Bifurcation structure of the generalized Henon map

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joint work with *V.S. Gonchenko* and *Hil Meijer*

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References

- *Gonchenko, V.S., Kuznetsov, Yu.A. and Meijer, H.G.E.* Generalized Hénon map and bifurcations of homoclinic tangencies. *SIAM J. Appl. Dyn. Sys.* **4** (2005) [to appear]
- *Kuznetsov, Yu.A., Meijer, H.G.E., and van Veen, L.* The fold-flip bifurcation. *IJBC* **14** (2004), 2253-2282

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Hénon maps

- Any quadratic planar map with constant Jacobian can be put by a linear coordinate transformation to the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} 1 + Y - aX^2 \\ bX \end{pmatrix} \qquad \text{(original Hénon map)}$$

[Hénon, 1976]

Hénon maps

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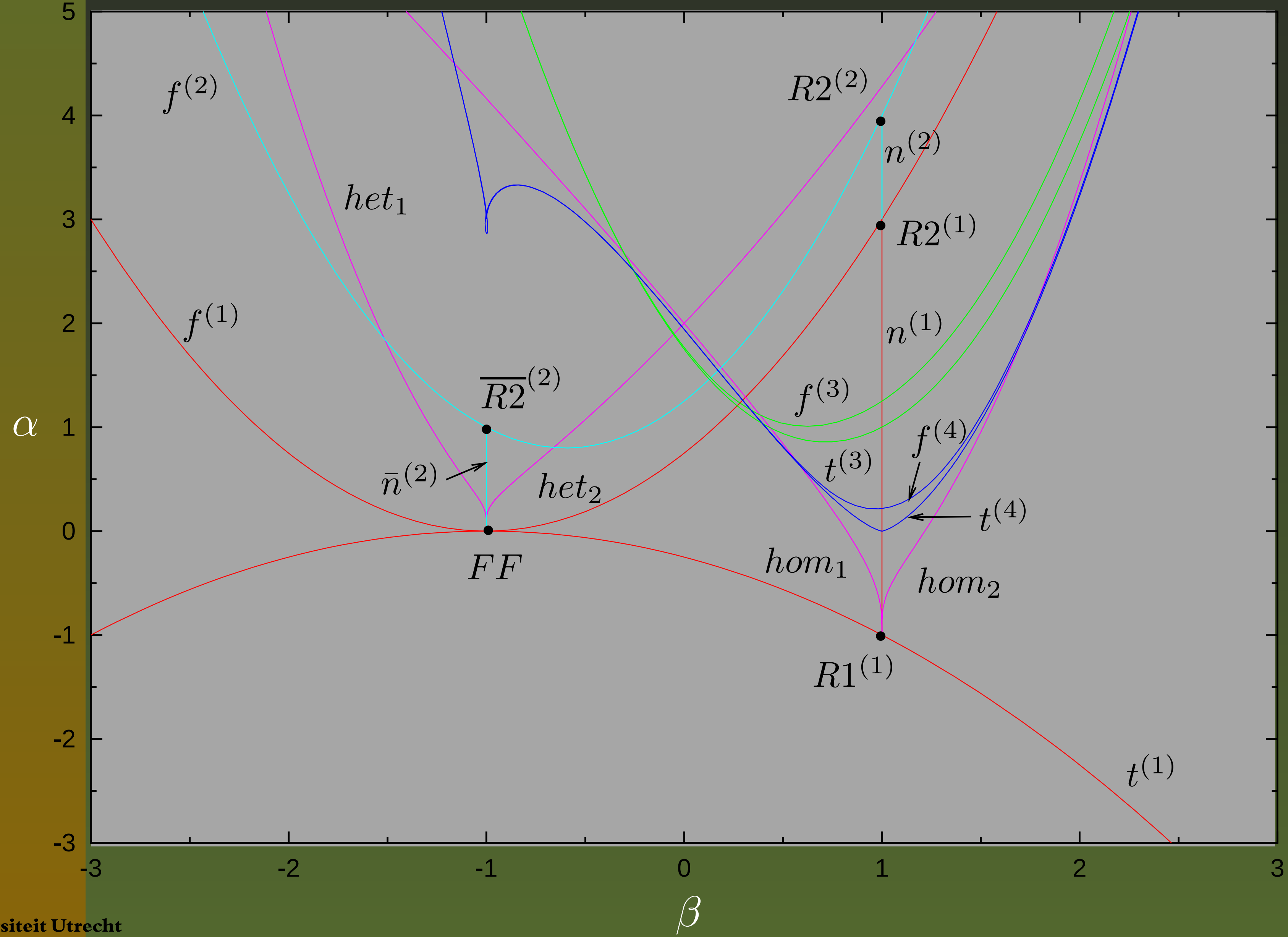
[Hénon, 1976]

- or

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \alpha - \beta x - y^2 \end{pmatrix} \qquad \text{(standard Hénon map)}$$

[Mira, 1987]

Bifurcation diagram of the standard Hénon map



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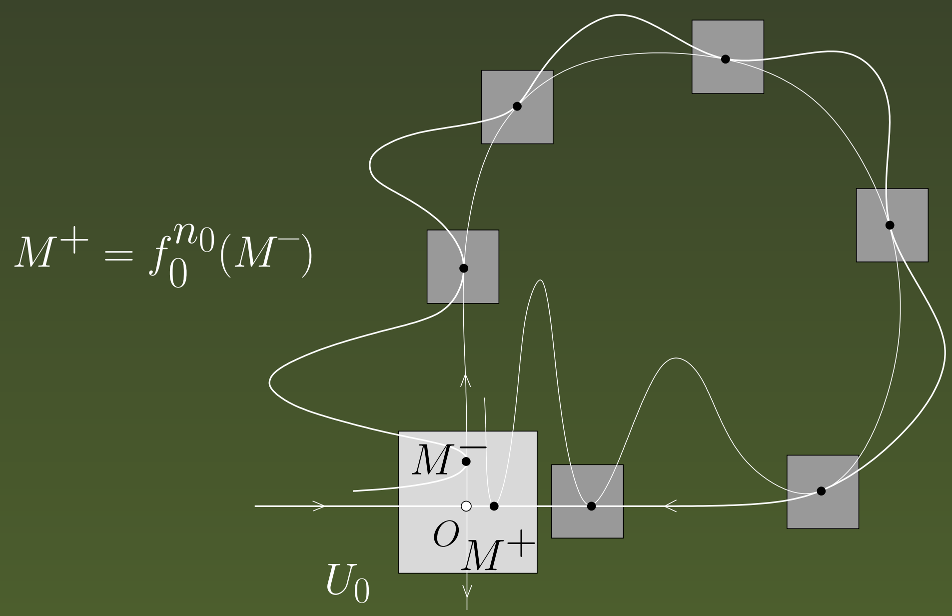
Homoclinic tangency of a neutral saddle in $\mathbb{R}^2$

Consider a two-parameter family f_μ of C^5-planar diffeomorphisms satisfying at $\mu = 0$:

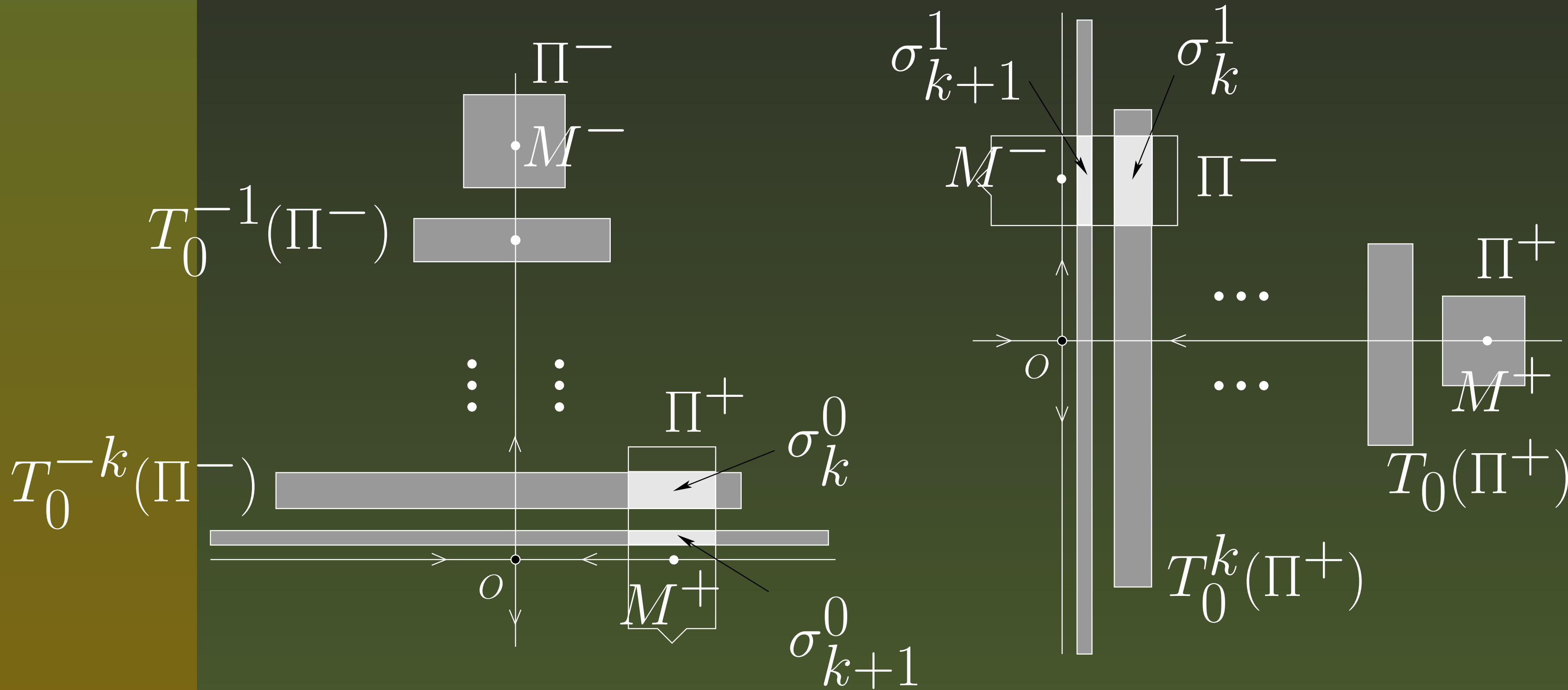
(A) f_0 has a saddle fixed point O with eigenvalues $0 < |\lambda| < 1 < |\gamma|$

(B) $\sigma := |\lambda\gamma| = 1$



(C) $W^u(O)$ and $W^s(O)$ have quadratic tangency at homoclinic orbit Γ

Domains of definition



$$\sigma_k^1 = f_0^k(\sigma_k^0)$$

Poincaré map $f_\mu^{k+n_0} = T_1 \circ T_0^k$

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Poincaré map $f_\mu^{k+n_0} = T_1 \circ T_0^k$

- Local map T_0^k: $\hat{\gamma} = \max\{|\lambda|^{-1}, |\gamma|\}$

$$\begin{cases} x_1 = \lambda^k(\mu) x_0 (1 + \hat{\gamma}^{-k} \xi_k(x_0, y_1, \mu)), \\ y_0 = \gamma^{-k}(\mu) y_1 (1 + \hat{\gamma}^{-k} \eta_k(x_0, y_1, \mu)). \end{cases}$$

Poincaré map $f_\mu^{k+n_0} = T_1 \circ T_0^k$

- Local map T_0^k: $\hat{\gamma} = \max\{|\lambda|^{-1}, |\gamma|\}$

$$\begin{cases} x_1 & = & \lambda^k(\mu)x_0(1+\hat{\gamma}^{-k}\xi_k(x_0,y_1,\mu)), \\ y_0 & = & \gamma^{-k}(\mu)y_1(1+\hat{\gamma}^{-k}\eta_k(x_0,y_1,\mu)). \end{cases}$$

- Global map T_1: $bcd \neq 0$

$$\begin{cases} \bar{x}_0 - x^+ & = & ax_1 + b(y_1 - y^-) \\ & + & e_{20}x_1^2 + e_{11}x_1(y_1-y^-) + e_{02}(y_1-y^-)^2 + \cdots \\ \bar{y}_0 & = & \mu_1 + cx_1 + d(y_1-y^-)^2 \\ & + & f_{20}x_1^2 + f_{11}x_1(y_1-y^-) + f_{03}(y_1-y^-)^3 + \cdots \end{cases}$$

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A rescaling result by Gonchenko S.V. & V.S. [2000]

For any $k > \bar{k}$ the map $f_\mu^{k+n_0}$ can be reduced by invertible affine transformations to the C^3-smooth map

$$\begin{cases} \bar{X} = Y, \\ \bar{Y} = \alpha - \beta X - Y^2 + R\lambda^k XY + S\gamma^{-k} Y^3 + o(\lambda^k), \end{cases}$$

where α, β, X, Y cover all finite values as $k \to \infty$, and

$$R = 2a - \frac{b}{d} f_{11} - 2\frac{c}{d} e_{02}, \quad S = \frac{1}{d^2} f_{03}.$$

Moreover

$$\alpha = -d\gamma^{2k}[\mu_1 - \gamma^{-k}(y^- + \dots) + c\lambda^k(x^+ + \dots)],$$
$$\beta = -bc(1 + \mu_2)^k(1 + \dots).$$

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Generalized Hénon map

General form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \alpha - \beta x - y^2 + Rxy + Sy^3 \end{pmatrix}$$

Noninvertibility line

$$y = \frac{\beta}{R}$$

Quadratic extension:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \alpha - \beta x - y^2 + Rxy \end{pmatrix}$$

Codim 1 bifurcations of fixed points when $S = 0$

Fold: If $R \neq 1$, there is a nondegenerate fold bifurcation for

$$\alpha = \frac{(\beta+1)^2}{4(R-1)}$$

with the critical fixed point at $x = y = \frac{\beta+1}{2(R-1)}$.

Flip: If $R \neq 3$, there is a nondegenerate flip bifurcation for $\alpha = \frac{1}{4}(\beta+1)^2(3-R)$ with the critical fixed point at $x = y = \frac{\beta+1}{2}$.

N–S: If $R \neq 0, 1, 2$, there is a Neimark-Sacker bifurcation for

$$\alpha = \frac{(\beta-1)(\beta-1+2R)}{R^2}$$

with the critical fixed point at $x = y = \frac{\beta-1}{R}$. The bifurcation is nondegenerate away from strong resonances.

Codim 2 bifurcations of fixed points when $S = 0$

FF : There is a fold-flip bifurcation of the fixed point $x = y = 0$ for $(\alpha, \beta) = (0, -1)$.

$R1$: There is a resonance 1:1 at $(\alpha, \beta) = \left(\frac{4(-1+R)}{(2-R)^2}, \frac{2-3R}{2-R} \right)$ for the fixed point $x = y = \frac{\beta - 1}{R}$.

$R2$: There is a resonance 1:2 at $(\alpha, \beta) = \left(\frac{4(3-R)}{(2-R)^2}, \frac{2+R}{2-R} \right)$ for the fixed point $x = y = \frac{\beta - 1}{R}$.

$R3$: There is a resonance 1:3 at $(\alpha, \beta) = \left(\frac{5-2R}{(2-R)^2}, \frac{2}{2-R} \right)$ for the fixed point $x = y = \frac{\beta - 1}{R}$.

$R4$: There is a resonance 1:4 for the fixed point $x = y = 0$ at $(\alpha, \beta) = (0, 1)$.

These codim 2 bifurcations are nondegenerate for small $R \neq 0$.

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Continuation of homo-/hetroclinic tangencies

Continuation of transverse homoclinic orbits in one parameter:

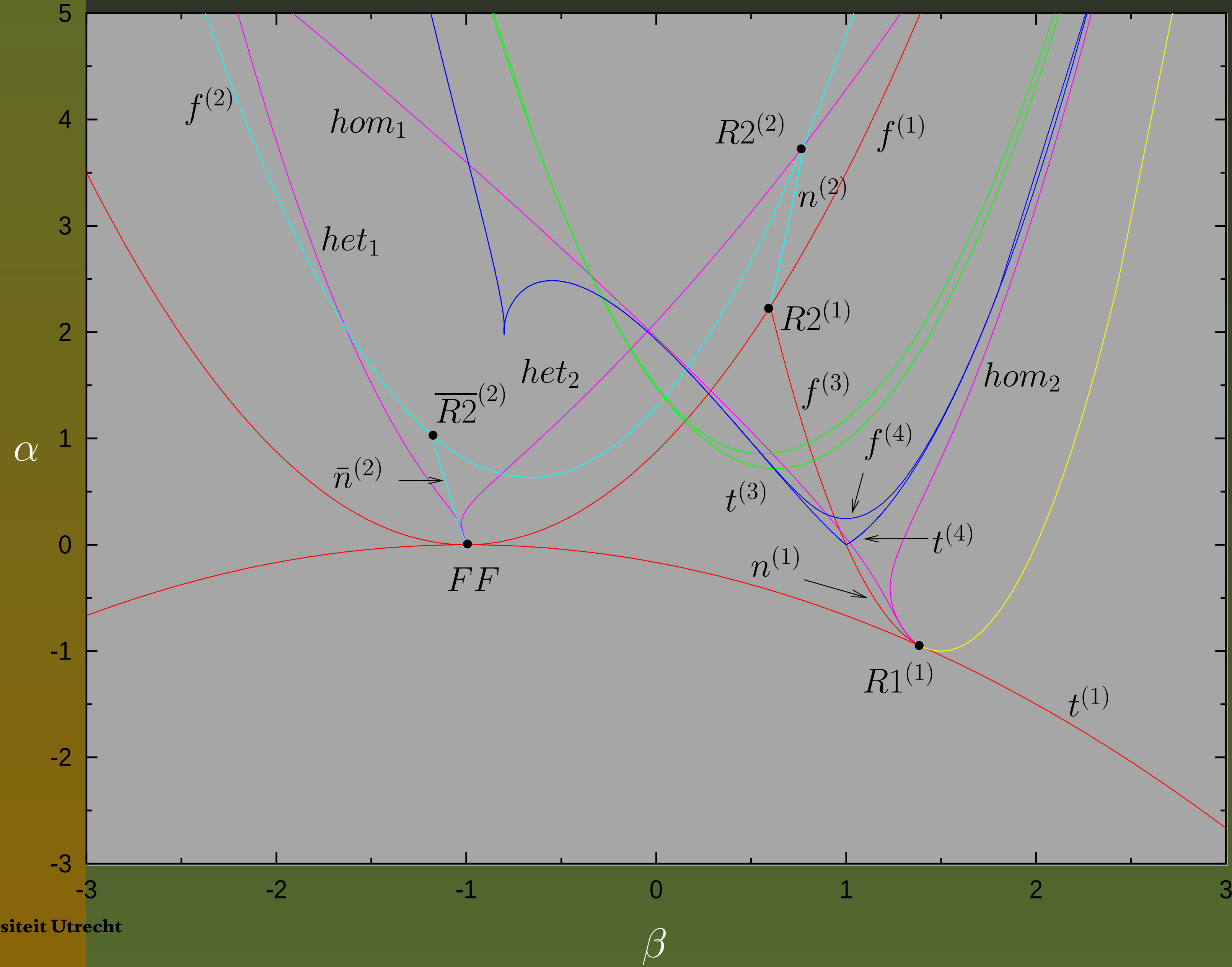
$$\begin{aligned} x_{i+1} - f(x_i) &= 0 \qquad \text{for } 1 \leq i \leq n-1, \\ \langle q_s, (x_1 - \xi) \rangle &= 0, \\ \langle q_u, (x_{n+1} - \xi) \rangle &= 0. \end{aligned}$$

Here $A = f_x(\xi)$ and

$$A^T q_s = \lambda q_s, \;\; A^T q_u = \gamma q_u, \;\; \|q_{s,u}\| = 1.$$

Homoclinic tangencies are continued in two parameters as *folds* of this system [Beyn & Kleinkauf, 1997].

Bifurcation diagram of GHM ($R = -0.5$)



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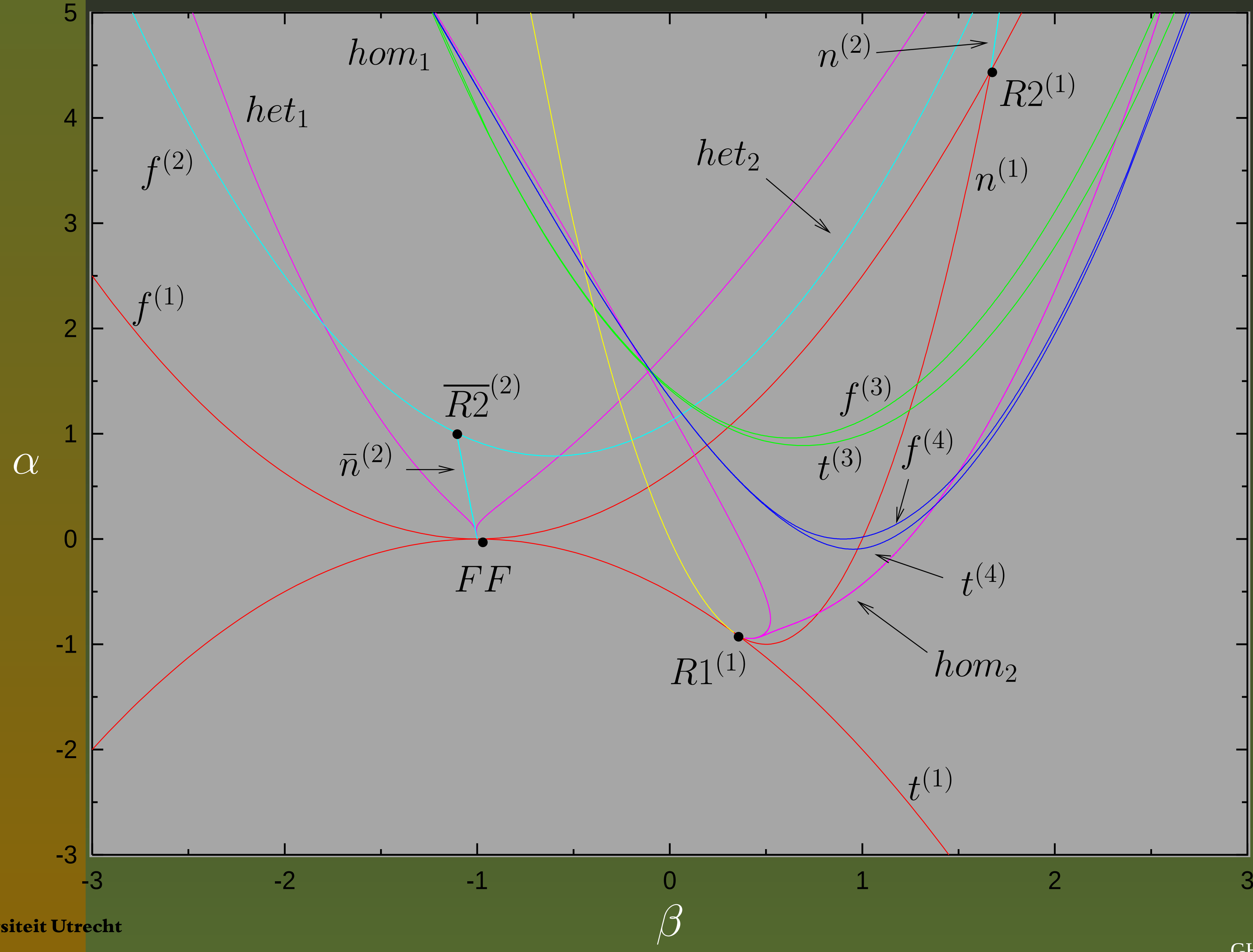
Bifurcation diagram of GHM ($R = 0.5$)



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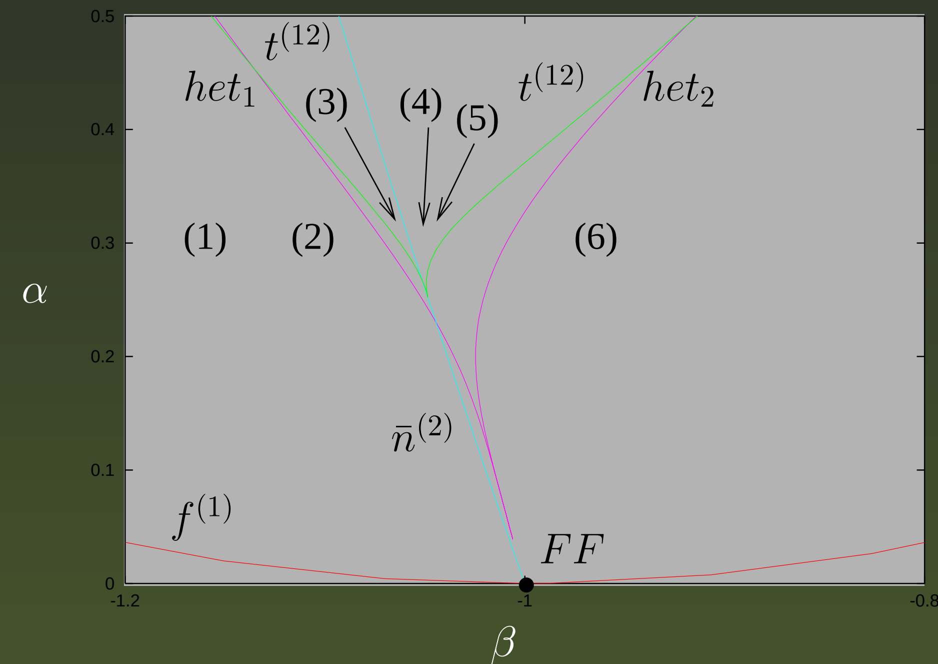
Bifurcation curves near FF point ($R = -0.5$)



$f^{(1)}$ – period-doubling of fixed points

$\bar{n}^{(2)}$ – Neimark Sacker bifurcation of 2-cycles

$het_{1,2}$ – heteroclinic tangencies

$t^{(12)}$ – fold of 12-cycles

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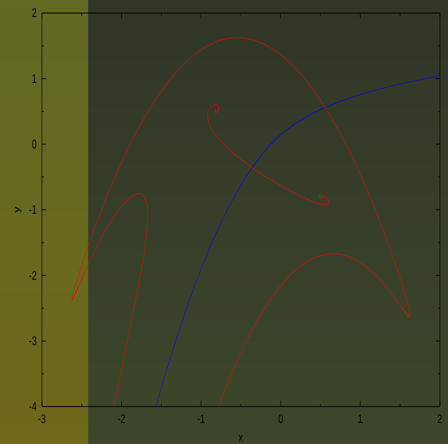
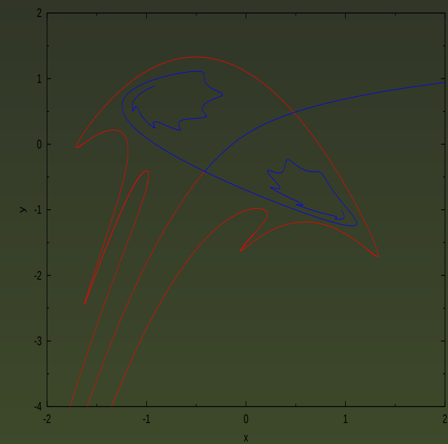
Dynamics near FF point ($R = -0.5$)



(1)



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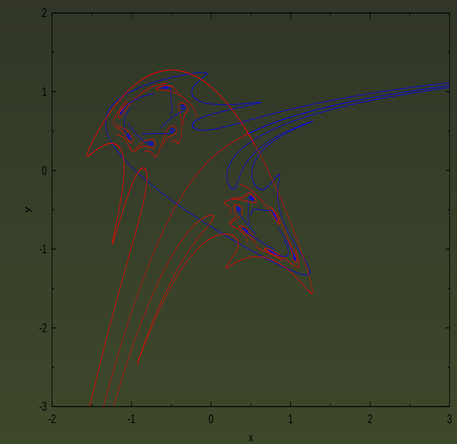

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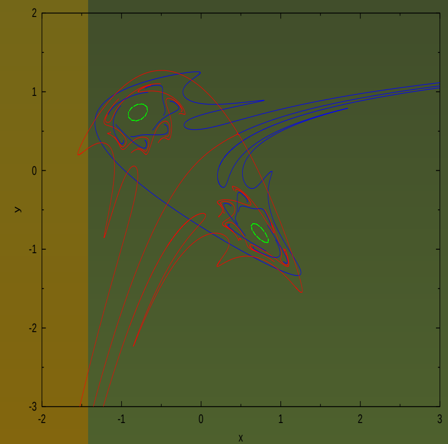

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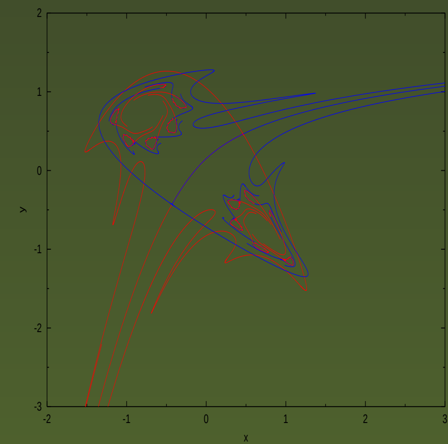

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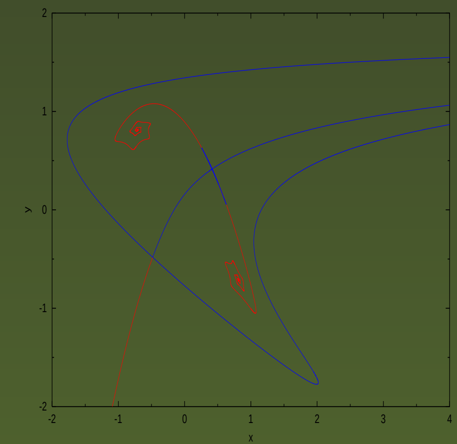

(6)

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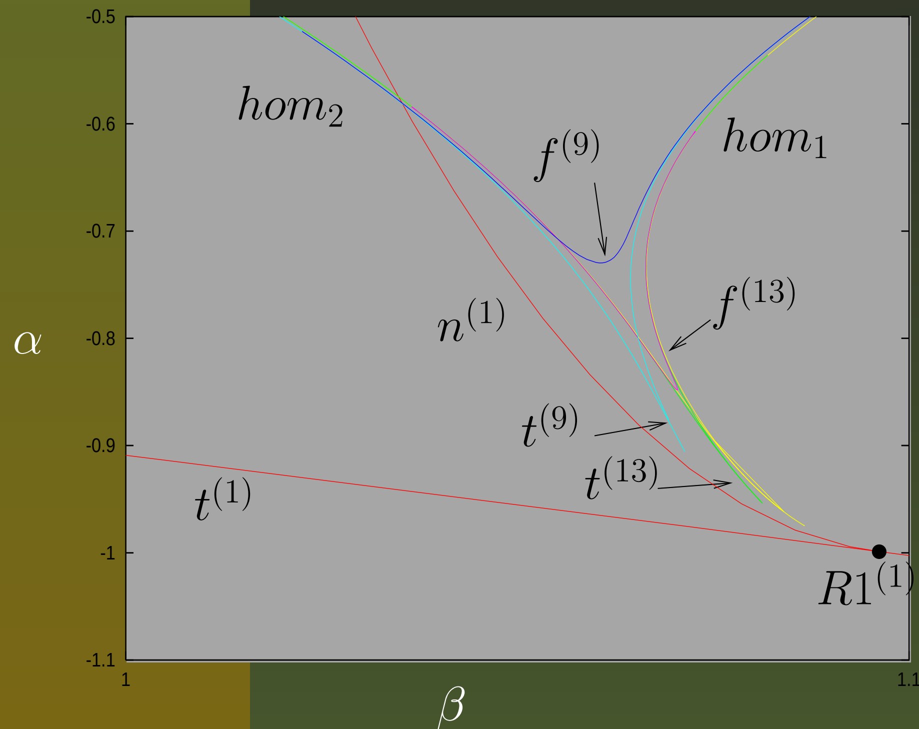
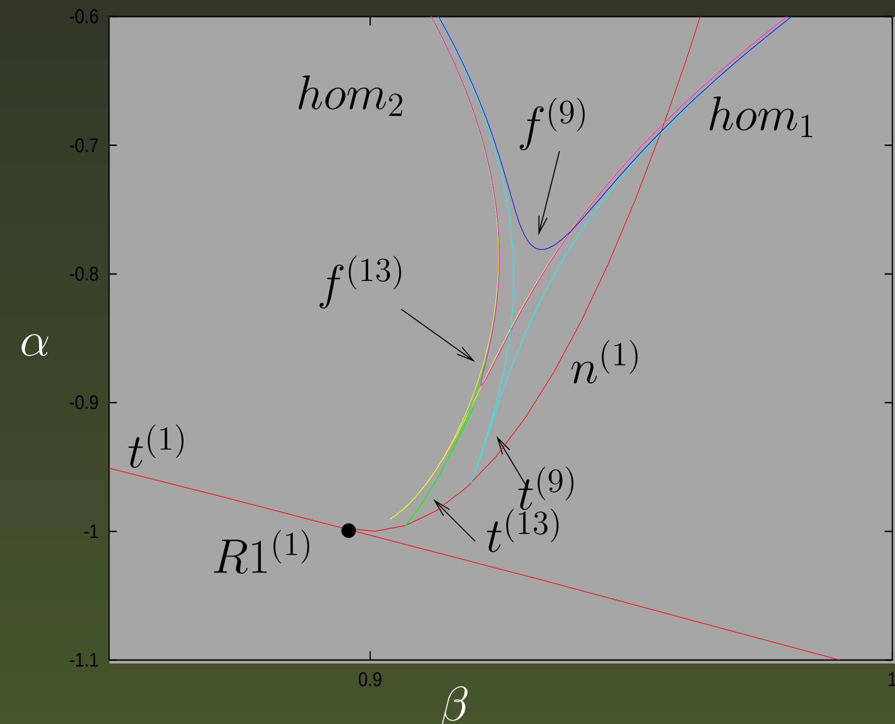
Arnold tongues near $1:1$ resonance



$R = -0.1$



$R = 0.1$

$hom_{1,2}$ – homoclinic tangencies

$t^{(9)}$ – fold of 9-cycles

$f^{(9)}$ – period-doubling of 9-cycles

$t^{(13)}$ – fold of 13-cycles

$f^{(13)}$ – period-doubling of 13-cycles

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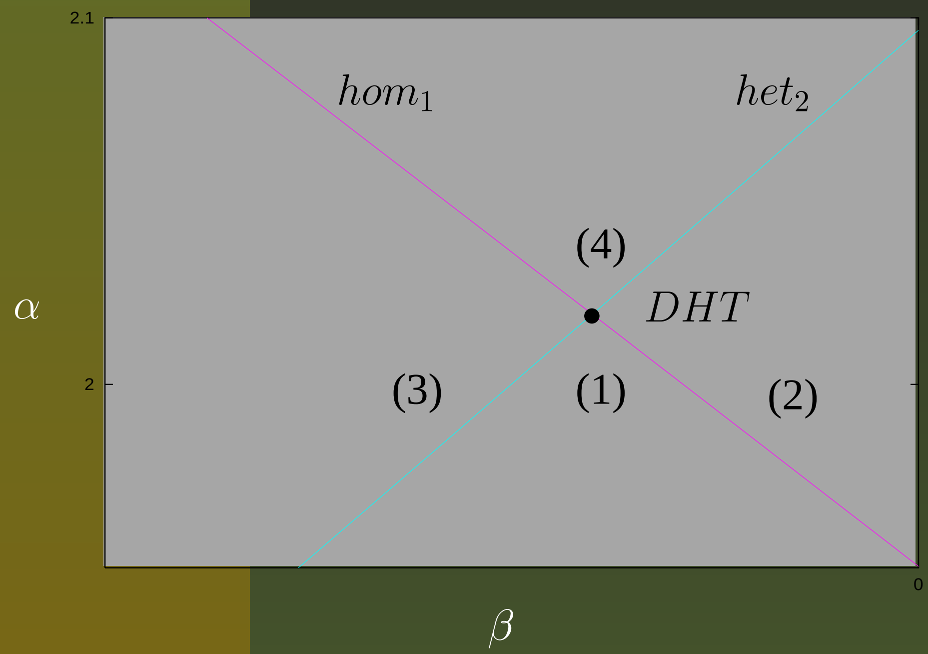
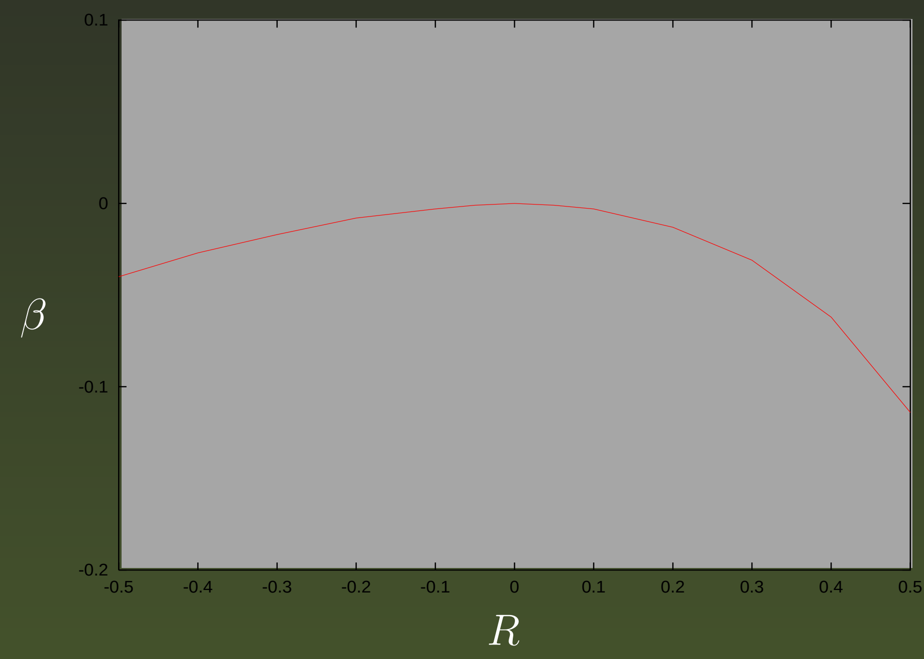
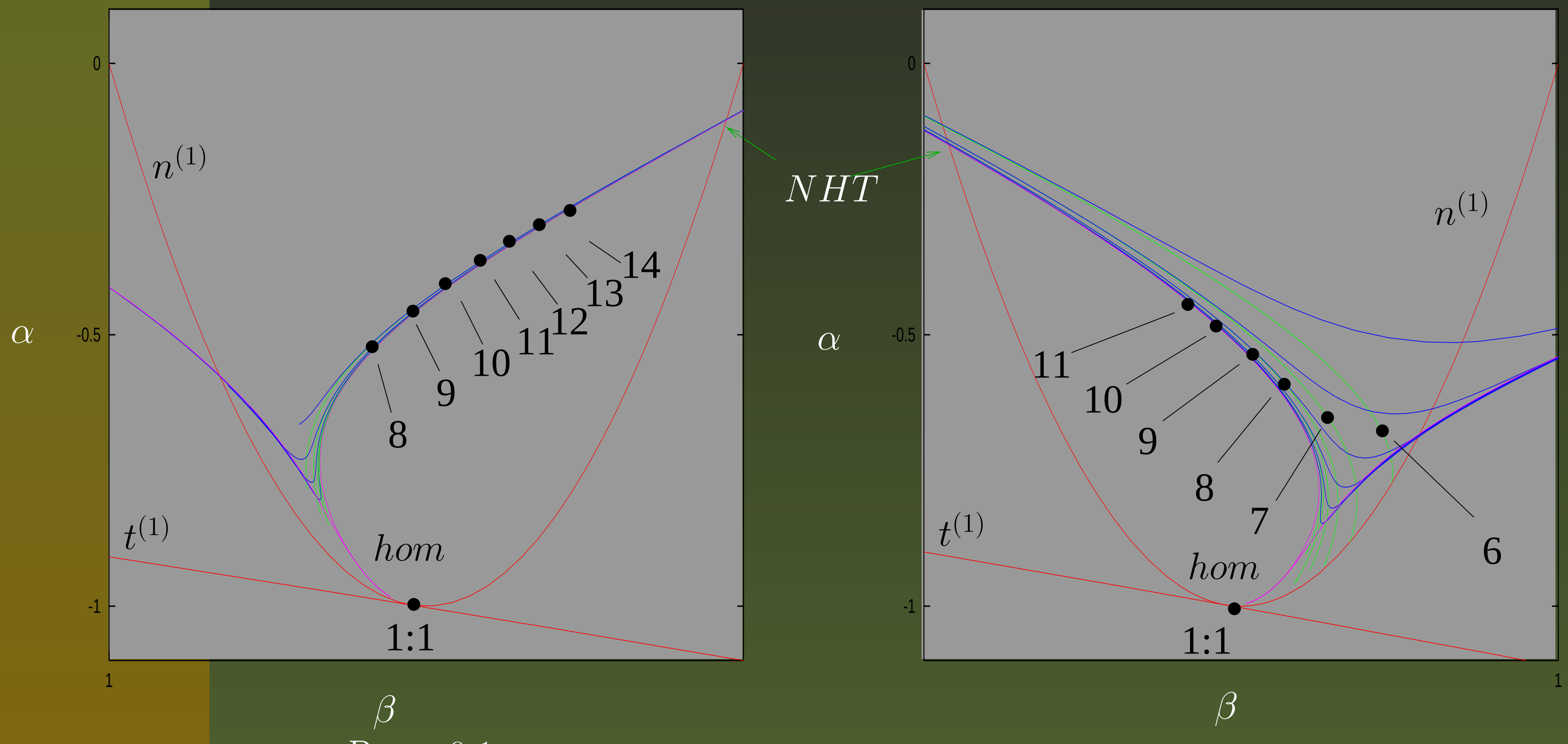
Double homoclinic/heteroclinic tangency





For $R = S = 0$, the curves hom_1 and het_2 intersect at $(\beta, \alpha) = (0, 2)$.

Neutral homoclinic tangency in GHM and 1:1 resonances



$R = -0.1$ $R = 0.1$

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Implications for Poincaré maps

$$\begin{cases} \bar{X} = Y, \\ \bar{Y} = \alpha - \beta X - Y^2 + R\lambda^k XY + S\gamma^{-k}Y^3 + o(\lambda^k), \end{cases}$$

with

$$\alpha = -d\gamma^{2k}[\mu_1 - \gamma^{-k}(y^- + \dots) + c\lambda^k(x^+ + \dots)],$$

$$\beta = -bc(1 + \mu_2)^k(1 + \dots).$$

Thus

$$\mu_2 \sim \left(-\frac{\beta}{bc}\right)^{\frac{1}{k}} - 1 \sim \frac{1}{k}\ln\left(-\frac{1}{bc}\right)$$

for $k \to \infty$ and in the (μ_2, μ_1)-plane we can see only one half of the (α, β)-plane of the GHM, depending on the sign of bc.

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Nondegeneracy of bifurcations of Poincaré maps for $k \to \infty$

codim 1: Fold, flip and NS-bifurcations are λ^k-nondegenerate when $R \neq 0$, in particular,

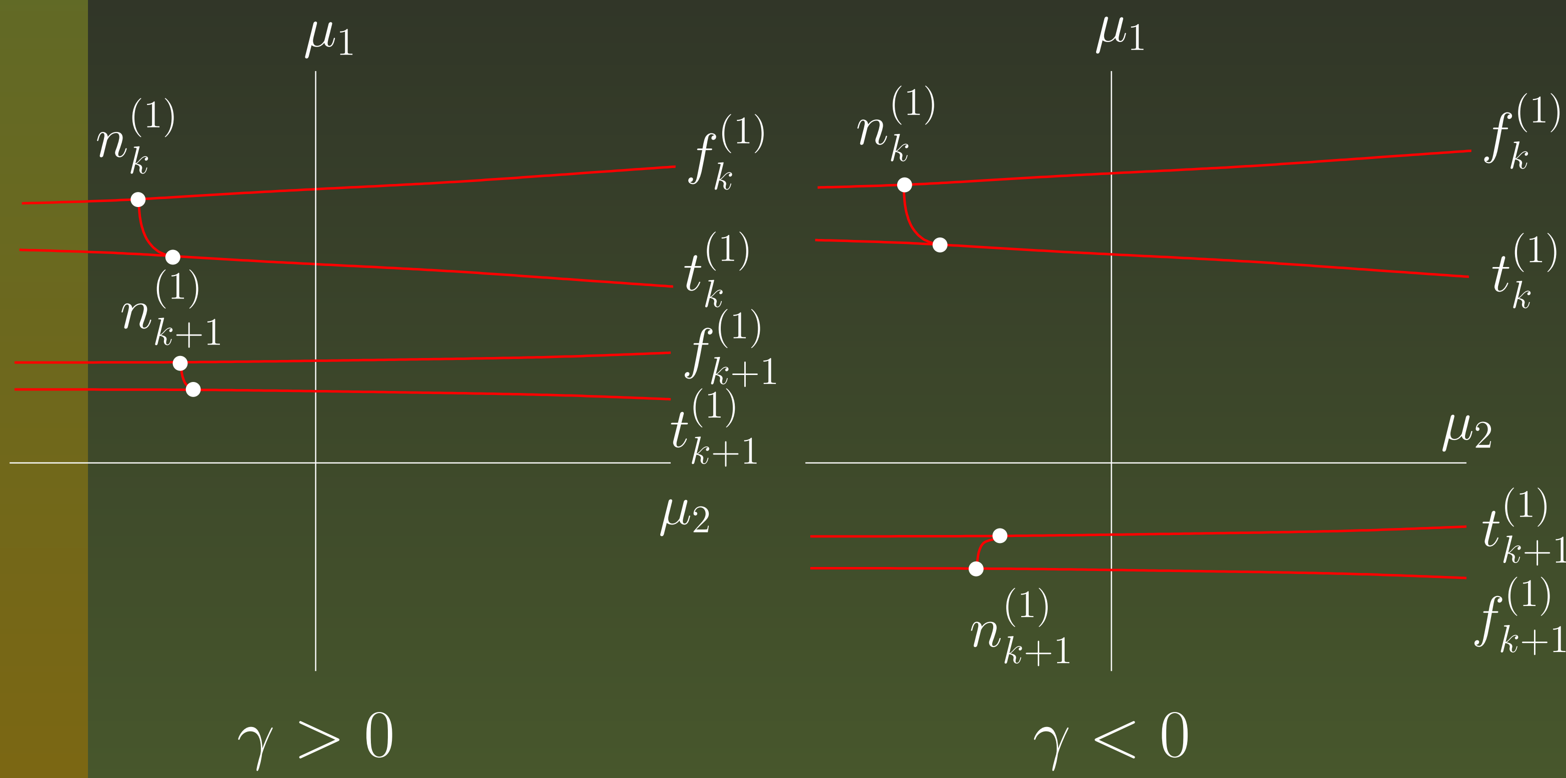
$$c_{NS} = \frac{R}{4\cos^2\left(\frac{\psi}{2}\right)}\lambda^k + o(\lambda^k).$$

codim 2: All strong resonances for fixed points are λ^k-nondegenerate when $R \neq 0$. The fold-flip bifurcation is *not* λ^k-nondegenerate, since

$$c_{NS} = (R^2 + 2bcRS + 8s_{21})\lambda^{2k} + o(\lambda^{2k}),$$

where s_{21} is the coefficient of the $\lambda^{2k}X^2Y$-term in the rescaled Poincaré map.

Accumulation of curves for orientation-preserving f_μ



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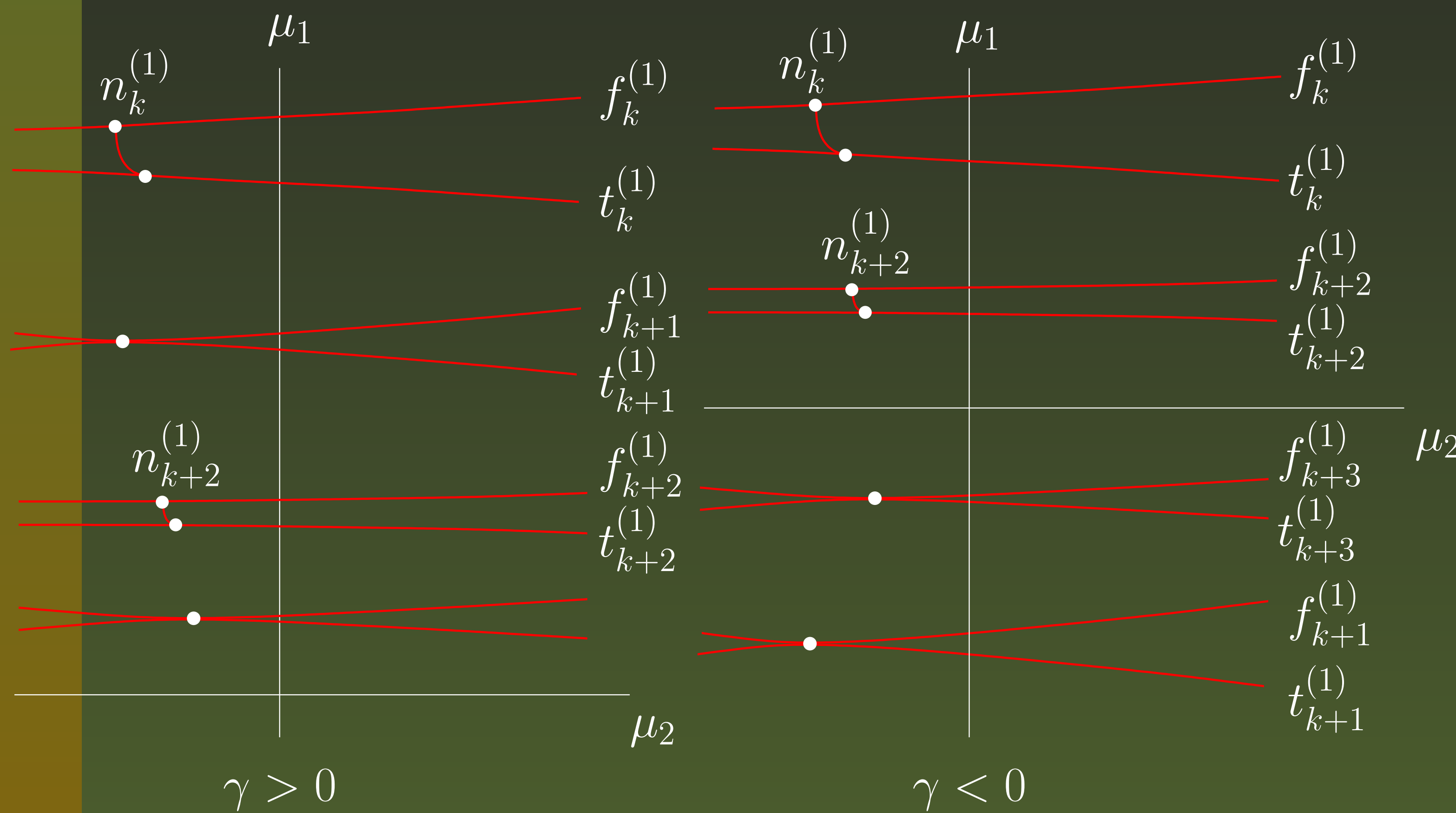
Accumulation of curves for orientation-reversing f_μ



Open research problems

- Explicit expression for s_{21}
- Dependence of DHT on R
- Accumulation of 1:2 resonances [Homburg, Kuznetsov, Meijer; work in progress]

Lyapunov chart