and $\underline{u}(\min(f,\,g))$ are finite, and $\overline{u}(\max(f,\,g)) - \underline{u}(\max(f,\,g)) \leq u(\max(\varphi,\,\psi)) - u(\max(\varphi',\psi'))$, $\overline{u}(\min(f,\,g)) - \underline{u}(\min(f,\,g)) \leq u(\min(\varphi,\,\psi)) - u(\min(\varphi',\psi'))$. The desired inequality follows because the infimum over all these φ and φ' of $u(\varphi) - u(\varphi')$ is equal to $\overline{u}(f) - \underline{u}(f)$, and the infimum over all these ψ and ψ' of $u(\psi) - u(\psi')$ is equal to $\overline{u}(g) - \underline{u}(g)$.

iii) Apply ii) with $g = -f$, when $\max(f,\,-f) = |f|$, $\min(f,\,-f) = -|f|$, $\overline{u}(-|f|) = -\underline{u}(|f|)$, $\underline{u}(-|f|) = -\overline{u}(|f|)$, $\overline{u}(-f) = -\underline{u}(f)$, and $\underline{u}(-f) = -\overline{u}(f)$. $\square$

4 The closure

The closure of E with respect to the linear form u is defined as the set $\overline{E} = \overline{E}^u$ of all $f : \Omega \to \overline{\mathbf{R}}$ such that $\overline{u}(f) = \underline{u}(f) \in \mathbf{R}$.

Lemma 4.1 *If $f \in E^{\uparrow}$, then $f \in \overline{E}$ if and only if $u(f) < \infty$, and in this case $\overline{u}(f) = u(f) = \underline{u}(f)$. If $f \in E^{\downarrow}$, then $f \in \overline{E}$ if and only if $u(f) > -\infty$, and in this case $\overline{u}(f) = u(f) = \underline{u}(f)$.*

Proof Let $f \in E^{\uparrow}$. It follows from v) in Lemma 3.1 and the definition of $\overline{E}$ that $u(f) = \overline{u}(f) = \underline{u}(f) \in \mathbf{R}$ if $f \in \overline{E}$.

Conversely, assume that $u(f) < \infty$. By definition, $\lim_{n\to\infty} u(f_n)$ for a sequence $f_n \in E$ such that $f_n \uparrow f$. It follows that $-f \leq -f_n \in E$, hence $\overline{u}(-f) \leq u(-f_n) = -u(f_n)$, and taking the limit for $n \to \infty$ we obtain that $-\underline{u}(f) = \overline{u}(-f) \leq -u(f)$, hence $\underline{u}(f) \geq u(f) = \overline{u}(f)$. In combination with i) in Lemma 3.2 we conclude that $\underline{u}(f) = u(f) = \overline{u}(f)$.

Replacing f by $-f$, the second statement in the lemma follows from the first one. $\square$

Lemma 4.1 implies that if, for every $f \in \overline{E}$ we define $u(f) = \overline{u}(f) = \underline{u}(f)$, then the function $u : \overline{E} \to \mathbf{R}$ agrees with the previously defined functions u on $E^{\uparrow} \cap \overline{E} = \{f \in E^{\uparrow} \mid u(f) < \infty\}$ and on $E^{\downarrow} \cap \overline{E} = \{f \in E^{\downarrow} \mid u(f) > -\infty\}$. Because $u(f) \in \mathbf{R}$ for every $u \in E$, Lemma 4.1 and the fact that $u : E^{\uparrow} \to \mathbf{R} \cup \{\infty\}$ was an extension of $u : E \to \mathbf{R}$, this implies that $u : \overline{E} \to \mathbf{R}$ is also an extension of $u : E \to \mathbf{R}$.

Lemma 4.2 *i) If $f \in \overline{E}$, $c \in \mathbf{R}$, $c \neq 0$, then $c\,f \in \overline{E}$ and $u(c\,f) = c\,u(f)$.*
ii) If $f,\,g \in \overline{E}$ and $f \leq g$, then $u(f) \leq u(g)$.
iii) If $f,\,g \in \overline{E}$, then $\max(f,\,g)$, $\min(f,\,g) \in \overline{E}$.
iv) If $f \in \overline{E}$ then $|f| \in \overline{E}$ and $|u(f)| \leq u(|f|)$.

Proof

i) The equations $\overline{u}(-f) = -\underline{u}(f)$ and $\underline{u}(-f) = -\overline{u}(f)$ show that if $f \in \overline{E}$, then $-f \in \overline{E}$ and $u(-f) = -u(f)$. If $c > 0$ then i) in Lemma 3.1 implies that $\overline{u}(c\,f) = c\,\overline{u}(f)$ and $\underline{u}(c\,f) = -\overline{u}(-(c\,f)) = -\overline{u}(c\,(-f)) = -(c\,\overline{u}(-f)) = c\,(-\overline{u}(-f)) = c\,\underline{u}(f)$, which implies the statement. When $c < 0$, we write $c\,f = (-c)\,(-f)$ with $-c > 0$, and observe that $f \in \overline{E}$ implies that $-f \in \overline{E}$, $c\,f = (-c)\,(-f) \in \overline{E}$, and $u(c\,f) = u((-c)\,(-f)) = (-c)\,u(-f) = (-c)\,(-u(f)) = c\,u(f)$.

ii) This follows from iii) in Lemma 3.1.

iii) This follows from ii) in Lemma 3.2.

iv) This follows from iii) with $g = -f$. □

For the "almost everywhere" version of the following theorem, see Theorem 5.3.

Theorem 4.3 *If f_n is a monotonous sequence in $\overline{E}$, $f : \Omega \to \overline{\mathbf{R}}$, $f_n \to f$ as $n \to \infty$, and the sequence $u(f_n)$ is bounded, then $f \in \overline{E}$ and $\lim_{n\to\infty} u(f_n) = u(f)$. If $\lim_{n\to\infty} u(f_n) = \pm\infty$, then $\underline{u}(f) = \overline{u}(f) = \infty$.*

Proof Suppose that $f - n \uparrow f$ as $n \to \infty$. For every n and every $\epsilon_N, \epsilon'_n \in \mathbf{R}_{>0}$ there exist $\varphi_n \in E^{\uparrow}$ and $\varphi'_n \in E^{\downarrow}$ such that $\varphi'_n \leq f_n \leq \varphi_n$, $u(\varphi_n) \leq u(f_n) + \epsilon_n$, and $u(\varphi'_n) \geq u(f_n) - \epsilon'_n$. Because $\varphi'_n \leq f$ we have $\underline{u}(f) \geq u(\varphi'_n) \geq u(f_n) - \epsilon'_n$, and choosing the ϵ'_n to converge to zero, it follows that $\underline{u}(f) \geq \lim_{n\to\infty} u(f_n)$, which also proves the last statement.

Now assume that $\lim_{n\to\infty} u(f_n) < \infty$. Define $\psi_n = \max_{1\leq m\leq n} \varphi_n$, that is, the ψ_n are defined by induction on n by $\psi_1 = \varphi_1$ and $\psi_n = \max(\psi_{n-1}, \varphi_n)$ for $n > 1$. It follows from iii) in Lemma 2.3 that $\psi_n \in E^{\uparrow}$, $\psi_n \geq \psi_{n-1}$, and $f_n \leq \varphi_n \leq \psi_n$. Furthermore $f_{n-1} \leq \psi_{n-1}$ and $f_{n-1} \leq f_n \leq \varphi_n$ imply that $f_{n-1} \leq \min(\psi_{n-1}, \varphi_n)$, hence $u(\psi_n) + u(f_{n-1}) \leq u(\max(\psi_{n-1}, \varphi_n)) + u(\min(\psi_{n-1}, \varphi_n)) = u(\psi_{n-1}) + u(\varphi_n) \leq u(\psi_{n-1}) + u(f_n) + \epsilon_n$, where in the middle identity we have used (3.1). It follows that $u(\psi_n) - u(f_n) \leq u(\psi_{n-1}) - u(f_{n-1}) + \epsilon_n$ for $n > 1$, where $u(\psi_1) - u(f_1) = u(\varphi_1) - u(f_1) \leq \epsilon_1$, hence we obtain by induction on n that

$$u(\psi_n) \leq u(f_n) + \sum_{m=1}^{n} \epsilon_m \tag{4.1}$$

for every n. For every $\epsilon \in \mathbf{R}_{>0}$ we can choose the sequence ϵ_n such that its sum is $\leq \epsilon$, when (4.1) implies that $u(\psi_n) \leq u(f_n) + \epsilon$ for every n. We have $\psi := \lim_{n\to\infty} \psi_n \in E^{+}$, $f = \lim_{n\to\infty} f_n \leq \lim_{n\to\infty} \psi_n = \psi$, hence $\overline{u}(f) \leq u(\psi) = \lim_{n\to\infty} u(\psi_n) \leq \lim_{n\to\infty} u(f_n) + \epsilon$. because this holds for every $\epsilon > 0$, the conclusion is that $\underline{u}(f) \leq \overline{u}(f) \leq \lim_{n\to\infty} u(f_n)$. In combination with the previously obtained inequality $\underline{u}(f) \geq \lim_{n\to\infty} u(f_n)$, this proves that $\underline{u}(f) = \overline{u}(f) = \lim_{n\to\infty} u(f_n)$, that is, $f \in \overline{E}$ and $u(f) = \lim_{n\to\infty} u(f_n)$.

The statements in the theorem for the decreasing sequence f_n can be proved by applying the previous to the increasing sequence $-f_n$. □

For the "almost everywhere" version of the following theorem, see Theorem 5.4.

Theorem 4.4 *If f_n is a sequence in $\overline{E}$ which converges pointwise to a function $f : \Omega \to \overline{\mathbf{R}}$, and there exists $\varphi \in \overline{E}$ such that $|f_n| \leq \varphi$ for every n, then $f \in \overline{E}$ and $u(f_n)$ converges to $u(f)$ as $n \to \infty$.*

Proof If follows from Lemma 4.2 that $\varphi,\ -\varphi \in \overline{E}$, $-\varphi \leq f_n \leq \varphi$, and $-u(\varphi) = u(-\varphi) \leq u(f_n) \leq u(\varphi)$ for every n.

Let $g_{n,m} := \max_{0 \leq l \leq m} f_{n+l}$ and $h_{n,m} := \min_{0 \leq l \leq m} f_{n+l}$. Then $g_{n,0} = f_n$, $g_{n,m} = \max(g_{n,m-1},\ f_{n+m})$ for $m > 0$, and it follows in view of iii) in Lemma 4.2 by induction on m that $g_{n,m} \in \overline{E}$ for every n and every $m \geq 0$. Because $-\varphi \leq f_n \leq \varphi$ for all n, we have $-\varphi \leq g_{n,m} \leq \varphi$ and $-\varphi \leq h_{n,m} \leq \varphi$ for all n, m, which in view of ii) in Lemma 4.2 implies that $-u(\varphi) \leq u(g_{n,m}) \leq u(\varphi)$ and $-u(\varphi) \leq u(h_{n,m}) \leq u(\varphi)$ for all n, m.

The sequence $m \mapsto g_{n,m}$ is increasing, let g_n denote its limit. The pointwise convergence of f_n to f implies that the sequence g_n converges pointwise to f as well. Because $-u(\varphi) \leq u(g_{n,m}) \leq u(\varphi)$ for all m, and Theorem 4.3 for increasing sequences implies that $g_n \in \overline{E}$ and $u(g_n) = \lim_{m \to \infty} u(g_{n,m}) \in [-u(\varphi),\ u(\varphi)$. The sequence g_n is decreasing, and Theorem 4.3 for decreasing sequences implies that $f = \lim_{n \to \infty} g_n \in \overline{E}$, and $u(f) = \lim_{n \to \infty} u(g_n)$. It follows that for every $\epsilon > 0$ there exists an N such that, for every $n \geq N$, $u(g_n) \leq u(f) + \epsilon$. Because $f_n \leq g_{n,m} \leq g_n$, it follows that $u(f_n) \leq u(g_n) \leq u(f) + \epsilon$. The analogous reasoning with the $g_{n,m}$ replaced by $h_{n,m}$ and alll inequalities reversed leads to the existence, for every $\eta \in \mathbf{R}_{>0}$, of an M such that $u(f_n) \geq u(f) - \eta$ for all $n \geq M$. $\square$

The following theorem yields an equivalent characterization of $\overline{E}$ as the closure of E in the space of functions $f : \Omega \to \overline{\mathbf{R}}$ with respect to the "distance" $\overline{u}(|f - \varphi|)$ of f to elements φ of E. The theorem also yields an equivalent characterization of $u(f)$ as the limit value of the $u(\varphi)$ as $\varphi \in E$ and $\overline{u}(|f - \varphi|) \to 0$. For the "almost everywhere version of Theorem 4.5, see Theorem 5.5.

Theorem 4.5 *Let $f : \Omega \to \overline{\mathbf{R}}$. Then $f \in \overline{E}$ if and only if for every $\epsilon \in \mathbf{R}_{>0}$ there exists $\varphi \in E$ such that $\overline{u}(|f - \varphi|) \leq \epsilon$. If $f \in \overline{E}$ and $\varphi \in E$, then $|f - \varphi| \in \overline{E}$ and $|u(f) - u(\varphi)| \leq u(|f - \varphi|)$.*

Proof If $\varphi \in E$, then $\varphi : \Omega \to \mathbf{R}$, hence $f - \varphi$, $\varphi - f$, and $|f - \varphi| = \max(f - \varphi,\ \varphi - f)$ are well-defined functions on Ω with values in $\overline{\mathbf{R}}$.

"Only if" Let $f \in \overline{E}$ and $\epsilon > 0$. There exists $g \in E^{\uparrow}$ such that $f \leq g$ and $u(g) \leq u(f) + \epsilon/2$. In turn there exists a sequence g_n in E such that $g_n \uparrow g$, when $u(g) = \lim_{n \to \infty} u(g_n)$. It follows that for some n, $\varphi = g_n$ satisfies $u(\varphi) \geq u(g) - \epsilon/2$. We have $f - \varphi \leq g - \varphi$ and $\varphi - f \leq g - f$, hence $|f - \varphi| \leq \max(g - \varphi,\ g - f)$. On the other hand $\min(g - \varphi,\ g - f) \geq 0$, and it follows from ii) in Lemma 3.1 that $\overline{u}(|f - \varphi|) \leq \overline{u}(\max(g - \varphi,\ g - f)) \leq u(g - \varphi) + u(g - f) \leq \epsilon/2 + \epsilon/2 = \epsilon$.

"If" For every $E^{\uparrow} \ni g \geq |f - \varphi|$ we have $E^{\uparrow} \ni \varphi + g \geq \varphi + |f - \varphi| \geq f$, hence, in view of ii) in Lemma 2.3 $\overline{u}(f) \leq u(\varphi + g) = u(\varphi) + u(g)$. Taking the infimum over all such g we obtain that $\overline{u}(f) \leq u(\varphi) + \overline{u}(|f - \varphi|)$. The substitution of f and φ by $-f$ and $-\varphi$, respectively, leads to $\underline{u}(f) = -\overline{u}(-f) \geq -(u(-\varphi) + \overline{u}(|-f + \varphi|) = u(\varphi) - \overline{u}(|f - \varphi|)$. Therefore, if $\overline{(}|f - \varphi|)$ can be made arbitrarily small, we conclude that $\overline{u}(f)$ and $\underline{u}(f)$ are finite and arbitrarily close to each other, that is, $f \in \overline{E}$.

If $f \in \overline{E}$ and $\varphi \in E$ then $f - \varphi \in \overline{E}$ and the above estimates take, in view of iv) in Lemma 4.2,the form $u(\varphi) - u(|f - \varphi|) \leq u(f) \leq u(\varphi) + u(|f - \varphi|)$, hence $|u(f) - u(\varphi)| \leq u(|f - \varphi|)$. $\square$

5 Null sets and equivalence classes of functions

If one wants to retain limit theorems like Theorem 4.3 and Theorem 4.4 in their full generality, then one has to allow, in general, $-\infty$ and $+\infty$ as values of the elements of $\overline{E}$, which causes problems in defining pointwise sums of elements of $\overline{E}$ and the pointwise scalar product of an element of $\overline{E}$ with zero. That is, in general $\overline{E}$ is not a vector space with the pointwise addition and scalar multiplication. In order to remedy this, we replace the elements of $\overline{E}$ by their equivalence classes for a suitable equivalence relation in $\overline{E}$.

A subset A of Ω is called a *null set* with respect to u if $\overline{u}(1_A) = 0$, where 1_A denotes the characteristic functio of A. Note that $1_A \geq 0$, hence $0 \leq \underline{u}(0) \leq \underline{u}(1_A) \leq \overline{u}(1_A) = 0$, which implies that $1_A \in \overline{E}$ and $u(1_A) = 0$.

Lemma 5.1 *i) The empty set is a null set.*
ii) If A is a null set and $B \subset A$, then B is a null set.
iii) If A_n is a sequence of null sets, then the union U of all the A_n is a null set.

Proof
i) $1_\emptyset = 0$ and $\overline{u}(0) = 0$.
ii) If $B \subset A$ then $0 \leq 1_B \leq 1_A$, hence $0 = \overline{u}(0) \leq \overline{u}(1_B) \leq \overline{u}(1_A)$ in view of ii) in Lemma 3.1.
iii) If A and B are null sets, then $1_{A \cup B} \leq 1_A + 1_B$, hence

$$0 \leq \overline{u}(1_{A \cup B}) \leq \overline{u}(1_A + 1_B) \leq \overline{u}(1_A) + \overline{u}(1_B) = 0 + 0 = 0$$

in view of iii) and ii) in Lemma 3.1, and therefore $A \cup B$ is a nulll set. Now let B_n be the union of the A_m, $1 \leq m \leq n$. Then the B_n form an increasing sequence of null sets with union equal to U. We have $1_{B_n} \uparrow 1_U$, hence, using iv) in Lemma 3.1, $\overline{u}(1_U) = \lim_{n \to \infty} \overline{u}(1_{B_n}) = 0$. $\square$

We say that a property $P(x)$ holds *for almost every* $x \in \Omega$ if there is a null set N such that $P(x)$ holds for every $x \in \Omega \setminus N$.

Lemma 5.2 *Let f, $g : \Omega \to \overline{\mathbf{R}}$ and $f(x) = g(x)$ for almost every $x \in \Omega$. Then $\overline{u}(f) = \overline{u}(g)$ and $\underline{u}(f) = \underline{u}(g)$. Therefore, if $f \in \overline{E}$, then $g \in \overline{E}$ and $u(f) = u(g)$.*

Proof We first The set $N = \{x \in \Omega \mid f(x) \neq g(x)\}$ is a null set, that is, $1_N \in \overline{E}$ and $u(1_N) = 0$. Let $\infty_N(x) = \infty$ and $\infty_N(x) = 0$ when $x \in N$ and $x \in \Omega \setminus N$, respectively. Then $\overline{E} \ni n\, 1_N \uparrow \infty_N$, $u(n\, 1_N) = n\, u(1_N) = n\, 0 = 0$ for every n, and therefore Theorem 4.3 implies that $\infty_N \in \overline{E}$ and $u(\infty_N) = 0$. it follows that for every $\epsilon > 0$ there exists $\chi \in E^\uparrow$

such that $\infty_N \leq \chi$ and $u(\chi) \leq \epsilon$. Let $f \leq \varphi \in E^\uparrow$. Then ii) in Lemma 2.3 implies that $\varphi + \chi \in E^\uparrow$ and $u(\varphi + \chi) = u(\varphi) + u(\chi) \leq u(\varphi) + \epsilon$. We have $g \leq f + \infty_N \leq \varphi + \chi$, and therefore $\overline{u}(g) \leq u(\varphi + \chi) \leq u(\varphi) + \epsilon$. Taking the infimum over all these φ in the right hand side, we obtain that $\overline{u}(g) \leq \overline{u} + \epsilon$, and because this holds for every $\epsilon > 0$, the conclusion is that $\overline{u}(g) \leq \overline{u}(f)$. The opposite inequality is obtained by interchanging f and g, hence $\overline{u}(f) = \overline{u}(g)$, which in turn implies $\underline{u}(f) = \underline{u}(g)$ by replacing f and g by $-f$ and $-g$, respectively. □

We call a function f with values in $\overline{\mathbf{R}}$ to be *defined almost everywhere*, if there is a null set N such that $f : \Omega \setminus N \to \overline{\mathbf{R}}$. Note that iii) implies that if $\mathcal{F}$ is a countable set of almost everywhere defined functions, then there is a null set N such that each $f \in \mathcal{F}$ is defined on $\Omega \setminus N$, that is, for almost every $x \in \Omega$ the value $f(x)$ is defined for every $f \in \mathcal{F}$. We say that the almost everywhere defined function f belongs almost everywhere to $\overline{E}$ if there exists $g \in \overline{E}$ such that for almost every $x \in \Omega$ we have $f(x) = g(x)$.

If f is an almost everywhere defined function, $g : \Omega \to \overline{\mathbf{R}}$, and $f(x) = g(x)$ for almost every $x \in \Omega$, then we define $\overline{u}(f) = \overline{u}(g)$. and $\underline{u}(f) = \underline{u}(g)$. Furthermore, we say that $f \in \overline{E}$ almost everywhere and $u(f) = u(g)$ if $g \in \overline{E}$. Lemma 5.2 implies that these definition do not depend on the choice of the everywhere defined function $g : \Omega \to \overline{E}$. The almost everywhere versions of Theorem 4.3, Theorem 4.4, and Theorem 4.5 now are the following.

Theorem 5.3 *If f_n is a sequence of almost everywhere defined functions and for almost every $x \in \Omega$ the sequence $f_n(x)$ is non-decreasing, with limit $f(x)$, and the sequence $u(f_n)$ is bounded, then $f \in \overline{E}$ almost everywhere, and $u(f_n) \uparrow u(f)$ as $n \to \infty$. If in the assumption we replace "non-decreasing" by "non-increasing", then the conclusion is that $f \in \overline{E}$ almost everywhere, and $u(f_n) \downarrow u(f)$ as $n \to \infty$.*

Theorem 5.4 *Let f_n be a sequence of almost everywhere defined functions, such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \Omega$. Assume that $\varphi \in \overline{E}$ almost everywhere, and for every n we have that $f_n(x) \leq \varphi(x)$ for almost every $x \in \Omega$. Then $f \in \overline{E}$ almost everywhere and $\lim_{n\to\infty} u(f_n) = u(f)$.*

Theorem 5.5 *Let f be an almost everywhere defined function. Then $f \in \overline{E}$ almost everywhere if and only if for every $\epsilon > 0$ there exists $\varphi \in E$ such that $\overline{u}(|f - \varphi|) \leq \epsilon$. If $f \in \overline{E}$ almost everywhere and $\varphi \in E$, then $|f - \varphi| \in \overline{E}$ almost everywhere and $|u(f) - u(\varphi)| \leq u(|f - \varphi|)$.*

We call two almost everywhere defined functions f, $g : \Omega \to \overline{\mathbf{R}}$ *equivalent*, notation $f \sim g$, if $f(x) = g(x)$ for almost every $x \in \Omega$. Lemma 5.1 immediately implies

Corollary 5.6 *i) $\sim$ is an equivalence relation in the set of all everywhere defined functions.*
ii) If $f \sim f'$ and $c \in \mathbf{R}$, $c \neq 0$, then $c\, f \sim c\, f'$.
iii) If $f \sim f'$ and $g \sim g'$, then $\max(f,\, g) \sim \max(f',\, g')$ and $\min(f,\, g) \sim \min(f',\, g')$.
iv) If f_n and f'_n are sequences such that $f_n \sim f'_n$ for every n, and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \Omega$, then $\lim_{n\to\infty} f'_n(x) = f(x)$ for almost every $x \in \Omega$.

If $[f]$ denotes the equivalence class of f, then Corollary 5.6 allow to define $c\,[f] := [c\,f]$ when $c \in \mathbf{R}$, $c \neq 0$, $\max([f],\,[g]) := [\max(f,\,g)]$, $\min([f],\,[g]) := [\min(f,\,g)]$, $[f] \leq [g]$ if and only if $\max([f],\,[g]) = [g]$ if and only if $f(x) \leq g(x)$ for almost every $x \in \Omega$, and $\lim_{n\to\infty}[f_n] = [f]$ if and only if $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \Omega$. The following lemma implies that if $f \in \overline{E}$ almost everywhere, then $f(x)$ is finite for almost every $x \in \Omega$

Lemma 5.7 *Let f be an almost everywhere defined function.*
i) If $f(x) > -\infty$ for almost every $x \in \Omega$, and $\overline{u}(f) < \infty$, then $f(x) < \infty$ for almost every $x \in \Omega$.
ii) If $f(x) < \infty$ for almost every $x \in \Omega$ and $\underline{u}(f) > -\infty$, then $f(x) > -\infty$ for almost every $x \in \Omega$.
iii) If $f \in \overline{E}$ almost everywhere, then $f(x) \in \mathbf{R}$ for almost every $x \in \Omega$.

Proof i) The assumption implies that there exists $g : \Omega \to \overline{\mathbf{R}}$ such that $\sim g$ and $-infty \notin g(\Omega)$. Let $N = \{x \in \Omega \mid g(x) = \infty\}$. If $\varphi \in E^{\uparrow}$, $g \leq \varphi$, then $\varphi(x) = \infty$ for every $x \in N$, and it follows that, for every positive real constant c, $f + c\,1_N \leq \varphi$. It follows that $\overline{u}(g + c\,1_N) \leq u(\varphi)$, and taking the infimum in the right hand side over all $E^{\uparrow} \ni \varphi \geq g$, we obtain that $\overline{u}(g + c\,1_N) \leq \overline{u}(g)$. Now iii) and i) in Lemma 3.1 yield that

$$0 \leq \overline{u}(1_N) \leq \overline{u}((1/c)\,g + 1_N) = \overline{u}((1/c)\,(g + c\,1_N) = (1/c)\,\overline{u}(g + c\,1_N) \leq (1/c)\,\overline{u}(f).$$

Taking the limit for $c \to \infty$ and using that $\overline{u}(g) = \overline{u}(f) < \infty$, we conclude that $\overline{u}(1_N) = 0$.

ii) follows immediately from the proof of i) and (3.2).

iii) There exists $g \in \overline{E}$ such that $f(x) = g(x)$ for every $x \in \Omega$. Then iii) in Lemma 4.2 implies that $g_+ := \max(g,\,0) \in \overline{E}$ and $g_- := \max(-g,\,0) \in \overline{E}$. We have $g(x) = \pm\infty$ if and only if $g_\pm(x) = \infty$ and because $\overline{u}(g_\pm) < \infty$, it follows that the sets $N_\pm = \{x \in \Omega \mid g(x) = \pm\infty\}$ are null sets. It follows now from iii) in Lemma 5.1 that $\{x \in \Omega \mid g(x) \notin \mathbf{R}\} = N_+ \cup N_-$ is a null set. $\square$

Let L denote the set of all equivalence classes of everywhere defined functions f such that $f \in \overline{E}$ almost everywhere. Note that it follows from Lemma 5.7 that each equivalence class contains a function f such that $f(x)$ is finite whenever $f(x)$ is defined. We denote the equivalence class of f by $[f]$.

Lemma 5.8 *Let f and g be almost everywhere defined functions such that, for almost every $x \in \Omega$ we have neither $f(x) = -\infty$ and $g(x) = \infty$, nor $f(x) = \infty$ and $g(x) = -\infty$, making $f(x) + g(x) \in \overline{\mathbf{R}}$ well-defined. Then $\underline{u}(f) + \underline{u}(g) \leq \underline{u}(f + g) \leq \overline{u}(f + g) \leq \overline{u}(f) + \overline{u}(g)$. Therefore, if $f,\,g \in \overline{E}$ almost everywhere, then $f + g \in \overline{E}$ almost everywhere, and $u(f + g) = u(f) + u(g)$.*

Proof By passing to suitable representant, we may assume that f and g are everywhere defined and that $f(x) + g(x)$ is well-defined for every $x \in \Omega$. If $E^{\uparrow} \ni \varphi \geq f$ and $E^{\uparrow} \ni \psi \geq g$, then it follows from ii) in Lemma 2.3 that $E^{\uparrow} \ni \varphi + \psi \geq f + g$ and $u(\varphi) + u(\psi) = u(\varphi + \psi) \geq u(f + g) \geq \overline{u}(f + g)$. Taking the infimum over all these φ and ψ in the left hand side we obtain that $u(f) + u(g) \geq \overline{u}(f+g)$. The inequality for $\underline{u}$ is proved analogously, and the other statements in the lemma follow. $\square$

It follows from Lemma 5.8 that for every $[f]$, $[g] \in L$ there is a unique $[h] \in L$ such that, for almost every $x \in \Omega$, $f(x)$, $g(x) \in \mathbf{R}$ and $h(x) = f(x) + g(x)$. We write $[h] = [f] + [g]$ in this case. Also, for every $[f] \in L$ and $c \in \mathbf{R}$, including $c = 0$, there is a unique $[g] \in L$ such that, for almost every $x \in \Omega$, $f(x) \in \mathbf{R}$ and $g(x) = c\, f(x)$. We write $[g] = c\,[f]$ in this case. With this "almost everywhere pointwise" addition and scalar multiplication, L is a vector space. Furthermore, Lemma 5.2 imply that if f and g are equivalent representants of an element of $\overline{E}$, then $u(f) = u(g)$, and therefore there is a unique function $u^{\sim} : L \to \mathbf{R}$ such that $u^{\sim}([f]) = [u(f)]$ for every $f \in \overline{E}$. It will cause no confusion to write $u^{\sim} = u$, and Lemma 5.8 together with i) in Lemma 4.2 imply that $u : L \to \mathbf{R}$ is a linear form on L. Finally there is a unique partial ordering $\leq$ in L such that $[f] \leq [g]$ if and only if $f(x) \leq g(x)$ for almost every $x \in \Omega$.

In the following theorem the allowance of almost everywhere defined functions and convergence is essential, even if the functions f_n are defined everywhere and $f_n(x) \in \mathbf{R}$ for every $x \in \Omega$.

Theorem 5.9 *Let f_n be a sequence of almost everywhere defined function such that, for each n, $f_n \in \overline{E}$ almost everywhere. Assume that $\sum_{n=1}^{\infty} u(|f_n|) < \infty$. Then, for almost every $x \in \Omega$, the series*

$$s(x) = \sum_{n=1}^{\infty} f_n(x) \tag{5.1}$$

is absolutely convergent, and $[s] \in L$. Furthermore, with the notation $s_m = \sum_{n=1}^{m} f_n$, we have

$$|u(s) - u(s_m)| \leq u(|s - s_m|) \leq \sum_{n=m+1}^{\infty} u(|f_n|) \to 0 \tag{5.2}$$

as $m \to \infty$.

Proof In view of the almost everywhere conclusion in (5.1) we may, by passing to suitable representants, assume that the functions f_n are defined everywhere and $f_n(x) \in \mathbf{R}$ for every $x \in \Omega$. According to iv) in Lemma 4.2 we have $|f_n| \in \overline{E}$, and Theorem 4.3, applied to the sequence $m \mapsto \sum_{n=1}^{m} |f_n|$, implies that its pointwise limit g belongs to $\overline{E}$. It follows from i) in Lemma 5.7 that for almost every $x \in \Omega$ we have $g(x) < \infty$, which implies that the series (5.1) converges absolutely. We have $|s_m| \leq g$ for all m and the almost everywhere version of Theorem 4.4 implies that $[s] \in L$ and $u(s_m) \to u(s)$ as $m \to \infty$. The first inequality in (5.2) follows from $\pm\,(u(s) - u(s_m)) = u(\pm\,(s - s_m)) \leq u(|s - s_m|)$, and the second equality follows from $|s - s_m| = |\sum_{n=m+1}^{\infty} f_n| \leq \sum_{n=m+1}^{\infty} |f_n|$. $\square$

Theorem 5.10 *The linear form $u : L \to \mathbf{R}$ is positive in the sense that $u(f) \geq 0$ whenever $f \in L$ and $f \geq 0$. $f \mapsto u(|f|)$ defines a norm on L, called the u-norm on L. With respect to* *the u-norm, L is complete, a Banach space. We have $|u(f)| \leq u(|f|)$ for every $f \in L$, which implies that the linear form u is continuous with repect to the topology in L defined by the u-norm.*

Proof The positivity of u follows from ii) in Lemma 4.2, and because $f \geq 0$ this also implies that $u(|f|) \geq 0$. Let $f \in \overline{E}$ and $u(|f|) = 0$ and let $N = \{x \in \Omega \mid f(x) \neq 0$. Then, for every positive integer n, $u(n\,|f|) = n\,u(|f|) = n\,0 = 0$, and therefore Theorem 4.3 implies that the limit ∞_N of the increasing sequence $n\,|f|$ in $\overline{E}$ belongs to $\overline{E}$, and $\overline{u}(1_N) \leq \overline{u}(\infty_N) = u(\infty_N) = \lim_{n\to\infty} u(n\,|f|) = 0$. This proves that N is a null set, that is $[f] = 0$ in L. This proves that if $f \in L$, $u(|f|) = 0$, then $f = 0$. Furthermore, if f, $g \in L$, then $|f + g| \leq |f| + |g|$, which in combination with the positivity, hence monotonicity, of the linear form u, implies that $u(|f + g|) \leq u(|f|) + u(|g|)$. This completes the proof that $f \mapsto u(|f|)$ is a norm on L. The inequality $|u(f)| \leq u(|f|)$ has already been observed in (5.2).

In order to prove the completeness of L with respect to the u-norm, let f_n be a sequence in L such that fir every $\epsilon > 0$ there exists N such that $u(|f_n - f_m|) \leq \epsilon$ whenever $n \geq N$ and $m \geq N$. Let ϵ_k be a sequence of strictly positive real numbers with finite sum, and take $N = N_k$ as above for $\epsilon = \epsilon_k$. Let n_k be a strictly increasing sequence of integers such that $n_k \geq N_k$ for every k. Then $n_k > n_{k-1} \geq N_{k-1}$, hence $u(|f_{n_k} - f_{n_{k-1}}|) \leq \epsilon_{k-1}$. It therefore follows from Theorem 5.9 with $n \mapsto f_n$ replaced by $k \mapsto f_{n_k} - f_{n_{k-1}}$ that, for almost every $x \in \Omega$,

$$f_{n_m}(x) = f_{n_0}(x) + \sum_{n=1}^{m} (f_{n_k}(x) - f_{n_{k-1}}(x))$$

converges to an $f(x) \in \mathbf{R}$, that the almost everywhere defined function f defines an element of L which we denote by the same letter, and that $u(|f - f_{n_m}|) \to 0$ as $m \to \infty$. Now $u(|f - f_l|) \leq u(|f - f_{n_m}|) + u(|f_{n_m} - f_l|) \leq u(|f - f_{n_m}|) + \epsilon_m$ if $l \geq n_m \geq N_m$. The right hand side in this inequality converges to zero as $m \to \infty$, and it follows that $u(|f - f_l|) \to 0$ as $l \to \infty$. This completes the proof of the completeness of L with respect to the u-norm on L. □

There are examples where f_n is a sequence in $\overline{E}$ such that $u(|f_n|) \to 0$, for no $x \in \Omega$ the sequence $f_n(x)$ converges. It follows that a u-Cauchy sequence in L need not converge at any point.

Also note that if for instance $u : E \to \mathbf{R}$ is identically zero, then $L = \{0\}$, Ω is a null set, and the mapping $f \mapsto [f] : E \to L$ is the null map. Although this example is not very interesting, it contains the warning that the mapping $f \mapsto [f] : E \to L$ need not be injective, and that $f \mapsto u(|f|)$ need not define a norm on E. However, if $u(|f|) > 0$ whenever $f \in E$ and $f \neq 0$, then L can be equivalently defined as the completion of E with respect to the u–norm on E.

6 Dropping the positivity assumption

In this section we prove Theorem 1.5.

If $f \in E$, $f \geq 0$, then $0 \leq 0 \leq f$, hence $v_+(f) \geq v(0) = 0$. If furthermore $c \in \mathbf{R}_{>0}$, then we have for any $g \in E$ that $0 \leq g \leq f$ if and only if $0 \leq c\,g \leq c\,f$, when $v(c\,g) = c\,v(g)$, and this shows that $v_+(c\,f) = c\,v_+(f)$.

If $e \in E$, $e \geq 0$, then $d \in E$, $0 \leq d \leq e$ implies in combination with $0 \leq g \leq f$ that $0 \leq d + g \leq e + f$, hence $v_+(e + f) \geq v(d + g) = v(d) + v(g)$, and taking the supremum in the right hand side over all these d, g, we obtain that $v_+(e + f) \geq v_+(e) + v_+(f)$. On the other hand, if $\varphi \in E$, $0 \leq \varphi \leq e + f$, then $\varphi - e \leq f$, $0 \leq \max(\varphi - e,\, 0) \leq f$, $\varphi - e = (\varphi - e) + 0 = \max(\varphi - e,\, 0) + \min(\varphi - e,\, 0) = \max(\varphi - e,\, 0) + \min(\varphi,\, e) - e$, hence $\varphi = \min(\varphi,\, e) + \max(\varphi - e,\, 0)$, $v(\varphi) = v(\min(\varphi,\, e)) + v(\max(\varphi - e,\, 0)) \leq v_+(e) + v_+(f)$, because also $0 \leq \min(\varphi,\, e) \leq e$. Taking the supremum in the left hand side over all these φ, we obtain $v_+(e + f) \leq v_+(f) + v_+(g)$, which in combination with the previous inequality yields $v_+(f + g) = v_+(f) + v_+(g)$. In particular $v_+(0) = v_+(0 + 0) = v_+(0) + v_+(0)$, hence $v_+(0) = 0$, and therefore also $v_+(0\,f) = v_+(0) = 0 = 0\,v_+(f)$ for every $f \in E$.

If φ, ψ, φ', ψ' are non-negative elements of E such that $\varphi - \psi = \varphi' - \psi'$, that is, $\varphi + \psi' = \varphi' + \psi$, then $v_+(\varphi) + v_+(\psi') = v_+(\varphi + \psi') = v_+(\varphi' + \psi) = v_+(\varphi') + v_+(\psi)$, hence $v_+(\varphi) - v_+(\psi) = v_+(\varphi') - v_+(\psi')$. If $f,\, g \in E$, then

$$
\begin{aligned}
& \max(f + g,\, 0) - \max(-f - g,\, 0) = f + g \\
= \;& \max(f,\, 0) - \max(-f,\, 0) + \max(g,\, 0) - \max(-g,\, 0) \\
= \;& (\max(f,\, 0) + \max(g,\, 0)) - (\max(-f,\, 0) + \max(-g,\, 0)),
\end{aligned}
$$

hence

$$
\begin{aligned}
v_+(f + g) &:= v_+(\max(f + g,\, 0)) - v_+(\max(-f - g,\, 0)) \\
&= v_+(\max(f,\, 0) + \max(g,\, 0)) - v_+(\max(-f,\, 0) + \max(-g,\, 0)) \\
&= v_+(\max(f,\, 0)) + v_+(\max(g,\, 0)) - v_+(\max(-f,\, 0)) - v_+(\max(-g,\, 0)) \\
&= (v_+(\max(f,\, 0)) - v_+(-f,\, 0))) + (v_+(\max(g,\, 0)) - v_+(-g,\, 0))) \\
&=: v_+(f) + v_+(g),
\end{aligned}
$$

proving the additivity of v_+. In turn it follows that, for every $f \in E$, $0 = v_+(0) = v_+(f + (-f)) = v_+(f) + v_+(-f)$, hence $v_+(-f) = -v_+(f)$, hence $v_+((-c)\,f)) = v_+(-(c\,f)) = -(c\,v_+(f)) = (-c)\,v_+(f)$ if $c \in \mathbf{R}_{>0}$, and we conclude that $v_+ : E \to \mathbf{R}$ is a linear form.

Let $f_n \in E$ and $f_n \downarrow 0$ as $n \to \infty$. Let $\epsilon \in \mathbf{R}_{>0}$ and choose a sequence $\epsilon \in \mathbf{R}_{>0}$ of which the sum is $\leq \epsilon$. There exist $\varphi_n \in E$ such that $0 \leq \varphi_n \leq f_n$ and $v(\varphi_n) \geq v_+(f_n) - \epsilon_n$. let $\psi_n = \min_{1 \leq m \leq n} \varphi_m$. Then it follows by induction on n as in the proof of iv) in Lemma 3.1, with u, $\overline{u}$ replaced by v, v_+, respectively, and all inequalities reversed, that

$$
v(\psi_n) \geq v_+(f_n) - \sum_{m=1}^{n} \epsilon_n \geq v_+(f_n) - \epsilon.
$$

Because $0 \leq \psi_n \leq f_n$, $\psi_{n+1} \leq \psi_n$, and $f_n \downarrow 0$ as $n \to \infty$, we have $\psi_n \downarrow 0$ as $n \to \infty$, hence $v(\psi_n) \to 0$ as $n \to \infty$ inview of the monotone convergence assumption for v, and it follows that $0 \leq \text{limsup}_{n \to \infty}\, v_+(f_n) \leq \epsilon$, which in turn implies that $v_+(f_n) \to 0$ as $n \to \infty$. That is, v_+ has the monotone convergence property.

Finally, if $f \in E$ and $f \geq 0$, then $0 \leq f \leq f$ shows that $v_+(f) \geq v(f)$, hence $v_+(f) - v(f) \geq 0$. It follows that the linear form $v_- := v_+ - v$ is positive, and it has the monotone convergence property because both v and $v_=$ have the monotone convergence property. This completes the proof of Theorem 1.5.

7 Remarks

Our E and $E^{\uparrow}$ are the spaces T_0 and T_1 of Daniell [2], respectively. Our set Ω is the set of points p of Daniell [2], who did not give a name to this set. Our set $\overline{E}$ is the set of all "summable functions" of Daniell [2], who did not give a name to this set either.

In [2, Sec. 9], Daniell wrote: "It is usual, though not necessary, to define the integral in terms of the measure of certain fundamental sets. Let us suppose that the measure of a certain class of elementary, or initial sets, or collections E, of the p are given. in connection with a collection E we can define a function $= 1$ when p belongs to E, $= 0$ otherwise. We can agree to call the measure of E, the integral of the corresponding function. The class T_0 is then taken as the class of all functions which are linear combinations of these elementary set-functions. It will then be closed with respect to pointwise scalar multiplication and pointwise addition, and the integral can be extended to a linear form on T_0. For any set E whatever we can say that it is measurable if the corresponding function is summable, and we can identify its measure with the integral of the function. This question requires however a seperate and careful consideration."

On the other hand Daniell [2] did not observe that if conversely a measure is given, then his closure is the standard way of defining the space of Lebesgue integrable functions with respect to the measure. Daniell opened [2, Introduction] with: "The idea of an integral has been extended by Radon, Young, Riesz and others ..., although many of the proofs given are mere translations into other language of methods already classical (particularly those due to Young), here and there ... new methods have been devised." I seen this as an admittance of Daniell that his closure procedure for a large part corresponds to the standard way of defining the space of Lebesgue integrable functions (Young [13] referred to Lebesgue [5], whereas Daniell did not mention the name of Lebesgue), but the opening remark in [2, Sec. 9] indicates a reluctance to talk about integrals defined by measures.

Nowadays the general feeling is that the most important contributions of Daniell are i) treating the extension of integrals to the space of all integrable functions in the framework of a positive linear form on a Riesz space of functions, using only the monotone convergence property, and ii) the splitting of a linear form, not necessarily positive, which has the monotone convergence property, as the diffenence two positive linear forms which have the monotone convergence property, see Theorem 1.5.

In [2, Sec. 5], Daniell allowed explicitly that the functions in his class $T_0 =$ our $E^{\uparrow}$ have

$+\infty$ as values, but in [2, Sec. 6, 7] it is done as if all functions only take finite values, which is quite incompatable with the generality of the limit theorems in these sections. In particular no attention is paid to the problem of the pointwise definition of the sum and scalar products of summable functions. As this is remedied by working modulo functions on null sets, one might conjecture that this gap is related to Daniell's reluctance to talk about subsets and their measures, even if one only needs null sets to define the equivalence relation.

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