Lecture 1. Filippov systems: Sliding solutions and bifurcations

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References:

- P.T. Piiroinen and Yu.A. Kuznetsov. An event-driven method to simulate Filippov systems with accurate computing of sliding motions. *ACM TOMS* **34** (2008), no.3, Article 13, 24p.
- Yu.A. Kuznetsov, S. Rinaldi, and A. Gragnani. One-parameter bifurcations in planar Filippov systems. *Int. J. Bifuraction & Chaos* **13**(2003), 2157-2188
- M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk. *Piecewise-smooth Dynamical Systems: Theory and Applications.* Springer-Verlag, London, 2008.
- A.F. Filippov. *Differential Equations with Discontinuous Right-Hand Sides.* Kluwer Academic, Dordrecht, 1988.

Contents

1. Standard and sliding solutions of Filippov systems

Consider a discontinuous system

$$\dot{x} = \begin{cases} f^{(1)}(x), & x \in S_1, \\ f^{(2)}(x), & x \in S_2, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$,

$$S_1 = \{x \in \mathbb{R}^n : H(x) < 0\}, \quad S_2 = \{x \in \mathbb{R}^n : H(x) > 0\},$$

$H : \mathbb{R}^n \to \mathbb{R}$ is smooth with $H_x(x) \neq 0$ on the **discontinuity boundary**

$$\Sigma = \{x \in \mathbb{R}^n : H(x) = 0\},$$

and $f^{(i)} : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions.

Orbits of (1) are defined by concatenation of **standard** and **sliding** orbit segments.

For $x \in \Sigma$, define

$$\sigma(x) = \langle H_x(x), f^{(1)}(x)\rangle \langle H_x(x), f^{(2)}(x)\rangle$$

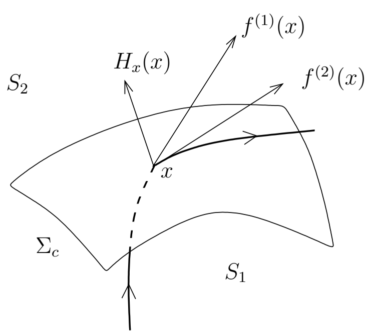
and introduce the sets of

- **crossing points:** $\Sigma_c = \{x \in \Sigma : \sigma(x) > 0\}$
- **sliding points:** $\Sigma_s = \{x \in \Sigma : \sigma(x) \leq 0\}$
- **regular sliding points:**

$$\widehat{\Sigma}_s = \{x \in \Sigma_s : \langle H_x(x), f^{(2)}(x) - f^{(1)}(x)\rangle \neq 0\}.$$

Crossing orbits:

At $x \in \Sigma_c$, concatenate the standard orbit of $f^{(1)}$ reaching x from S_1 with the standard orbit of $f^{(2)}$ departing from x into S_2, or vice versa.



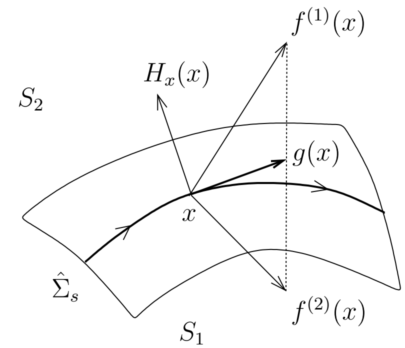
Sliding orbits:

For $x \in \widehat{\Sigma}_s$ define the **Filippov vector**

$$g(x) = \lambda(x) f^{(1)}(x) + (1 - \lambda(x)) f^{(2)}(x),$$

where

$$\lambda(x) = \frac{\langle H_x(x), f^{(2)}(x)\rangle}{\langle H_x(x), f^{(2)}(x) - f^{(1)}(x)\rangle}.$$



Utkin's equivalent control method:

One can write

$$g(x) = \frac{f^{(1)}(x) + f^{(2)}(x)}{2} + \frac{f^{(2)}(x) - f^{(1)}(x)}{2}\mu(x),$$

where

$$\mu(x) = -\frac{\langle H_x(x), f^{(1)}(x) + f^{(2)}(x)\rangle}{\langle H_x(x), f^{(2)}(x) - f^{(1)}(x)\rangle}.$$

It follows that

$$\lambda(x) = \frac{1 - \mu(x)}{2},$$

so that $g = f^{(1)}$ if $\mu = -1$ ($\lambda = 1$) and $g = f^{(2)}$ if $\mu = 1$ ($\lambda = 0$).

This gives the **sliding system**

$$\dot{x} = g(x), \quad x \in \hat{\Sigma}_s. \tag{2}$$

At $x \in \hat{\Sigma}_s$, concatenate the standard orbit of $f^{(i)}$ reaching x from S_i with the maximal sliding orbit of g in $\hat{\Sigma}_s$ departing from x. The sliding orbit can reach a point x at the boundary of $\hat{\Sigma}_s$ that is composed of **singular sliding points**, **boundary equilibria**, and **tangent points**.

An equilibrium of (2) satisfies $g(X) = 0$. We could have

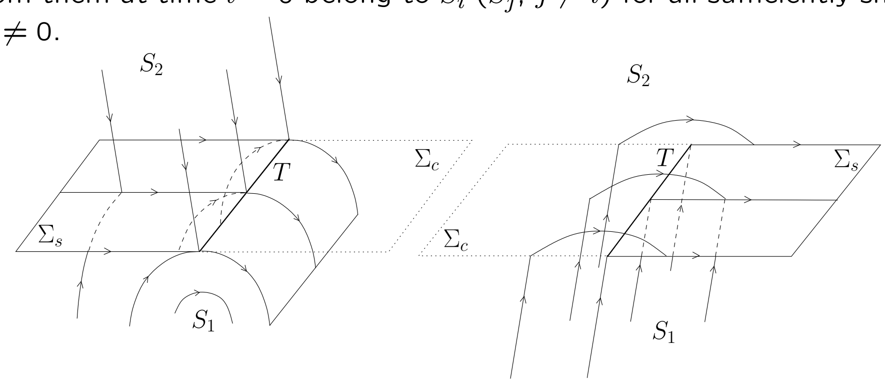
- **pseudo-equilibria** where $f^{(i)}(X)$ are both transversal to Σ_s and anti-collinear;
- **boundary equilibria** where
$$f^{(1)}(X) = 0 \quad \text{or} \quad f^{(2)}(X) = 0.$$

If both $f^{(i)}(T) \neq 0$ but

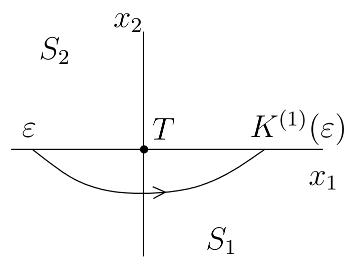
$$\langle H_x(T), f^{(1)}(T) \rangle = 0 \quad \text{or} \quad \langle H_x(T), f^{(2)}(T) \rangle = 0,$$

point $T \in \hat{\Sigma}_s$ is called a **tangent point**. Note that $\mu(T) = \pm 1$, while $\lambda(T) = 0$ or 1.

Tangent points are called **visible** (**invisible**) if the orbits of $f^{(i)}$ starting from them at time $t = 0$ belong to S_i (S_j, $j \neq i$) for all sufficiently small $|t| \neq 0$.



Quadratic tangent point in 2D



$$x_2 = \frac{1}{2}\nu x_1^2 + O(x_1^3)$$

If $f^{(1)}(x) = \begin{pmatrix} p + ax_1 + bx_2 + \cdots \\ cx_1 + dx_2 + \frac{1}{2}qx_1^2 + rx_1x_2 + \frac{1}{2}sx_2^2 + \cdots \end{pmatrix}$,

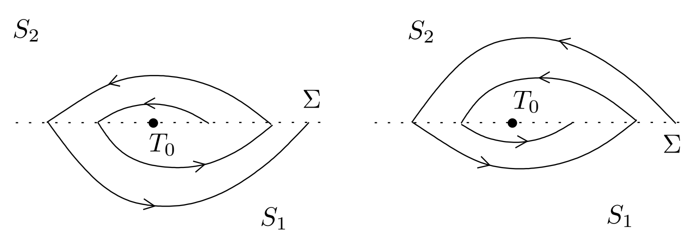
then $\nu = \dfrac{c}{p}$ and

$$K^{(1)}(\varepsilon) = -\varepsilon + k_2^{(1)}\varepsilon^2 + O(\varepsilon^3), \quad k_2^{(1)} = \frac{2}{3}\left(\frac{a+c}{p} - \frac{q}{2c}\right).$$

Fused focus in 2D (a singular sliding point)

When two invisible tangent points coincide, define the Poincaré map:

$$P(\varepsilon) = \varepsilon + k_2\varepsilon^2 + O(\varepsilon^3), \quad k_2 = k_2^{(1)} - k_2^{(2)}.$$



(a) (b)

2. Numerical integration of sliding solutions

The regular sliding set $\widehat{\Sigma}_s$ is a **neutral** invariant manifold for the Filippov vector field

$$g(x) = \frac{f^{(1)}(x) + f^{(2)}(x)}{2} + \frac{f^{(2)}(x) - f^{(1)}(x)}{2}\mu(x), \quad x \in \mathbb{R}^n,$$

where

$$\mu(x) = -\frac{\langle H_x(x), f^{(1)}(x) + f^{(2)}(x)\rangle}{\langle H_x(x), f^{(2)}(x) - f^{(1)}(x)\rangle}.$$

Moreover, $\widehat{\Sigma}_s$ is an **attracting** invariant manifold for the **modified Filippov vector field**:

$$G(x) = g(x) - H(x)H_x(x), \quad x \in \mathbb{R}^n, \tag{3}$$

so that the sliding orbits on it can be merely integrated forward in time using $\dot{x} = G(x)$ with $x \in \mathbb{R}^n$ and $x_0 \in \widehat{\Sigma}_s$.

Event functions and variables

$$\begin{aligned} e_1(t) &= H(x(t)) \\ e_2(t) &= \langle H_x(x(t)), f^{(1)}(x(t))\rangle \\ e_3(t) &= \langle H_x(x(t)), f^{(2)}(x(t))\rangle \end{aligned}$$

Define domains

$$M = \{x \in \mathbb{R}^n : |\mu(x)| \geq 1\}, \quad \widehat{M} = \{x \in \mathbb{R}^n : |\mu(x)| < 1\}$$

$x \in$	v	e_1	e_2	e_3
$S_1 \cup M$	$(1,-1,-1,1,-1)$	$e_1^- = 0$	$e_2^\pm = 0$	$e_3^\pm = 0$
$S_1 \cup \widehat{M}$	$(1,-1,-1,-1,1)$	$e_1^- = 0$	$e_2^\pm = 0$	$e_3^\mp = 0$
$S_2 \cup M$	$(-1,1,-1,1,-1)$	$e_1^+ = 0$	$e_2^\pm = 0$	$e_3^\pm = 0$
$S_2 \cup \widehat{M}$	$(-1,1,-1,-1,1)$	$e_1^+ = 0$	$e_2^\pm = 0$	$e_3^\mp = 0$
$\widehat{\Sigma}_s$	$(-1,-1,1,-1,1)$	$-$	$e_2^\pm = 0$	$e_3^\mp = 0$

Use $f^{(1)}(x)$ if $v_1 = 1$, $f^{(2)}(x)$ if $v_2 = 1$, and $G(x)$ if $v_3 = 1$.

3. Codim 1 bifurcations of 2D Filippov systems

$$\dot{x} = \begin{cases} f^{(1)}(x,\alpha), & H(x,\alpha) < 0, \\ f^{(2)}(x,\alpha), & H(x,\alpha) > 0, \end{cases} \tag{4}$$

where $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is smooth with $H_x(x,\alpha) \neq 0$ on the discontinuity boundary

$$\Sigma(\alpha) = \{x \in \mathbb{R}^n : H(x,\alpha) = 0\},$$

and $f^{(i)} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are smooth functions.

Two systems (4) corresponding to different parameter values are called **topologically equivalent** if there is a homeomorphism of $\mathbb{R}^n$ that maps any standard/sliding orbit segment of the first system onto the standard/sliding orbit segment of the second system, preserving the direction of time.

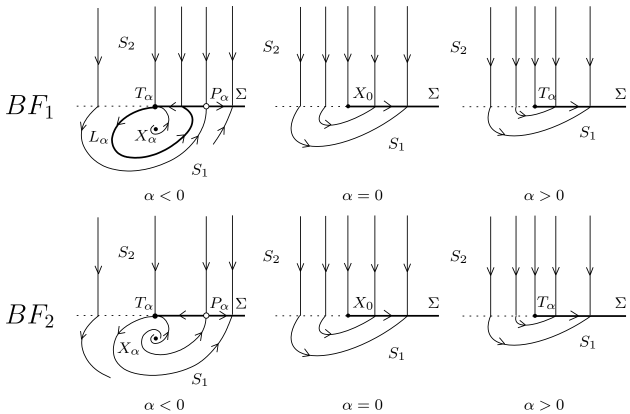
Bifurcations are changes of the topological equivalence class under parameter variations.

3.1 Codim 1 local bifurcations in 2D

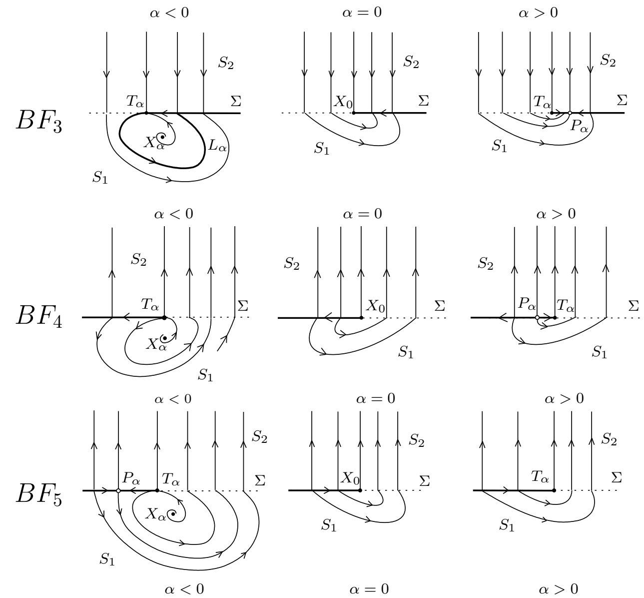
Collisions of

- standard equilibria with Σ
- tangent points
- pseudo-equilibria

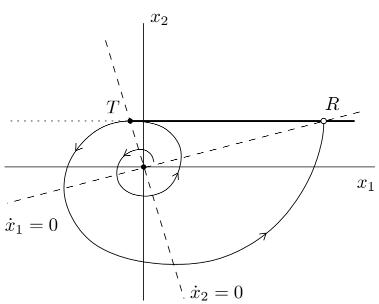
Boundary focus cases



Boundary focus cases (continue)



Degenerate boundary focus: DBF



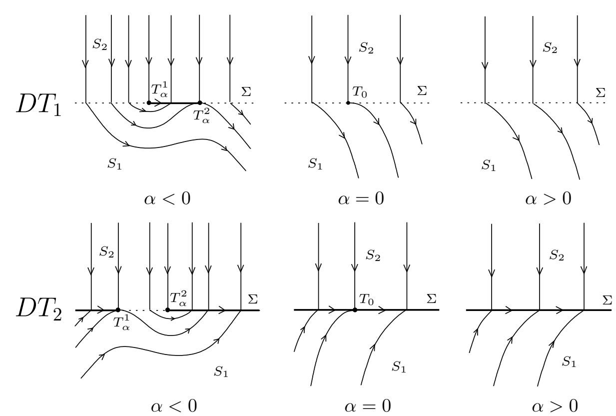
$$\begin{cases} \dot{x}_1 &= ax_1 + bx_2, \\ \dot{x}_2 &= cx_1 + dx_2, \end{cases}$$

$$T = \left(-\frac{d}{c}, 1\right), \quad R = \left(-\frac{b}{a}, 1\right).$$

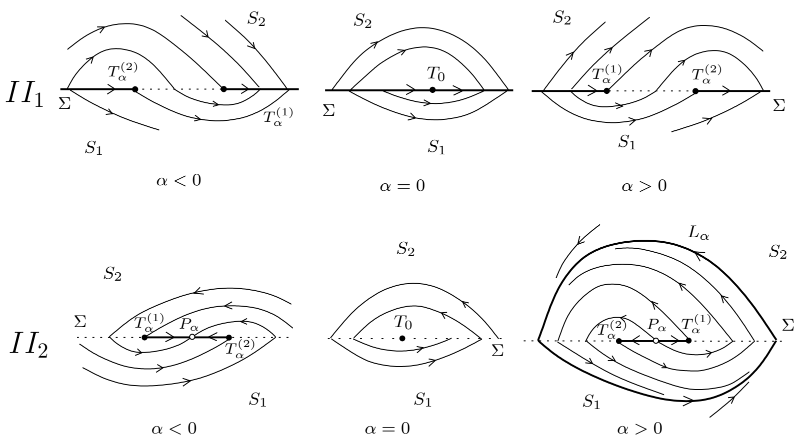
$$\frac{d-a}{2\omega} \operatorname{tg}\left[\frac{\omega}{a+d} \ln\left(-\frac{bc}{a^2}\right)\right] = 1,$$

where $\omega = \frac{1}{2}\sqrt{-(a-d)^2 - 4bc}$.

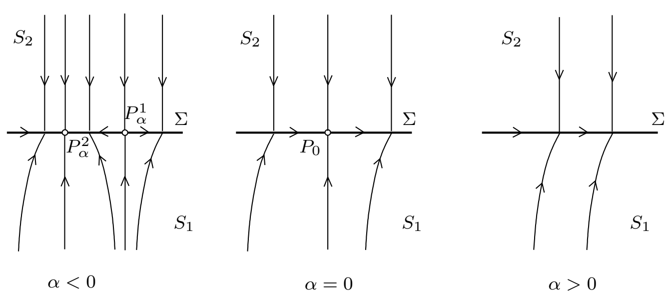
Double tangency



Collision of two invisible tangencies



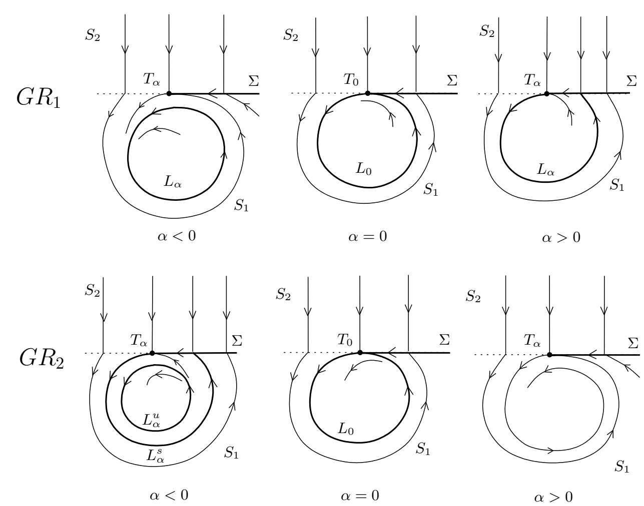
Pseudo-saddle-node: PSN



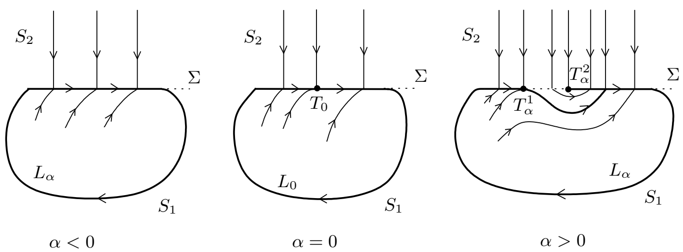
3.2 Codim 1 global bifurcations in 2D

- Bifurcations of sliding cycles:
  - Grazing-sliding
  - Adding-sliding
  - Switching-sliding
  - Crossing-sliding
- Pseudo-homoclinic bifurcations:
  - Homoclinic orbit to a pseudo-saddle-node
  - Homoclinic orbit to a pseudo-saddle
- Sliding homoclinic orbit to a saddle
- Pseudo-heteroclinic bifurcations

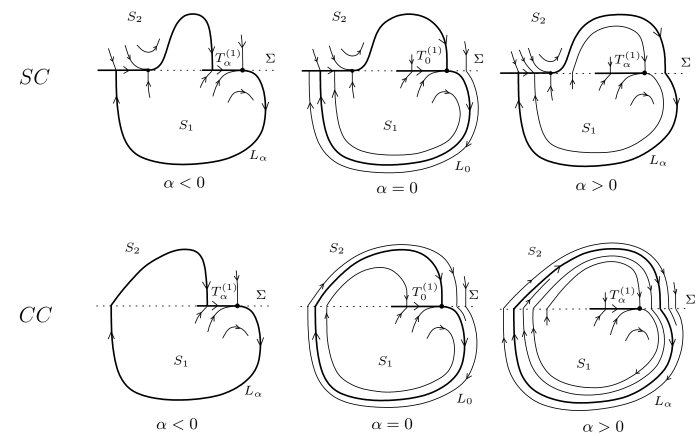
Grazing-sliding cases



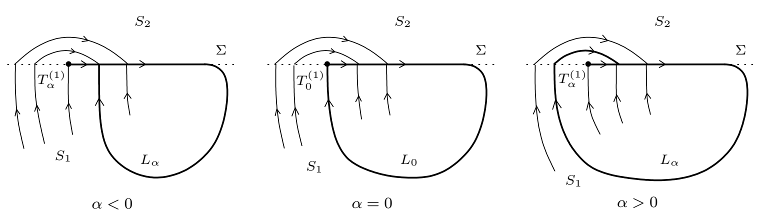
Adding-sliding: DT_2 with global reinjection



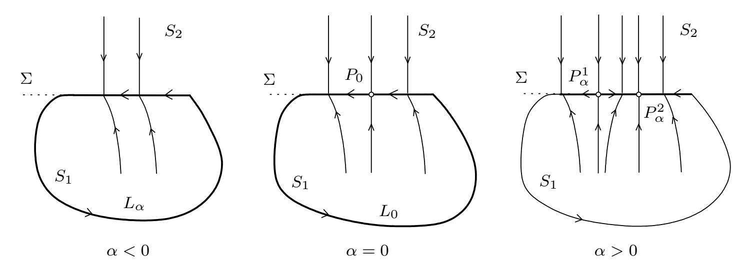
Crossing-sliding cases



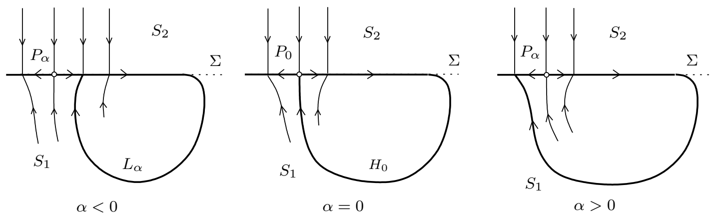
Switching-sliding



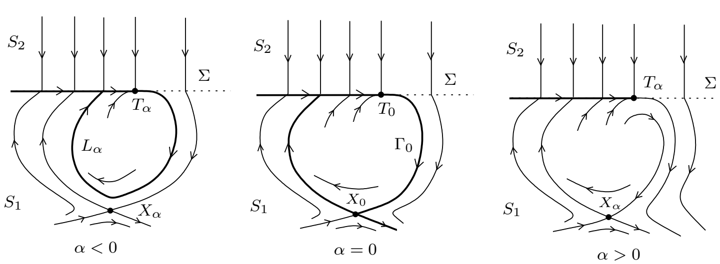
Homoclinic orbit to a pseudo-saddle-node



Homoclinic orbit to a pseudo-saddle: HPS



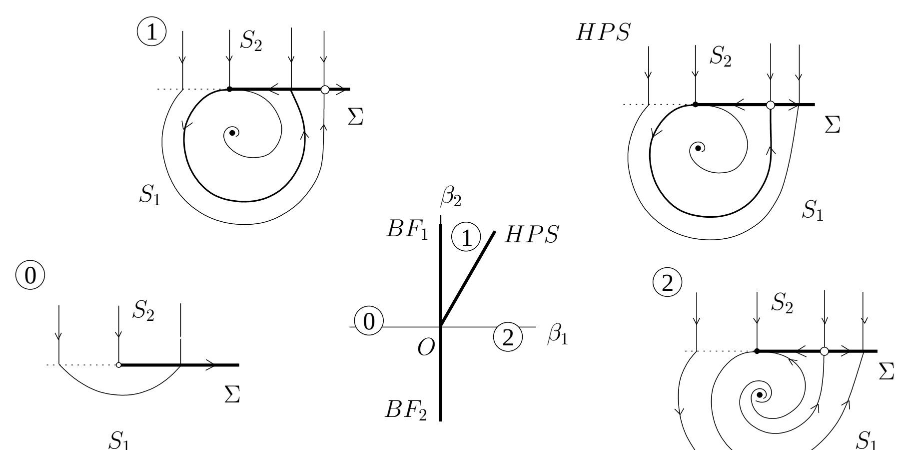
Sliding homoclinic orbit to a saddle



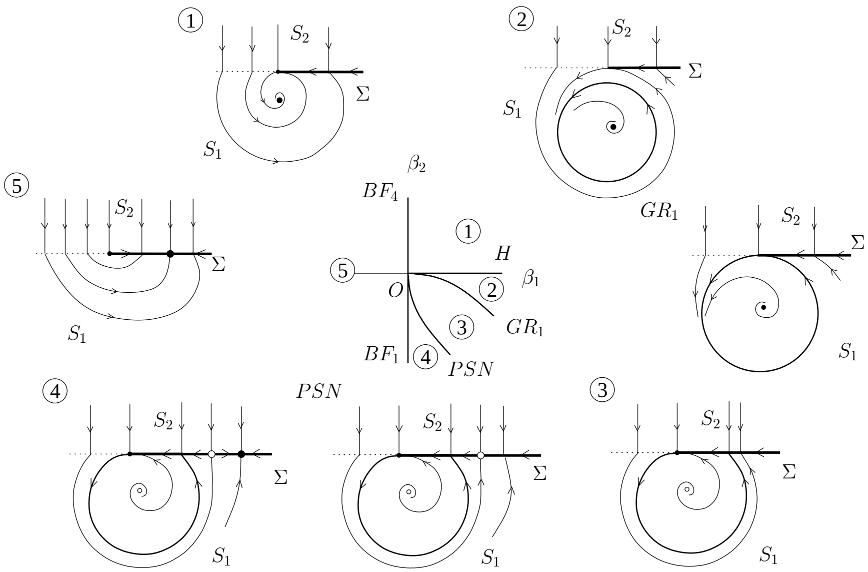
4. Codim 2 bifurcations in 2D Filippov systems

- Local bifurcations:
  - Degenerate boundary focus
  - Boundary Hopf
- Global bifurcations:
  - Sliding-grazing of a nonhyperbolic cycle (fold-grazing)

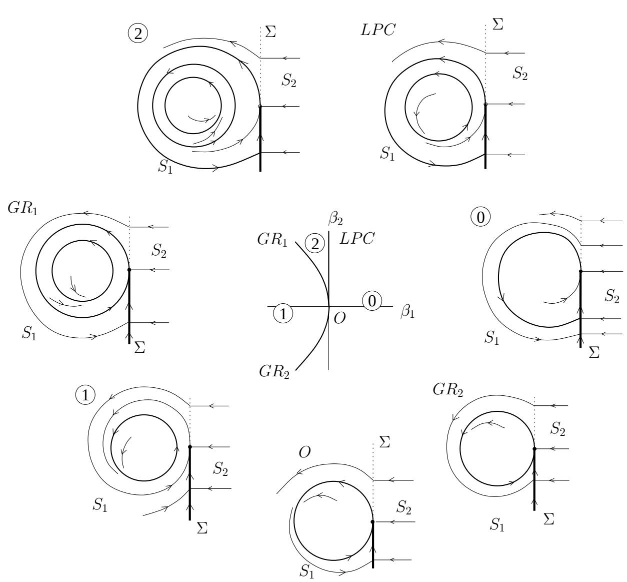
Degenerate boundary focus: DBF



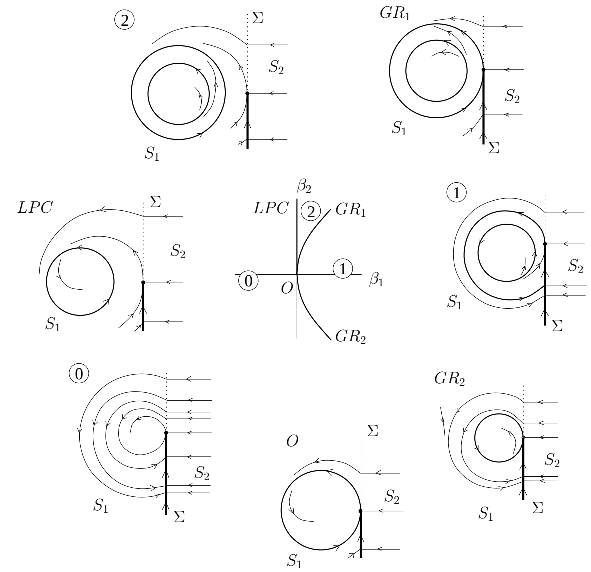
Boundary Hopf: BHP



Fold-grazing: FG_1



Fold-grazing: FG_2



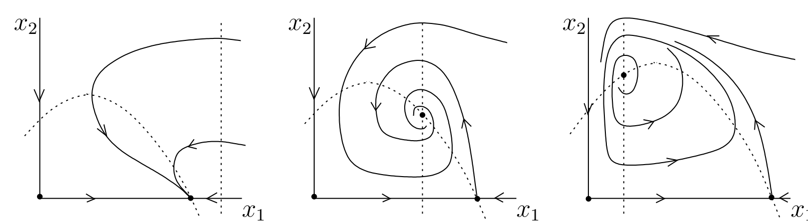
5. Example: Controlled harvesting a prey-predator community

Rosenzweig-MacArthur-Holling model:

$$\begin{cases} \dot{x}_1 &= x_1(1-x_1) - \dfrac{a x_1 x_2}{\alpha_2 + x_1} \\ \dot{x}_2 &= \dfrac{a x_1 x_2}{\alpha_2 + x_1} - c x_2 \end{cases}$$

Nontrivial zero-isoclines:

$$x_2 = \frac{1}{a}(\alpha_2 + x_1)(1 - x_1), \quad x_1 = \frac{\alpha_2 c}{a - c}.$$



(a) (b) (c)

Relay control by harvesting

Assume that the predator population is harvested at constant effort $e > 0$ only when abundant ($x_2 > \alpha_5$). This leads to a planar Filippov system:

$$\dot{x} = \begin{cases} f^{(1)}(x), & x_2 - \alpha_5 < 0, \\ f^{(2)}(x), & x_2 - \alpha_5 > 0, \end{cases}$$

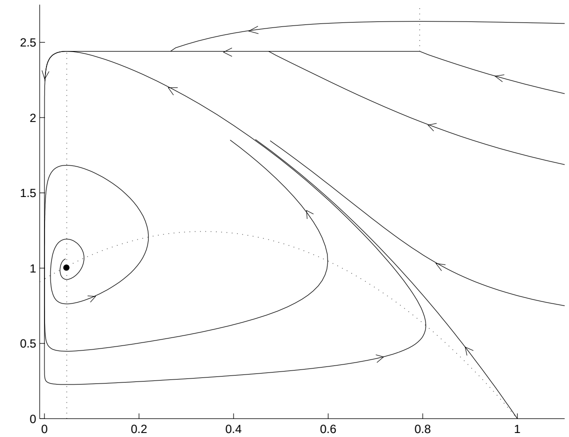
where

$$f^{(1)} = \begin{pmatrix} x_1(1-x_1) - \psi(x_1)x_2 \\ \psi(x_1)x_2 - dx_2 \end{pmatrix}, \quad f^{(2)} = \begin{pmatrix} x_1(1-x_1) - \psi(x_1)x_2 \\ \psi(x_1)x_2 - dx_2 - ex_2 \end{pmatrix},$$

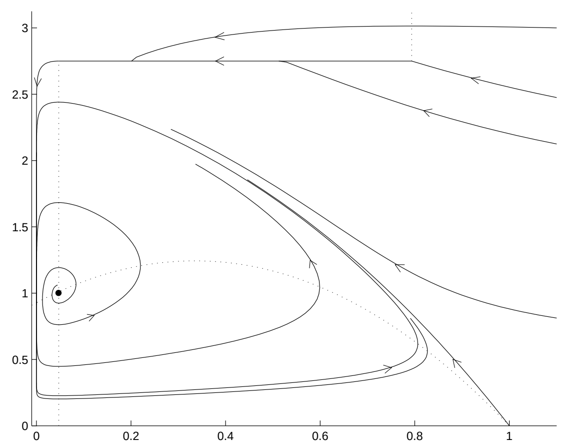
$$\psi(x_1) = \frac{ax_1}{\alpha_2 + x_1}.$$

Fix $a = 0.3556$, $d = 0.0444$, $e = 0.2067$ and $\alpha_2 = 0.33$

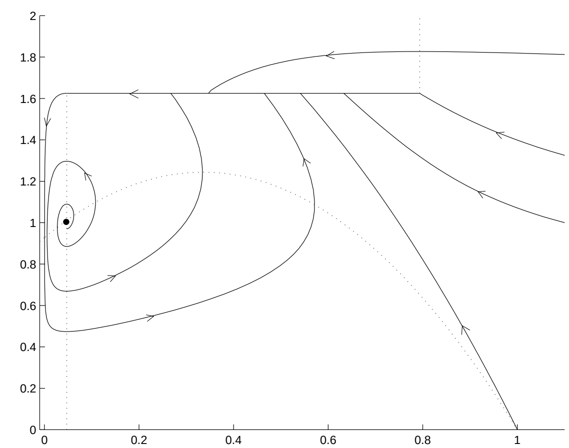
Grazing-sliding GR_1: $\alpha_5 \approx 2.440$



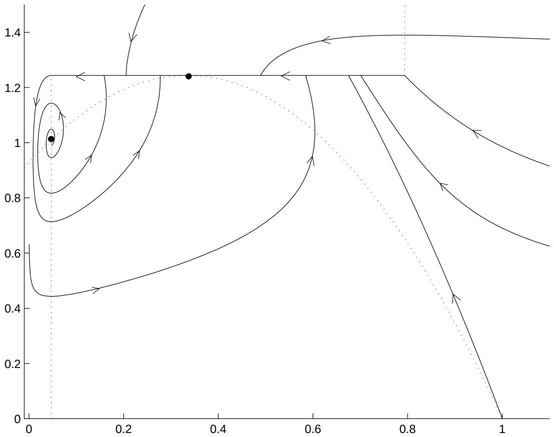
Stable standard cycle: $\alpha_5 = 2.75$



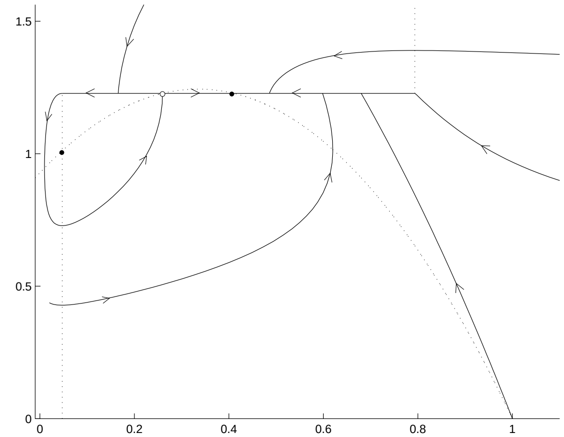
Stable sliding cycle: $\alpha_5 = 1.625$



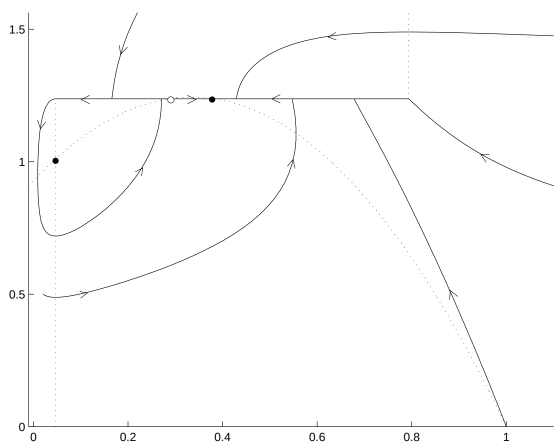
Pseudo-saddle-node PSN: $\alpha_5 \approx 1.2437$



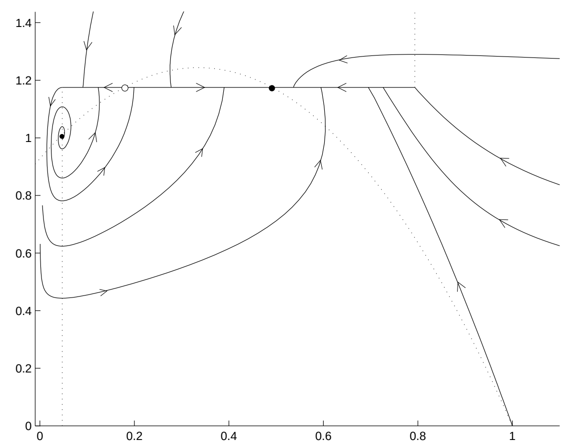
Homoclinic orbit to pseudo-saddle HPS: $\alpha_5 \approx 1.2277$



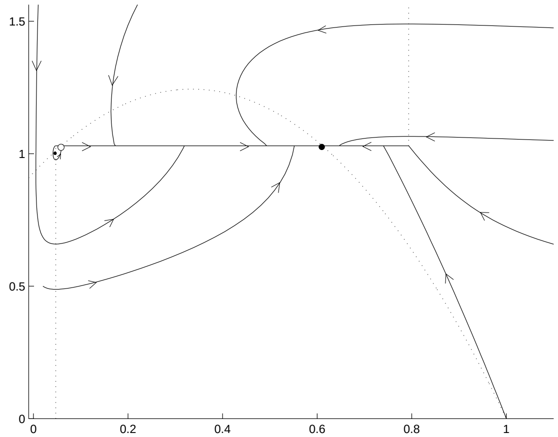
Stable sliding cycle and pseudo-node: $\alpha_5 \approx 1.2375$



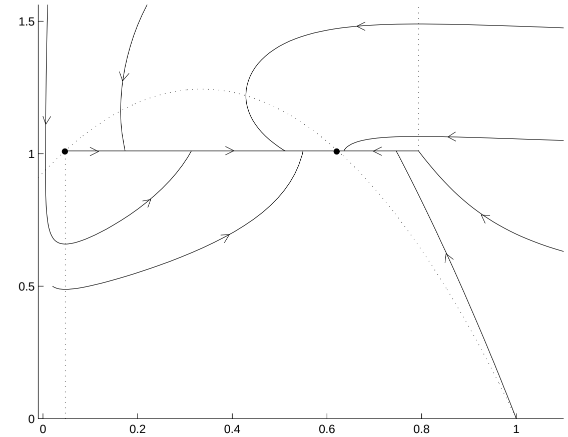
Stable pseudo-node: $\alpha_5 \approx 1.175$



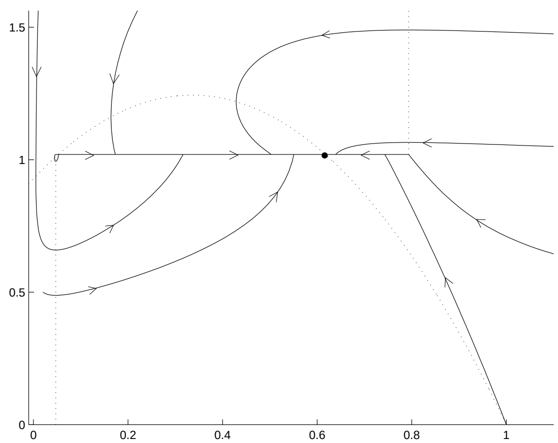
Homoclinic orbit to pseudo-saddle HPS: $\alpha_5 \approx 1.03$



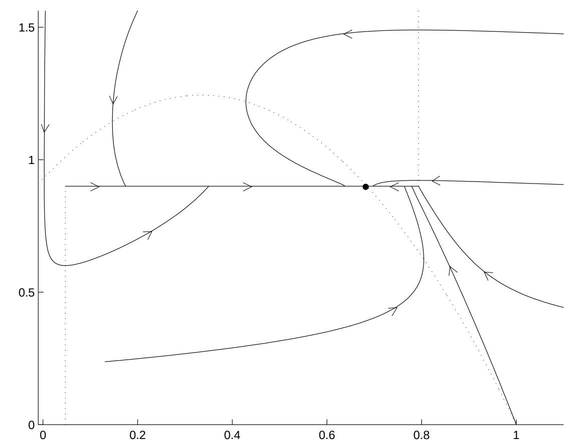
Boundary focus BF_1: $\alpha_5 \approx 1.01017$



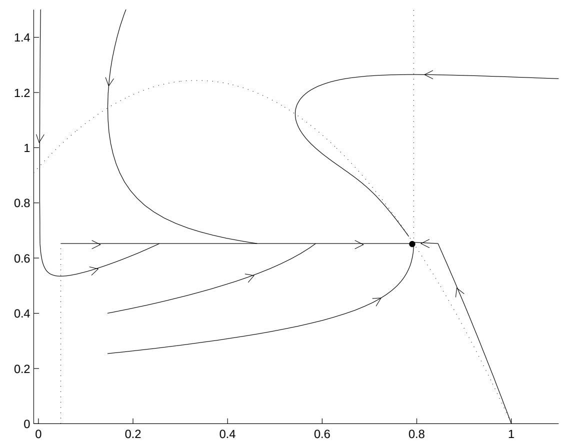
Stable sliding cycle (small) and pseudo-node: $\alpha_5 = 1.02$



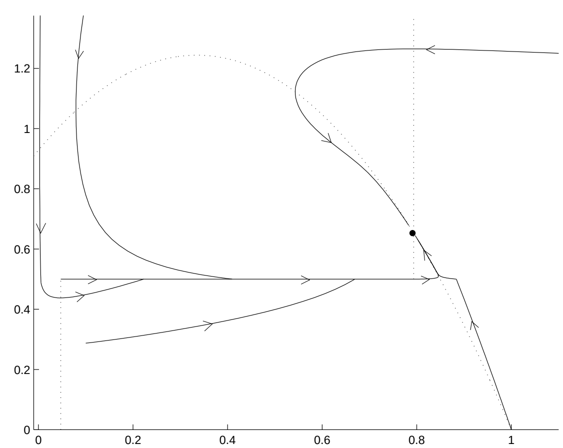
Stable pseudo-node: $\alpha_5 = 0.9$



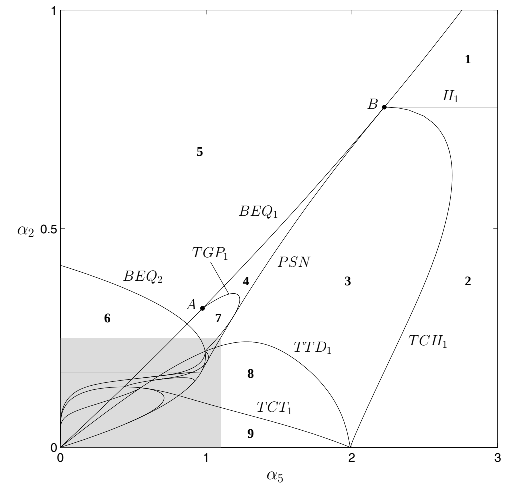
Boundary node BN_1: $\alpha_5 \approx 0.6527$



Stable node: $\alpha_5 = 0.5$



Two-parameter bifurcation diagram



A - degenerate boundary focus (DBF); B - boundary Hopf (BHP).