Bialternate matrix product

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1. Introduction

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \lambda_1, \lambda_2, \lambda_3$$

Characteristic polynomial:

$$\lambda^3 - \sigma\lambda^2 + \rho\lambda - \Delta = 0,$$

where

$$\begin{aligned} \sigma &= a_{11} + a_{22} + a_{33} = \mathrm{Tr}(A), \\ \rho &= a_{11}a_{22} + a_{11}a_{33} - a_{12}a_{21} + \\ &\quad a_{22}a_{33} - a_{13}a_{31} - a_{23}a_{32}, \\ \Delta &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + \\ &\quad a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22} \\ &= \det(A). \end{aligned}$$

If $\lambda_{1,2} = \pm i\omega,\ \omega \neq 0$, then the **Hurwitz condition** holds:

$$\sigma\rho - \Delta = 0.$$

Amazingly,

$$\sigma\rho - \Delta = \det \underbrace{\begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}}_{2A \odot I_3}.$$

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2. Wedge product of vectors

Def. 1 *Two index pairs* $(i, j), (m, n)$ *are listed in the* **lexicographic order** *if either* $i < m$ *or* $(i = m$ *and* $j < n)$.

Def. 2 *The* **wedge product** *of two vectors,*

$v = (v_1, v_2, \ldots, v_n)^\mathsf{T}$, $w = (w_1, w_2, \ldots, w_n)^\mathsf{T} \in \mathbb{C}^n$,

is a vector

$$v \wedge w \in \mathbb{C}^m, \quad m = \frac{1}{2}n(n-1),$$

with the components:

$$(v \wedge w)_{(i,j)} = v_i w_j - v_j w_i, \quad n \geq i > j \geq 1,$$

listed in the lexicographic order of their index pairs.

For example, if

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix},$$

then

$$v \wedge w = \begin{pmatrix} (v \wedge w)_{(2,1)} \\ (v \wedge w)_{(3,1)} \\ (v \wedge w)_{(3,2)} \end{pmatrix} = \begin{pmatrix} v_2 w_1 - v_1 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_3 w_2 - v_2 w_3 \end{pmatrix}.$$

Lemma 1 *For any* $v, w, w_{1,2} \in \mathbb{C}^n,\ \lambda \in \mathbb{C}:$
(i) $v \wedge w = -w \wedge v$;
(ii) $v \wedge (\lambda w) = \lambda(v \wedge w)$;
(iii) $v \wedge (w_1 + w_2) = v \wedge w_1 + v \wedge w_2$.

Lemma 2 *If* $e^i \in \mathbb{C}^n,\ n \geq i \geq 1$, *form a basis in* $\mathbb{C}^n$, *then* $e^i \wedge e^j \in \mathbb{C}^m,\ n \geq i > j \geq 1$, *form a basis in* $\mathbb{C}^m$.

Lemma 3 *Fix a basis* $e^i \in \mathbb{C}^n,\ n \geq i \geq 1$, *in* $\mathbb{C}^n$ *and consider two vectors* $v, w \in \mathbb{C}^n:$

$$v = \sum_i v_i e^i, \quad w = \sum_j w_j e^j,$$

and a vector $u \in \mathbb{C}^m$*:*

$$u = \sum_{n \geq i > j \geq 1} u_{(i,j)} e^i \wedge e^j.$$

If $u = v \wedge w$, *then*

$$u_{(i,j)} = v_i w_j - v_j w_i, \quad n \geq i > j \geq 1.$$

Proof: For $n \geq i, j \geq 1$:

$$\begin{aligned}
v \wedge w &= \sum_i v_i e^i \wedge \sum_j w_j e^j \\
&= \sum_{i,j}^{n} v_i w_j \ (e^i \wedge e^j) \\
&= \sum_{i>j} v_i w_j \ (e^i \wedge e^j) \ + \ \sum_{j>i} v_i w_j \ (e^i \wedge e^j) \\
&= \sum_{i>j} v_i w_j \ (e^i \wedge e^j) \ - \ \sum_{j>i} v_i w_j \ (e^j \wedge e^i) \\
&= \sum_{i>j} v_i w_j \ (e^i \wedge e^j) \ - \ \sum_{i>j} v_j w_i \ (e^i \wedge e^j) \\
&= \sum_{i>j} (v_i w_j - v_j w_i)(e^i \wedge e^j) \\
&= \sum_{i>j} u_{(i,j)} e^i \wedge e^j.
\end{aligned}$$

3. Bialternate product of matrices

Consider two linear transformations of $\mathbb{C}^n$:

$$v \mapsto Av, \quad w \mapsto Bw,$$

where $A, B \in \mathbb{C}^{n\times n}$.

Def. 3 *The transformation of $\mathbb{C}^m$ defined as*

$$(v \wedge w) \mapsto (A \odot B)(v \wedge w) = \frac{1}{2}(Av \wedge Bw - Aw \wedge Bv)$$

is called the **bialternate product** *of the above transformations.*

In particular,

$$(A \odot A)(v \wedge w) = Av \wedge Aw$$

and

$$(2A \odot I_n)(v \wedge w) = Av \wedge w + v \wedge Aw.$$

Th. 1 *The bialternate product is a linear transformation of $\mathbb{C}^m$. Its matrix $A \odot B$ in the basis $e^i \wedge e^j$, $n \geq i > j \geq 1$, has the elements*

$$(A \odot B)_{(p,q),(r,s)} = \frac{1}{2}\left\{ \left| \begin{array}{cc} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{array} \right| + \left| \begin{array}{cc} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{array} \right| \right\},$$

where $n \geq p > q \geq 1$ and $n \geq r > s \geq 1$.

Proof:

$$(A \odot B)(e^r \wedge e^s) \;=\; \frac{1}{2}(Ae^r \wedge Be^s - Ae^s \wedge Be^r)$$

$$= \frac{1}{2}\left[\sum_p a_{pr}e^p \wedge \sum_q b_{qs}e^q - \sum_p a_{ps}e^p \wedge \sum_q b_{qr}e^q\right]$$

$$= \frac{1}{2}\left[\sum_{p,q} a_{pr}b_{qs}(e^p \wedge e^q) \;-\; \sum_{p,q} a_{ps}b_{qr}(e^p \wedge e^q)\right]$$

$$= \frac{1}{2}\left[\sum_{p>q} a_{pr}b_{qs}(e^p \wedge e^q) - \sum_{p<q} a_{qr}b_{ps}(e^p \wedge e^q)\right.$$

$$\left. - \sum_{p>q} a_{ps}b_{qr}(e^p \wedge e^q) + \sum_{p<q} a_{qs}b_{pr}(e^p \wedge e^q)\right]$$

$$= \frac{1}{2}\sum_{p>q}(a_{pr}b_{qs} - a_{ps}b_{qr} + a_{qs}b_{pr} - a_{qr}b_{ps})(e^p \wedge e^q)$$

$$= \sum_{p>q}(A \odot B)_{(p,q),(r,s)}(e^p \wedge e^q).$$

Thus,

$$(A \odot B)_{(p,q),(r,s)} = \frac{1}{2}\left\{\begin{vmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{vmatrix} + \begin{vmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{vmatrix}\right\},$$

where $n \geq p > q \geq 1$ and $n \geq r > s \geq 1$.

Biproduct:

$$(2A \odot I_n)_{(p,q),(r,s)} = \begin{cases} -a_{ps} & r = q, \\ a_{pr} & r \neq p \text{ and } s = q, \\ a_{pp} + a_{qq} & r = p \text{ and } s = q, \\ a_{qs} & r = p \text{ and } s \neq q, \\ -a_{qr} & s = p, \\ 0 & \text{otherwise.} \end{cases}$$

Examples:

$$2A \odot I_2 = a_{11} + a_{22}$$

$$2A \odot I_3 = \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}$$

$$2A \odot I_4 =$$

$$\begin{pmatrix} a_{11}+a_{22} & a_{23} & -a_{13} & a_{24} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{12} & a_{34} & 0 & -a_{14} \\ -a_{31} & a_{21} & a_{22}+a_{33} & 0 & a_{34} & -a_{24} \\ a_{42} & a_{43} & 0 & a_{11}+a_{44} & a_{12} & a_{13} \\ -a_{41} & 0 & a_{43} & a_{21} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & -a_{42} & a_{31} & a_{32} & a_{33}+a_{44} \end{pmatrix}$$

Lemma 4 *For any* $A, B, B_{1,2} \in \mathbb{C}^{n\times n},\ \lambda \in \mathbb{C}$:
(i) $A \odot B = B \odot A$;
(ii) $A \odot (\lambda B) = \lambda(A \odot B)$;
(iii) $A \odot (B_1 + B_2) = A \odot B_1 + A \odot B_2$;

Lemma 5 *For any* $A, B \in \mathbb{C}^{n\times n}$:

$$(A \odot A)(B \odot B) = (AB \odot AB).$$

Lemma 6 *For any* $A \in \mathbb{C}^{n\times n}$ *and any nonsingular* $P \in \mathbb{C}^{n\times n}$:
(i) $(P \odot P)^{-1} = P^{-1} \odot P^{-1}$;
(ii) $(PAP^{-1}) \odot (PAP^{-1}) =$
$$(P \odot P)(A \odot A)(P \odot P)^{-1};$$
(iii) $2(PAP^{-1}) \odot I_n = (P \odot P)(2A \odot I_n)(P \odot P)^{-1}$.

Th. 2 (Stéphanos, 1900) *If* $A \in \mathbb{C}^{n\times n}$ *has eigenvalues* $\mu_1, \mu_2, \ldots, \mu_n$, *then*

(i) $A \odot A$ *has eigenvalues* $\mu_i\mu_j$,
(ii) $2A \odot I_n$ *has eigenvalues* $\mu_i + \mu_j$,

where $n \geq i > j \geq 1$.

Proof: Suppose, all μ_i are simple, then the corresponding eigenvectors v^i, $n \geq i \geq 1$, compose a basis in $\mathbb{C}^n$. Thus, $v^i \wedge v^j$, $n \geq i > j \geq 1$, form a basis in $\mathbb{C}^m$. In particular, they are all nonzero vectors.

Compute

$$
\begin{aligned}
(A \odot A)(v^i \wedge v^j) &= Av^i \wedge Av^j = \mu_i v^i \wedge \mu_j v^j \\
&= \mu_i \mu_j (v^i \wedge v^j).
\end{aligned}
$$

This means that $v^i \wedge v^j$ is an eigenvector of $A \odot A$ corresponding to the eigenvalue $\mu_i \mu_j$.

Similarly

$$
\begin{aligned}
(2A \odot I_n)(v^i \wedge v^j) &= Av^i \wedge v^j + v^i \wedge Av^j \\
&= \mu_i v^i \wedge v^j + \mu_j v^i \wedge v^j \\
&= (\mu_i + \mu_j)(v^i \wedge v^j).
\end{aligned}
$$

This means that $v^i \wedge v^j$ is an eigenvector of $2A \odot I_n$ corresponding to the eigenvalue $\mu_i + \mu_j$.

The theorem follows now from the continuity if the eigenvalues as functions of the matrix elements.

4. Applications to bifurcation analysis

Def. 4 *A fixed point* x_0 *of a map*

$$x \mapsto f(x,\alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R},$$

exhibits a **Neimark-Sacker bifurcation** *at* α_0 *if* $A = f_x(x_0,\alpha_0)$ *has a pair of eigenvalues on the unit circle:*

$$\mu_{1,2} = \cos\theta \pm i \sin\theta, \quad -\pi < \theta < 0.$$

Lemma 7 *The function*

$$\varphi_{NS}(x,\alpha) = \det(f_x(x,\alpha) \odot f_x(x,\alpha) - I_m)$$

vanishes at a Neimark-Sacker bifurcation.

Def. 5 *An equilibrium* x_0 *of an ODE system*

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R},$$

exhibits a **Hopf bifurcation** *at* α_0 *if* $A = f_x(x_0,\alpha_0)$ *has a pair of eigenvalues on the imaginary axis:*

$$\lambda_{1,2} = \pm i\omega, \quad \omega > 0.$$

Lemma 8 *The function*

$$\varphi_H(x,\alpha) = \det(2f_x(x,\alpha) \odot I_n)$$

vanishes at a Hopf bifurcation.

Lemma 9 *Suppose that* $A \in \mathbb{R}^{n\times n}$ *has a single pair of eigenvalues with* $\mu_1\mu_2 = 1$*, then*

$$\frac{\mu_1+\mu_2}{2} = \frac{\langle v,v\rangle\langle w,Aw\rangle + \langle w,w\rangle\langle v,Av\rangle - \langle v,w\rangle\langle w,Av\rangle - \langle w,v\rangle\langle v,Aw\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2},$$

where $(A \odot A - I_m)(v \wedge w) = 0$*.*

If $\mu_{1,2} = \cos\theta \pm i \sin\theta$, then $\frac{1}{2}(\mu_1+\mu_2) = \cos\theta$.

Lemma 10 *Suppose* $A \in \mathbb{R}^{n\times n}$ *has a single pair or eigenvalues with* $\lambda_1 + \lambda_2 = 0$*, then*

$$-\lambda_1\lambda_2 = \frac{\langle v,Av\rangle\langle w,Aw\rangle - \langle w,Av\rangle\langle v,Aw\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2},$$

where $(2A \odot I_n)(v \wedge w) = 0$*.*

If $\lambda_{1,2} = \pm i\omega$, then $-\lambda_1\lambda_2 = \omega^2$.

5. Relation to tensor products

Def. 6 *The* **tensor product** *of two vectors,*

$v = (v_1, v_2, \ldots, v_n)^\mathsf{T}$, $w = (w_1, w_2, \ldots, w_n)^\mathsf{T} \in \mathbb{C}^n$,

is a vector

$$v \otimes w \in \mathbb{C}^{n^2}, \quad ,$$

with the components:

$$(v \otimes w)_{(i,j)} = v_i w_j, \quad n \geq i, j \geq 1,$$

listed in the lexicographic order of their index pairs.

Lemma 11 *Fix a basis* $e^i \in \mathbb{C}^n$, $n \geq i \geq 1$, *in* $\mathbb{C}^n$ *and consider two vectors* $v, w \in \mathbb{C}^n$:

$$v = \sum_i v_i e^i, \quad w = \sum_j w_j e^j,$$

and a vector $u \in \mathbb{C}^{n^2}$*:*

$$u = \sum_{n \geq i,j \geq 1} u_{(i,j)} e^i \otimes e^j.$$

If $u = v \otimes w$*, then*

$$u_{(i,j)} = v_i w_j, \quad n \geq i, j \geq 1.$$

Consider two linear transformations of $\mathbb{C}^n$: $v \mapsto Av, \quad w \mapsto Bw$, where $A, B \in \mathbb{C}^{n\times n}$.

Def. 7 *The transformation of* $\mathbb{C}^{n^2}$ *defined as*

$$(v \otimes w) \mapsto (A \otimes B)(v \otimes w) = Av \otimes Bw$$

is called the **tensor product** *of the above transformations.*

Th. 3 *The tensor product is a linear transformation of* $\mathbb{C}^{n^2}$*. Its matrix* $A \otimes B$ *in the basis* $e^i \otimes e^j,\ n \geq i, j \geq 1$*, has the elements*

$$(A \otimes B)_{(p,q),(r,s)} = a_{pr} b_{qs},$$

where $n \geq p, q \geq 1$ *and* $n \geq r, s \geq 1$*.*

Lemma 12 *For any* $A_1, A_2, B_1, B_2 \in \mathbb{C}^{n\times n}$

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \otimes (B_1 B_2).$$

Lemma 13 *For any nonsingular matrices* $A, B \in \mathbb{C}^{n^2}$*,*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

Def. 8 *Denote by* E_a *a subspace of* $\mathbb{C}^{n^2}$ *spanned by the vectors*

$$\zeta^{ij} = e^i \otimes e^j - e^j \otimes e^i, \quad n \geq i > j \geq 1.$$

Note that $\dim E_a = \frac{1}{2}n(n-1)$.

Lemma 14 *The subspace* E_a *is an invariant subspace of* $(A \otimes B + B \otimes A)$*.*

Proof:

$$(A \otimes B + B \otimes A)\zeta^{rs} = \sum_{p,q}(a_{pr}b_{qs} + a_{qs}b_{pr})\zeta^{pq}.$$

Th. 4 *Let* $A, B \in \mathbb{C}^{n\times n}$*, then*

$$A \odot B = \frac{1}{2}(A \otimes B + B \otimes A)\Big|_{E_a},$$

in particular,

$$\begin{aligned} A \odot A &= (A \otimes A)|_{E_a}, \\ 2A \odot I_n &= (A \otimes I_n + I_n \otimes A)|_{E_a}. \end{aligned}$$