

Lecture 2. Filippov systems: Continuation of sliding bifurcations

Yuri A. Kuznetsov

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References:

- Dercole, F. and Kuznetsov, Yu.A. User Guide to SclideCont 2.0, 2005. <http://www.math.uu.nl/people/kuznet/cm/slidecont.pdf>
- Dercole, F. and Kuznetsov, Yu.A. SlideCont: An AUTO97 driver for bifurcation analysis of Filippov systems. *ACM TOMS* **31** (2005), no.1, 95-119.
- Kowalczyk, P., di Bernardo, M., Champneys, A. R., Hogan, S. J., Homer, M., Piiroinen, P. T., Kuznetsov, Yu. A., and Nordmark, A. Two-parameter discontinuity-induced bifurcations of limit cycles: Classification and open problems. *Int. J. Bifurcation & Chaos* **16** (2006), 601-629.

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1. Reminder on Filippov systems

$$\dot{x} = \begin{cases} f^{(1)}(x, \alpha), & x \in S_1(\alpha), \\ f^{(2)}(x, \alpha), & x \in S_2(\alpha), \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$,

$$S_1(\alpha) = \{x \in \mathbb{R}^n : H(x, \alpha) < 0\}, \quad S_2(\alpha) = \{x \in \mathbb{R}^n : H(x, \alpha) > 0\},$$

$H : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth with $H_x(x, \alpha) \neq 0$ on the **discontinuity boundary**

$$\Sigma(\alpha) = \{x \in \mathbb{R}^n : H(x, \alpha) = 0\}.$$

For $x \in \Sigma(\alpha)$, define $\sigma(x, \alpha) = \langle H_x(x, \alpha), f^{(1)}(x, \alpha) \rangle \langle H_x(x, \alpha), f^{(2)}(x, \alpha) \rangle$ and introduce the **crossing set**:

$$\Sigma_c(\alpha) = \{x \in \Sigma : \sigma(x, \alpha) > 0\},$$

and the **sliding set**:

$$\Sigma_s(\alpha) = \{x \in \Sigma : \sigma(x, \alpha) \leq 0\}.$$

Points $x \in \Sigma_s(\alpha)$, where

$$\langle H_x(x, \alpha), f^{(2)}(x, \alpha) - f^{(1)}(x, \alpha) \rangle = 0$$

are called **singular sliding points**.

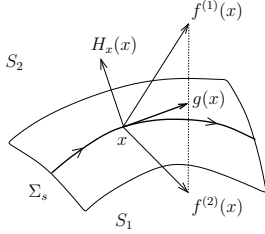
Crossing orbits: At $x \in \Sigma_c(\alpha)$ cross.

Sliding orbits: For a regular sliding point $x \in \Sigma_s(\alpha)$ define the **Filippov vector**

$$g(x, \alpha) = \lambda(x, \alpha)f^{(1)}(x, \alpha) + (1 - \lambda(x, \alpha))f^{(2)}(x, \alpha),$$

where

$$\lambda(x, \alpha) = \frac{\langle H_x(x, \alpha), f^{(2)}(x, \alpha) \rangle}{\langle H_x(x, \alpha), f^{(2)}(x, \alpha) - f^{(1)}(x, \alpha) \rangle}.$$



This gives the **sliding system**:

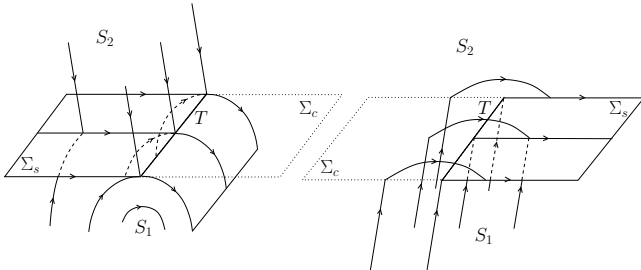
$$\dot{x} = g(x, \alpha), \quad x \in \Sigma_s(\alpha). \quad (2)$$

- **pseudo-equilibrium:** $g(P, \alpha) = 0$ and $f^{(i)}(P, \alpha)$ are transversal to $\Sigma_s(\alpha)$ and anti-collinear;

- **boundary equilibrium:** $f^{(i)}(X, \alpha) = 0$;

- **tangent point:** Both $f^{(i)}(T, \alpha) \neq 0$ but

$$\langle H_x(T, \alpha), f^{(i)}(T, \alpha) \rangle = 0.$$

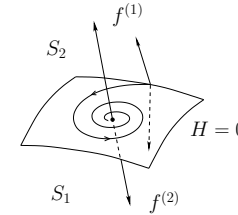


2. Supported n -dimensional cases

2.1. Local objects

Pseudo-equilibrium:

$$\begin{cases} H(x, \alpha) = 0, \\ \lambda_1 f^{(1)}(x, \alpha) + \lambda_2 f^{(2)}(x, \alpha) = 0, \\ \lambda_1 + \lambda_2 - 1 = 0. \end{cases}$$



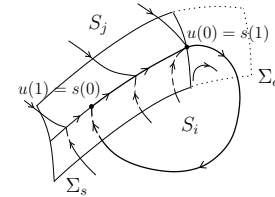
Boundary equilibrium:

$$\begin{cases} f^{(i)}(x, \alpha) = 0, \\ H(x, \alpha) = 0. \end{cases}$$

2.2. Global objects

Sliding cycle with one standard segment:

$$\begin{cases} \dot{u} - T_i f^{(i)}(u, \alpha) = 0, \\ \dot{s} - T_0 g(s, \alpha) = 0, \\ H(u(0), \alpha) = 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle = 0, \\ s(0) - u(1) = 0, \\ s(1) - u(0) = 0. \end{cases}$$



Notice that the condition $H(u(1), \alpha) = 0$ holds automatically due to the invariance of Σ_s .

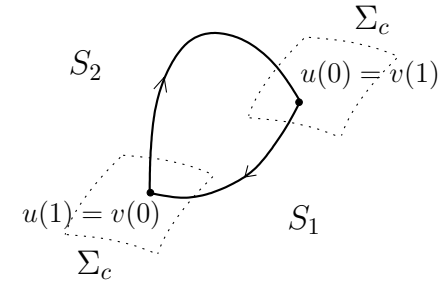
Singularities and test functions:

$$t(u, \alpha) = \begin{pmatrix} H_{x_2}(u, \alpha) \\ -H_{x_1}(u, \alpha) \end{pmatrix} \quad (n = 2)$$

- boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n = 2$): $\langle t(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle$
- tangent point of $f^{(j)}$ at $u(0)$: $\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha) \rangle$
- tangent point of $f^{(i)}$ at $u(1)$: $\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha) \rangle$
- tangent point of $f^{(j)}$ at $u(1)$: $\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha) \rangle$
- pseudo-equilibrium at $u(1)$ ($n = 2$): $\langle t(u(1), \alpha), g(u(1), \alpha) \rangle$

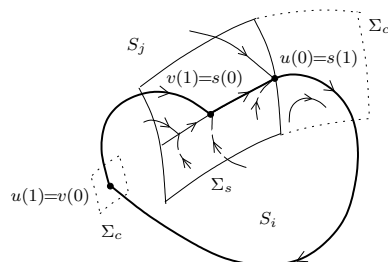
Crossing cycle:

$$\begin{cases} \dot{u} - T_1 f^{(1)}(u, \alpha) = 0, \\ \dot{v} - T_2 f^{(2)}(v, \alpha) = 0, \\ H(u(0), \alpha) = 0, \\ H(u(1), \alpha) = 0, \\ u(1) - v(0) = 0, \\ v(1) - u(0) = 0. \end{cases}$$



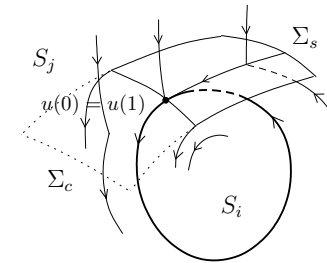
Sliding cycle with two standard segments:

$$\begin{cases} \dot{u} - T_i f^{(i)}(u, \alpha) = 0, \\ \dot{v} - T_j f^{(j)}(v, \alpha) = 0, \\ \dot{s} - T_0 g(s, \alpha) = 0, \\ H(u(0), \alpha) = 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle = 0, \\ H(u(1), \alpha) = 0, \\ v(0) - u(1) = 0, \\ s(0) - v(1) = 0, \\ s(1) - u(0) = 0. \end{cases}$$



Grazing cycle:

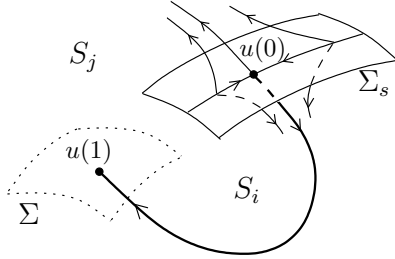
$$\begin{cases} \dot{u} - T f^{(i)}(u, \alpha) = 0, \\ H(u(0), \alpha) = 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle = 0, \\ u(0) - u(1) = 0. \end{cases}$$



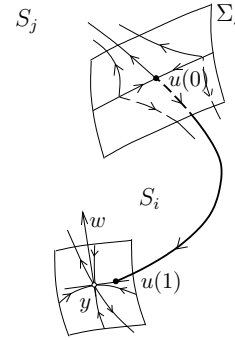
Grazing in multidimensional ($n \geq 3$) Filippov systems can create chaos.

Pseudo-equilibrium to boundary connection:

$$\left\{ \begin{array}{l} \dot{u} - Tf^{(i)}(u, \alpha) = 0, \\ H(u(0), \alpha) = 0, \\ \lambda_i f^{(i)}(u(0), \alpha) + \lambda_j f^{(j)}(u(0), \alpha) = 0, \\ \lambda_i + \lambda_j - 1 = 0, \\ H(u(1), \alpha) = 0. \end{array} \right.$$



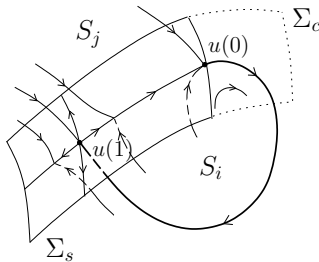
Pseudo-equilibrium to saddle connection:



$$\left\{ \begin{array}{l} \dot{u} - Tf^{(i)}(u, \alpha) = 0, \\ H(u(0), \alpha) = 0, \\ \lambda_i f^{(i)}(u(0), \alpha) + \lambda_j f^{(j)}(u(0), \alpha) = 0, \\ \lambda_i + \lambda_j - 1 = 0, \\ f^{(i)}(y, \alpha) = 0, \\ [f_x^{(i)}(y, \alpha)]^T w - \nu_k w = 0, \\ \langle w, w \rangle - 1 = 0, \\ \langle w, y - u(1) \rangle = 0. \end{array} \right.$$

Tangent point to pseudo-equilibrium connection:

$$\left\{ \begin{array}{l} \dot{u} - Tf^{(i)}(u, \alpha) = 0, \\ H(u(0), \alpha) = 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle = 0, \\ H(u(1), \alpha) = 0, \\ \lambda_i f^{(i)}(u(1), \alpha) + \lambda_j f^{(j)}(u(1), \alpha) = 0, \\ \lambda_i + \lambda_j - 1 = 0. \end{array} \right.$$



3. Introduction to AUTO97

• **Algebraic continuation problems:**

$$F(x, \alpha, \beta) = 0, \quad F: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_d} \rightarrow \mathbb{R}^{n_d}$$

with $n + m + m_d = n_d + 1$.

• **Boundary-value continuation problems:**

$$\left\{ \begin{array}{l} \dot{u}(\tau) - F(u(\tau), \alpha, \beta) = 0, \quad \tau \in [0, 1], \\ B(u(0), u(1), \alpha, \beta) = 0, \end{array} \right.$$

where

$$F: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_d} \rightarrow \mathbb{R}^{n_d}, \quad B: \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \times \mathbb{R}^m \times \mathbb{R}^{m_d} \rightarrow \mathbb{R}^{n_b}$$

with $m + m_d = n_b - n_d + 1$.

q.exp

```

1 19 5 3 1 1 21 3 59 5 4 100
0.0000000000E+00 -2.5400193050E-28 4.0001091088E+00
5.0413306178E-02 1.9688222065E-01 3.8041759126E+00
1.0082661236E-01 3.8286645953E-01 3.5668745507E+00
1.5123991853E-01 5.5573216144E-01 3.2829394737E+00
2.0165322471E-01 7.1301770175E-01 2.9480725088E+00
2.5018727968E-01 8.4725515962E-01 2.5751571496E+00
2.9872133464E-01 9.6218085712E-01 2.1527534092E+00
3.4725538961E-01 1.0554613113E+00 1.6840075999E+00
3.9578944457E-01 1.1249993513E+00 1.1755113926E+00
4.4423319077E-01 1.1690295031E+00 6.3828072652E-01
4.9267693697E-01 1.1865703437E+00 8.4311078622E-02
5.4112068316E-01 1.1771612119E+00 -4.7191551630E-01
5.8956442936E-01 1.1410487251E+00 -1.0156810991E+00
6.4075124035E-01 1.0749264131E+00 -1.5617190061E+00
6.9193805133E-01 9.8188498909E-01 -2.0659892005E+00
7.4312486232E-01 8.6429393184E-01 -2.5198546141E+00
7.9431167330E-01 7.2485488175E-01 -2.9189739029E+00
8.4573375498E-01 5.6563915198E-01 -3.2643082784E+00
8.9715583665E-01 3.9005236537E-01 -3.5564406234E+00
9.4857791833E-01 2.0071768661E-01 -3.7998531288E+00
1.0000000000E+00 3.8411471564E-28 -4.0001091083E+00

```

```

1 18 0 0 3.49429E+00 6.87210E-01 1.06466E+00 3.64921E+00
1 19 5 3 3.51383E+00 7.61924E-01 1.18657E+00 4.00011E+00
1 20 0 0 3.36507E+00 9.95030E-01 1.56701E+00 5.05340E+00
1 21 0 0 3.04297E+00 1.21473E+00 1.93274E+00 5.99852E+00
1 22 0 0 2.66628E+00 1.42154E+00 2.28360E+00 6.85569E+00
1 23 0 0 2.27928E+00 1.62442E+00 2.63283E+00 7.67335E+00
1 24 -4 4 2.00000E+00 1.77497E+00 2.89525E+00 8.26877E+00
1 25 0 0 1.59587E+00 2.01109E+00 3.30800E+00 9.18846E+00
1 26 0 0 1.24155E+00 2.25066E+00 3.73614E+00 1.01095E+01
1 27 0 0 9.44505E-01 2.49327E+00 4.17518E+00 1.10349E+01
1 28 0 0 7.04832E-01 2.73800E+00 4.62595E+00 1.19649E+01
1 29 0 0 5.17569E-01 2.98382E+00 5.08064E+00 1.28981E+01
1 30 0 0 3.75005E-01 3.22992E+00 5.53973E+00 1.38332E+01
1 31 0 0 2.68618E-01 3.47599E+00 6.00855E+00 1.47701E+01
1 32 0 0 1.90639E-01 3.72139E+00 6.47350E+00 1.57073E+01
1 33 0 0 1.34202E-01 3.96616E+00 6.93983E+00 1.66451E+01
1 34 0 0 9.37538E-02 4.21081E+00 7.39289E+00 1.75856E+01
1 35 0 0 6.51128E-02 4.45446E+00 7.85956E+00 1.85257E+01
1 36 0 0 4.49720E-02 4.69748E+00 8.32664E+00 1.94668E+01
1 37 0 0 3.09138E-02 4.93989E+00 8.79679E+00 2.04102E+01
1 38 0 0 2.11478E-02 5.18190E+00 9.26479E+00 2.13541E+01
1 39 0 0 1.44066E-02 5.42347E+00 9.73303E+00 2.22993E+01
1 40 0 0 9.77843E-03 5.66370E+00 1.02097E+01 2.32321E+01
1 41 0 0 6.60585E-03 5.90505E+00 1.06802E+01 2.41780E+01
1 42 0 0 4.44930E-03 6.14592E+00 1.11505E+01 2.51238E+01
1 43 0 0 2.97625E-03 6.38949E+00 1.15971E+01 2.60397E+01
1 44 0 0 1.98818E-03 6.63144E+00 1.20692E+01 2.69823E+01
1 45 0 0 1.32706E-03 6.87210E+00 1.25394E+01 2.79196E+01
1 46 0 0 8.82361E-04 7.11263E+00 1.30506E+01 2.88833E+01
1 47 0 0 5.85165E-04 7.35381E+00 1.35249E+01 2.98277E+01
1 48 0 0 3.87693E-04 7.59414E+00 1.39979E+01 3.07690E+01
1 49 0 0 2.56194E-04 7.83481E+00 1.44740E+01 3.17179E+01
1 50 9 5 1.69018E-04 8.07507E+00 1.49476E+01 3.26599E+01

```

p.exp

```

0 0.0000E+00 4.0000E+00 0.0000E+00 5.0000E+01
0 EPSL= 1.0000E-06 EPSU = 1.0000E-06 EPSS = 1.0000E-04
0 DS = 1.0000E-02 DSMIN= 1.0000E-03 DSMAX= 1.0000E+00
0 NDIM= 2 IPS = 4 IRS = 0 ILP = 1
0 NTST= 5 NCOL= 4 IAD = 3 ISP = 1
0 ISW = 1 IPLT= 3 NEC = 2 NINT= 0
0 NMX= 50 NPR = 50 MXBF= 10 IID = 2
0 ITMX= 8 ITNW= 5 NWTN= 3 JAC= 0 NUZR= 1
0 User-specified parameter: 1
0 Active continuation parameter: 1
0
0 PT TY LAB PAR(1) INTEGRAL U(1) MAX U(1) MAX U(2)
1 1 9 1 0.00000E+00 0.00000E+00 0.00000E+00 0.00000E+00
1 2 0 0 9.57018E-03 7.98279E-04 1.19747E-03 4.78891E-03
1 3 0 0 2.39225E-02 1.99833E-03 2.99780E-03 1.19852E-02
1 4 0 0 4.54445E-02 3.80437E-03 5.70590E-03 2.28089E-02
1 5 0 0 7.77124E-02 6.52697E-03 9.79066E-03 3.91108E-02
1 6 0 0 1.26080E-01 1.06418E-02 1.59662E-02 6.37150E-02
1 7 0 0 1.98551E-01 1.68849E-02 2.53402E-02 1.00969E-01
1 8 0 0 3.07069E-01 2.64144E-02 3.96607E-02 1.57655E-01
1 9 0 0 4.69391E-01 4.10981E-02 6.17532E-02 2.44583E-01
1 10 0 0 7.11732E-01 6.40690E-02 9.63921E-02 3.79569E-01
1 11 0 0 1.07219E+00 1.00924E-01 1.52120E-01 5.93634E-01
1 12 0 0 1.42795E+00 1.41155E-01 2.13188E-01 8.23876E-01
1 13 0 0 1.77733E+00 1.85476E-01 2.80806E-01 1.07350E+00
1 14 -4 2 2.00000E+00 2.16936E-01 3.28952E-01 1.24822E+00
1 15 0 0 2.49304E+00 2.99469E-01 4.55976E-01 1.69721E+00
1 16 0 0 2.94174E+00 4.01371E-01 6.13199E-01 2.23398E+00
1 17 0 0 3.30042E+00 5.30089E-01 8.14985E-01 2.88665E+00

```

4. SlideCont 2.0: Makefile

```

#-----
#
# *****
# *
# * Linear oscillator with normal and dry friction *
# *
# *****
#-----
#
PGM      = dryf
RM       = rm -f
#
all: superclean run
#
run: 1 2 3 4 5
#
1:
@echo " "
@echo "1: Forward continuation of a sliding cycle with one standard segment"
@cp sc.$(PGM).1 sc.$(PGM)
@cp $(PGM).f.1 $(PGM).f
@cp $(PGM).dat.1 $(PGM).dat
@"@scdat" $(PGM)
@cp q.$(PGM) q.dat.1
@"@sc" $(PGM)
@"@sv" $(PGM).1
@rm fort.* sc.$(PGM) q.$(PGM) $(PGM).f $(PGM).dat
#

```

sc.dfyf.1

```

2 132 1 1          NDIM,IPS,IRS,ILP
3 2 60 61         NICP,(ICP(I),I=1,NICP)
35 4 3 1 1 0 0 0  NTST,NCOL,IAD,ISP,ISW,IPLT,NBC,NINT
50 0.0 3.0 0.0 100.0 NMX,RLO,RL1,A0,A1
10 0 2 8 7 5 0    NPR,MXBF,IID,ITMX,ITNW,NWTN,JAC
1.e-8 1.e-8 1.e-7  EPSP,EPSP,EPSS
-1.e-3 1.e-10 1.0 1 DS,DSMIN,DSMAX,IADS
0                NTHL,((ITHL(I),THL(I)),I=1,NTHL)
0                NTHU,((ITHU(I),THU(I)),I=1,NTHU)
0                NUZR,((IUZR(I),UZR(I)),I=1,NUZR)
0 0 2            SCISTART,SCIDIFF
2 3 4           SCNPSI,(SCIPSI(I),I=1,SCNPSI)
0              SCNFIXED,(SCIFIXED(I),I=1,SCNFIXED)

```

```

C-----
C-----
C                               dryf.f
C-----
C
C   SlideCont
C
C   An AUTO97 driver for sliding bifurcation analysis
C
C   BY   Fabio Dercole & Yuri Kuznetsov
C
C   VERSION 2.0 (last revision 4/2004)
C
C-----
C
C *****
C *
C * Self-excited autoparametric system with dry friction (planar) *
C *
C *****
C-----
C
C   SUBROUTINE SCFUNC(SCNDIM,X,PAR,SCIDIFF,FI,
+   DFIDX,DFIDP,DFIDXDX,DFIDXDP,I)
C-----
C
C   IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C

```

```

INCLUDE 'dryf.h'
C
INTEGER SCNDIM,SCIDIFF
C
DIMENSION X(SCNDIM),PAR(*),FI(SCNDIM)
DIMENSION DFIDX(SCNDIM,SCNDIM),DFIDP(SCNDIM,*)
DIMENSION DFIDXDX(SCNDIM,SCNDIM,SCNDIM)
DIMENSION DFIDXDP(SCNDIM,SCNDIM,*)
C
C   Debug
C   IF (DEBUG) THEN
C       IF (DBGLEV.GE.DBGL2) THEN
C           PRINT *, 'enter in SCFUNC',I
C       END IF
C   END IF
C
C-----
C   Vector field selection
C-----
C   IF (I.EQ.1) THEN
C
C-----
C   Vector field 1
C-----
C
C   Right-hand side
C-----
C   FI(1)=X(2)
C   FI(2)=-X(1)+PAR(1)*PAR(3)-PAR(2)*PAR(4)*(PAR(5)-X(2))
C
C   First derivatives
C-----
C
C   IF (SCIDIFF.GE.1) THEN
C
C       with respect to state
C       DFIDX(1,2)=1.0D0
C       DFIDX(2,1)=-1.0D0
C       DFIDX(2,2)=PAR(2)*PAR(4)
C
C       with respect to parameters
C       DFIDP(2,1)=PAR(3)
C       DFIDP(2,2)=-PAR(4)*(PAR(5)-X(2))
C       DFIDP(2,3)=PAR(1)
C       DFIDP(2,4)=-PAR(2)*(PAR(5)-X(2))
C       DFIDP(2,5)=-PAR(2)*PAR(4)
C
C   END IF
C
C   Second derivatives
C-----
C   IF (SCIDIFF.GE.2) THEN
C
C       with respect to state twice
C
C       with respect to state and parameters
C       DFIDXDP(2,2,2)=PAR(4)
C       DFIDXDP(2,2,4)=PAR(2)
C
C   END IF
C
C   ELSE IF (I.EQ.2) THEN
C
C-----
C   Vector field 2

```

```

c -----
c
c Right-hand side
c -----
c FI(1)=X(2)
c FI(2)=-X(1)-PAR(1)*PAR(3)-PAR(2)*PAR(4)*(PAR(5)-X(2))
c
c First derivatives
c -----
c IF (SCIDIFF.GE.1) THEN
c
c with respect to state
c DFIDX(1,2)=1.0D0
c DFIDX(2,1)=-1.0D0
c DFIDX(2,2)=PAR(2)*PAR(4)
c
c with respect to parameters
c DFIDP(2,1)=-PAR(3)
c DFIDP(2,2)=-PAR(4)*(PAR(5)-X(2))
c DFIDP(2,3)=-PAR(1)
c DFIDP(2,4)=-PAR(2)*(PAR(5)-X(2))
c DFIDP(2,5)=-PAR(2)*PAR(4)
c
c END IF
c
c Second derivatives
c -----
c IF (SCIDIFF.GE.2) THEN
c
c with respect to state twice
c
c with respect to state and parameters
c
c
c
c DFIDXDP(2,2,2)=PAR(4)
c DFIDXDP(2,2,4)=PAR(2)
c
c END IF
c
c ELSE
c PRINT *, 'SCFUNC error: I=', I
c STOP
c END IF
c
c Debug
c IF (DEBUG) THEN
c IF (DBGLEV.GE.DBGL2) THEN
c PRINT *, 'exit from SCFUNC', I
c END IF
c END IF
c
c RETURN
c END
c
c -----
c SUBROUTINE SCBOUND(SCNDIM,X,PAR,SCIDIFF,
+ H,DHDX,DHDP,DHDXDX,DHDXDP)
c -----
c
c IMPLICIT DOUBLE PRECISION (A-H,O-Z)
c
c INCLUDE 'dryf.h'
c
c INTEGER SCNDIM,SCIDIFF

```

```

DIMENSION X(SCNDIM),PAR(*),DHDX(SCNDIM),DHDP(*)
DIMENSION DHDXDX(SCNDIM,SCNDIM),DHDXDP(SCNDIM,*)
c
c Debug
c IF (DEBUG) THEN
c IF (DBGLEV.GE.DBGL2) THEN
c PRINT *, 'enter in SCBOUND'
c END IF
c END IF
c
c -----
c Discontinuity boundary
c -----
c
c boundary function
c -----
c H=X(2)-PAR(5)
c
c First derivatives
c -----
c IF (SCIDIFF.GE.1) THEN
c
c with respect to state
c DHDX(2)=1.0D0
c
c with respect to parameters
c DHDP(5)=-1.0D0
c
c END IF
c
c Second derivatives
c -----
c
c IF (SCIDIFF.GE.2) THEN
c
c with respect to state twice
c
c with respect to state and parameters
c
c END IF
c
c Debug
c IF (DEBUG) THEN
c IF (DBGLEV.GE.DBGL2) THEN
c PRINT *, 'exit from SCBOUND'
c END IF
c END IF
c
c RETURN
c END
c
c -----
c SUBROUTINE SCSTPNT(SCNDIM,X,PAR,T,I)
c -----
c
c IMPLICIT DOUBLE PRECISION (A-H,O-Z)
c
c INCLUDE 'dryf.h'
c
c INTEGER SCNDIM
c
c DIMENSION X(SCNDIM),PAR(*)
c
c Debug

```



```

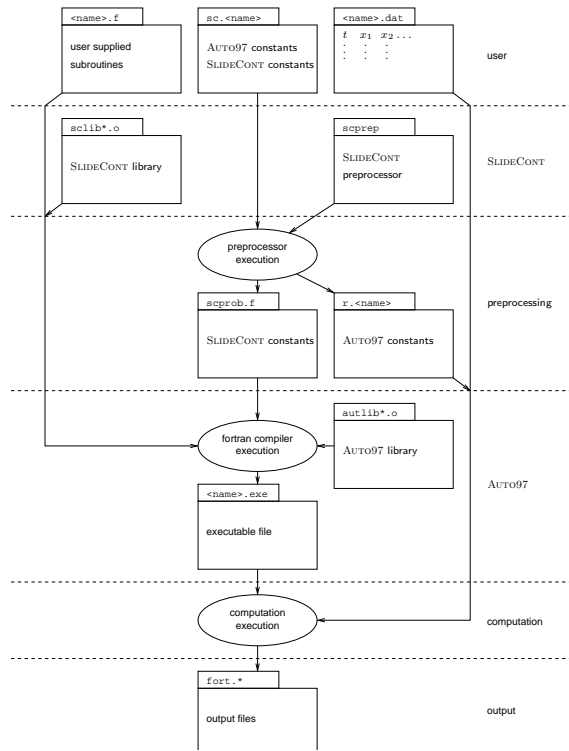
IF (DEBUG) THEN
  IF (DBGLEV.GE.DBGL2) THEN
    PRINT *, 'enter in SCSTPNT', I
  END IF
END IF
c
c   Initialize parameters
c   -----
c
PAR(1)=0.1D0
PAR(2)=0.03D0
PAR(3)=4.0D0
PAR(4)=4.0D0
PAR(5)=0.5D0
PAR(60)=1.02448D0
PAR(61)=5.46477D0
c
c   Initialize solution
c   -----
c
IF (I.EQ.1) THEN
c
  Init state of vector field 1
  -----
c
ELSE IF (I.EQ.2) THEN
c
  Init state of vector field 2
  -----
c
ELSE IF (I.EQ.3) THEN
c
c
c   Init state
c   -----
c
END IF
c
c   Debug
c   IF (DEBUG) THEN
c     IF (DBGLEV.GE.DBGL2) THEN
c       PRINT *, 'exit from SCSTPNT', I
c     END IF
c   END IF
c
RETURN
END
c
c   -----
c
SUBROUTINE SCPVLS(SCNDIM,X,PAR,I)
c
c   -----
c
  IMPLICIT DOUBLE PRECISION (A-H,O-Z)
c
  INCLUDE 'dryf.h'
c
  INTEGER SCNDIM
c
  DIMENSION X(SCNDIM), PAR(*)
c
c   Debug
c   IF (DEBUG) THEN
c     IF (DBGLEV.GE.DBGL2) THEN
c       PRINT *, 'enter in SCPVLS'

```

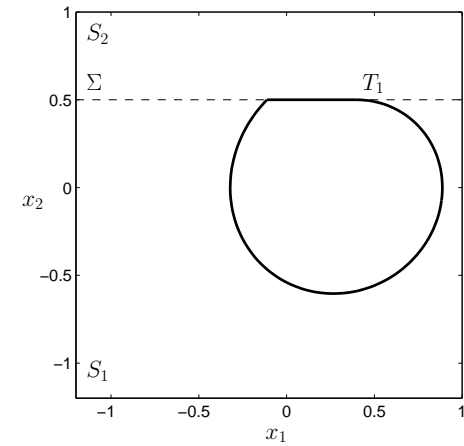
```

END IF
END IF
c
c   User functions
c   -----
c
PAR(?)=?
c
c   Debug
c   IF (DEBUG) THEN
c     IF (DBGLEV.GE.DBGL2) THEN
c       PRINT *, 'exit from SCPVLS'
c     END IF
c   END IF
c
RETURN
END
c
c   -----
c
SUBROUTINE SCBCND
c
c   -----
c
  Dummy subroutine
c
RETURN
END
c
c   -----
c
SUBROUTINE SCICND
c
c   -----
c
c
c   Dummy subroutine
c
RETURN
END
c
c   -----
c
SUBROUTINE SCFOPT
c
c   -----
c
  Dummy subroutine
c
RETURN
END

```



A sliding cycle:



$$\varepsilon = 0.1, \mu = 0.03, \alpha_0 = \alpha_1 = 4, v = 0.5$$

Continuation of the sliding cycle in $\text{PAR}(2)=\mu$

BR	PT	TY	LAB	PAR(2)	...	PAR(11)	PAR(43)	...
1	50	2		2.65719E-02	...	5.50782E+00	4.73382E-01	...
1	100	3		2.30473E-02	...	5.55611E+00	4.32653E-01	...
1	150	4		1.93686E-02	...	5.61187E+00	3.88868E-01	...
1	200	5		1.58932E-02	...	5.67093E+00	3.45696E-01	...
1	250	6		1.23734E-02	...	5.73921E+00	2.99217E-01	...
1	300	7		8.80398E-03	...	5.82108E+00	2.47461E-01	...
1	350	8		5.12502E-03	...	5.92802E+00	1.84932E-01	...
1	400	9		9.60983E-04	...	6.12808E+00	7.81608E-02	...
1	423	UZ	10	1.13542E-15	...	6.28319E+00	-1.55730E-08	...
1	425	EP	11	7.34401E-06	...	6.29677E+00	-6.79324E-03	...

Label 10: **grazing bifurcation (O)**

BR	PT	TY	LAB	PAR(2)	...	PAR(11)	...	PAR(44)
1	50	12		3.00584E-02	...	5.46407E+00	...	2.87101E-01
1	100	13		3.38037E-02	...	5.42081E+00	...	2.45180E-01
1	150	14		3.76239E-02	...	5.38008E+00	...	2.02723E-01
1	200	15		4.14958E-02	...	5.34182E+00	...	1.59781E-01
1	250	16		4.53988E-02	...	5.30595E+00	...	1.16404E-01
1	300	17		4.93144E-02	...	5.27237E+00	...	7.26398E-02
1	350	18		5.32267E-02	...	5.24098E+00	...	2.85347E-02
1	383	UZ	19	5.57355E-02	...	5.22189E+00	...	8.24871E-09
1	400	EP	20	5.83763E-02	...	5.20261E+00	...	-3.02850E-02

Label 19: **switching (buckling) bifurcation (SW)**

4. Examples

4.1. A simple planar example [Tondl, 1970]

$$\ddot{x}_1 + x_1 = \varepsilon \alpha_0 \text{sign}(v - \dot{x}_1) - \mu \alpha_1 (v - \dot{x}_1).$$

It is equivalent to (1) with $x_2 = \dot{x}_1$,

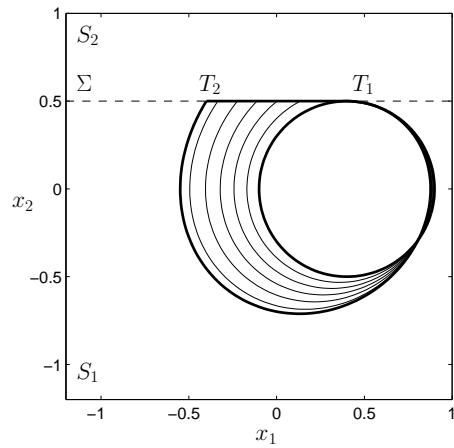
$$S_1 = \{x \in \mathbb{R}^2 : x_2 < v\}, S_2 = \{x \in \mathbb{R}^2 : x_2 > v\}$$

and

$$f^{(1)}(x) = \begin{pmatrix} x_2 \\ -x_1 + \varepsilon \alpha_0 - \mu \alpha_1 (v - x_2) \end{pmatrix},$$

$$f^{(2)}(x) = \begin{pmatrix} x_2 \\ -x_1 - \varepsilon \alpha_0 - \mu \alpha_1 (v - x_2) \end{pmatrix}.$$

The family of sliding cycles without crossings:



Continuation of the crossing cycle:

BR	PT	TY	LAB	PAR(2)	...	PAR(12)	PAR(41)
1	20		2	7.16260E-02	...	2.55611E+00	1.07609E+00
1	40		3	7.65905E-02	...	2.50756E+00	9.33090E-01
1	60		4	8.18743E-02	...	2.45264E+00	7.93320E-01
1	80		5	8.73065E-02	...	2.39121E+00	6.58564E-01
1	100		6	9.25997E-02	...	2.32381E+00	5.30933E-01
1	120		7	9.73634E-02	...	2.25165E+00	4.12404E-01
1	140		8	1.01168E-01	...	2.17637E+00	3.04205E-01
1	160		9	1.03627E-01	...	2.09946E+00	2.06403E-01
1	180	LP	10	1.04455E-01	...	2.02413E+00	1.20594E-01
1	200		11	1.03528E-01	...	1.94532E+00	3.94702E-02
1	211	UZ	12	1.02312E-01	...	1.90398E+00	-1.02862E-11
1	220	EP	13	1.00829E-01	...	1.86710E+00	-3.35819E-02

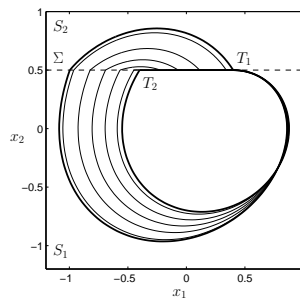
Label 10: **limit point of crossing cycles (LP)**

Label 12: **crossing bifurcation (CC)**

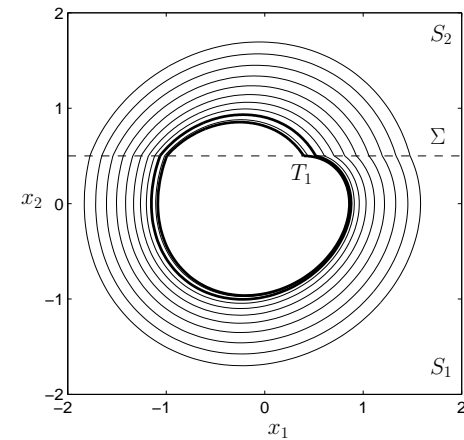
Continuation of the sliding cycle with a crossing:

BR	PT	TY	LAB	PAR(2)	...	PAR(11)	PAR(12)	PAR(45)
1	60		2	5.90369E-02	...	5.19791E+00	1.52241E-01	7.61637E-01
1	120		3	6.24130E-02	...	5.17465E+00	3.10107E-01	7.20834E-01
1	180		4	6.60618E-02	...	5.15083E+00	4.81592E-01	6.74597E-01
1	240		5	7.00156E-02	...	5.12647E+00	6.66284E-01	6.21526E-01
1	300		6	7.44073E-02	...	5.10105E+00	8.67216E-01	5.58318E-01
1	360		7	7.93300E-02	...	5.07446E+00	1.08361E+00	4.81218E-01
1	420		8	8.48984E-02	...	5.04663E+00	1.31315E+00	3.84667E-01
1	480		9	9.18210E-02	...	5.01510E+00	1.57172E+00	2.48336E-01
1	540		10	1.01313E-01	...	4.97688E+00	1.87536E+00	2.60810E-02
1	547	UZ	11	1.02312E-01	...	4.97317E+00	1.90398E+00	2.02644E-12
1	600	EP	12	1.10002E-01	...	4.94653E+00	2.10506E+00	-2.20614E-01

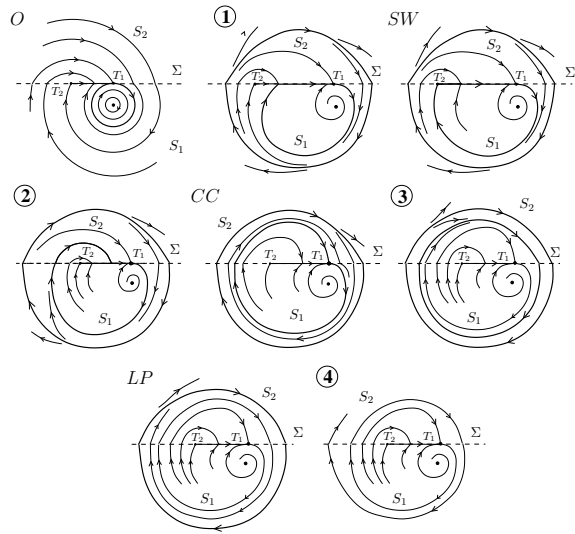
Label 11: **crossing bifurcation (CC)**



The family of crossing cycles:

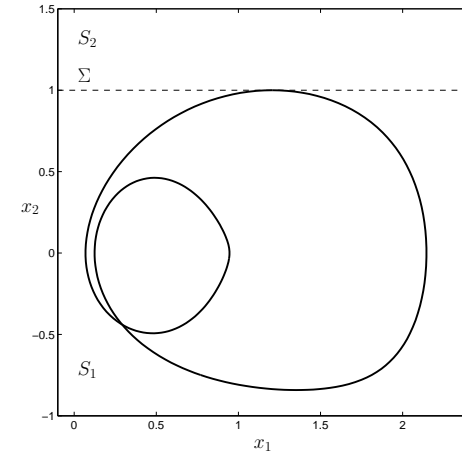


Phase portraits

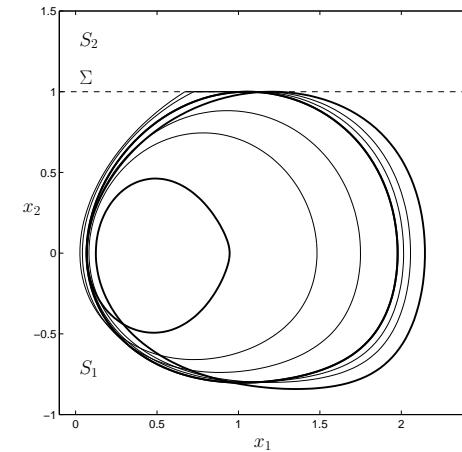


There is a grazing $\frac{8\pi}{\alpha_5}$ -cycle at

$$\alpha_1 = \alpha_2 = 1.5, \alpha_3 = 0.45, \alpha_4 = 0.1, \alpha_5 = 1.7078\dots$$



A family of sliding cycles originated from the $\frac{8\pi}{\alpha_5}$ -periodic grazing cycle:



4.2. Forced dry-friction oscillations

$$\ddot{x}_1 + x_1 = \alpha_1 \operatorname{sgn}(1 - \dot{x}_1) - \alpha_2(1 - \dot{x}_1) + \alpha_3(1 - \dot{x}_1)^3 + \alpha_4 \cos(\alpha_5 t).$$

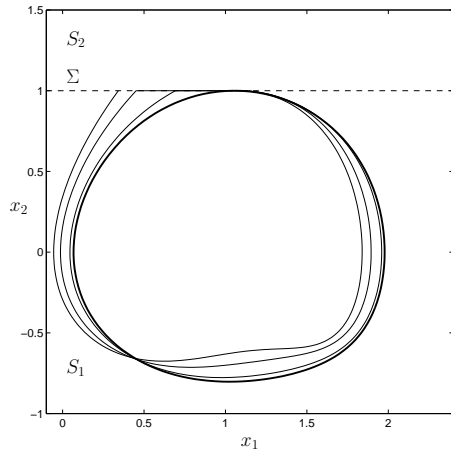
[Yoshitake & Sueoka, 2002]

It is equivalent to (1) with $x \in \mathbb{R}^4$, $H(x, \alpha) = x_2 - 1$,

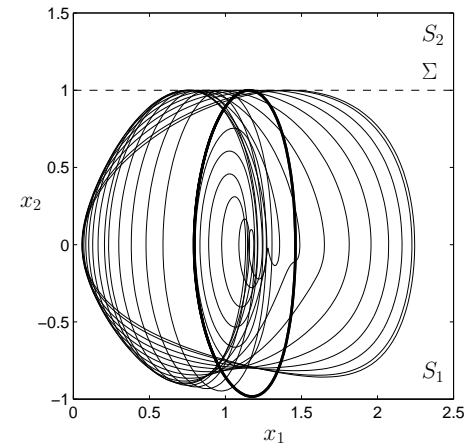
$$f^{(1,2)}(x, \alpha) = \begin{pmatrix} x_2 \\ -x_1 + g_{1,2}(x, \alpha) + \alpha_4 x_3 \\ x_3 - \alpha_5 x_4 - x_3(x_3^2 + x_4^2) \\ \alpha_5 x_3 + x_4 - x_4(x_3^2 + x_4^2) \end{pmatrix},$$

where $g_{1,2}(x, \alpha) = \pm \alpha_1 - \alpha_2(1 - x_2) + \alpha_3(1 - x_2)^3$.

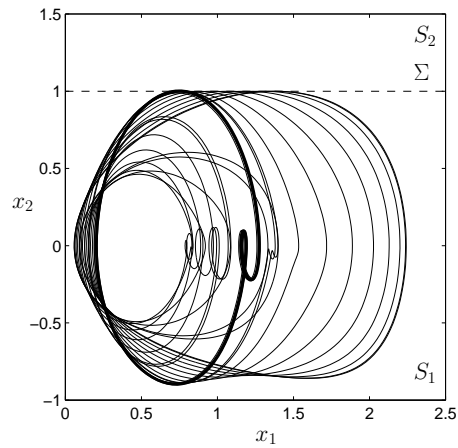
A family of sliding cycles originated from the $\frac{4\pi}{\alpha_5}$ -periodic grazing cycle:



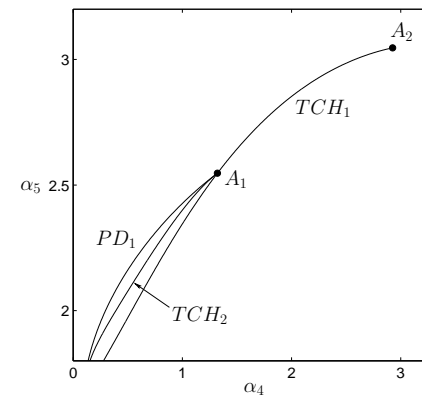
A family of $\frac{4\pi}{\alpha_5}$ -periodic grazing cycles:



A family of $\frac{8\pi}{\alpha_5}$ -periodic grazing cycles:



Sliding bifurcation curves:



TCH_1 - grazing bifurcation of the $\frac{4\pi}{\alpha_5}$ -cycle
 TCH_2 - grazing bifurcation of the $\frac{8\pi}{\alpha_5}$ -cycle
 $A_{1,2}$ - codimension-2 flip-grazing points