Lecture 5

Numerical continuation of connecting orbits of iterated maps and ODEs

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Contents

1. Point-to-point connections

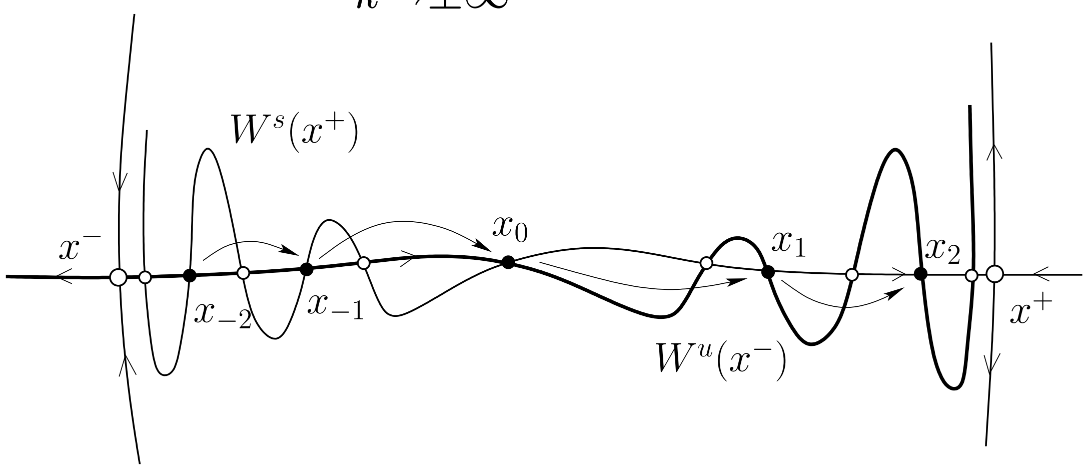
- Consider a **diffeomorphism**

$$x \mapsto f(x), \quad x \in \mathbb{R}^n,$$

having fixed points x^- and x^+, $f(x^\pm) = x^\pm$.

Def. 1 *An orbit* $\Gamma = \{x_k\}_{k\in\mathbb{Z}}$ *where* $x_{k+1} = f(x_k)$*, is called* **heteroclinic** *between* x^- *and* x^+ *if*

$$\lim_{k\to\pm\infty} x_k = x^\pm.$$



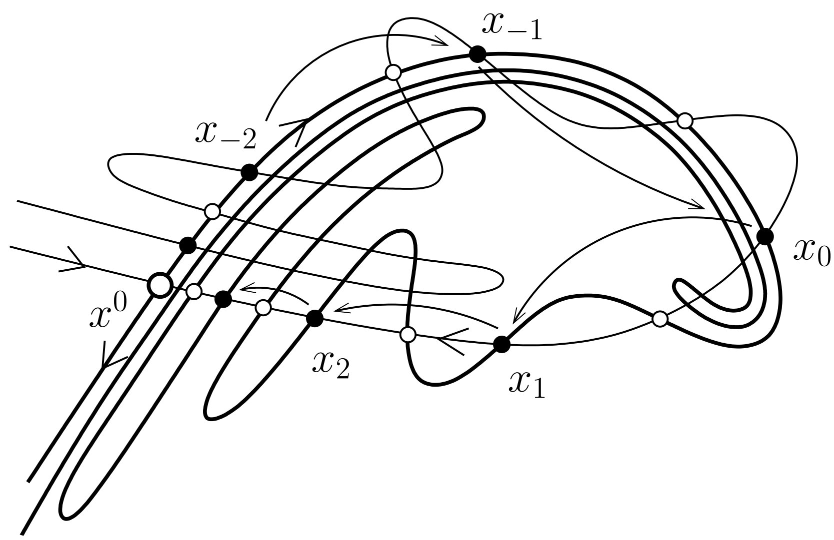
If $x^\pm = x^0$, it is called **homoclinic** to x^0.

- Introduce **unstable** and **stable invariant sets**

$$W^u(x^-) = \{x \in \mathbb{R}^n : \lim_{k\to\infty} f^{-k}(x) = x^-\},$$
$$W^s(x^+) = \{x \in \mathbb{R}^n : \lim_{k\to+\infty} f^{k}(x) = x^+\}.$$

Then $\Gamma \subset W^u(x^-) \cap W^s(x^+)$.

- **Def. 2** *A homoclinic orbit* Γ *is called* **regular** *if* $f_x(x^0)$ *has no eigenvalues with* $|\mu| = 1$ *and the intersection of* $W^u(x^0)$ *and* $W^s(x^0)$ *along* Γ *is transversal.*



- The presence of a regular homoclinic orbit implies the existence of infinite number of cycles of f nearby (**Poincaré–Birkhoff–Smale–Shilnikov Theorem**).

- In families of diffeomorphisms

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R},$$

regular homoclinic orbts exist in open parameter intervals.

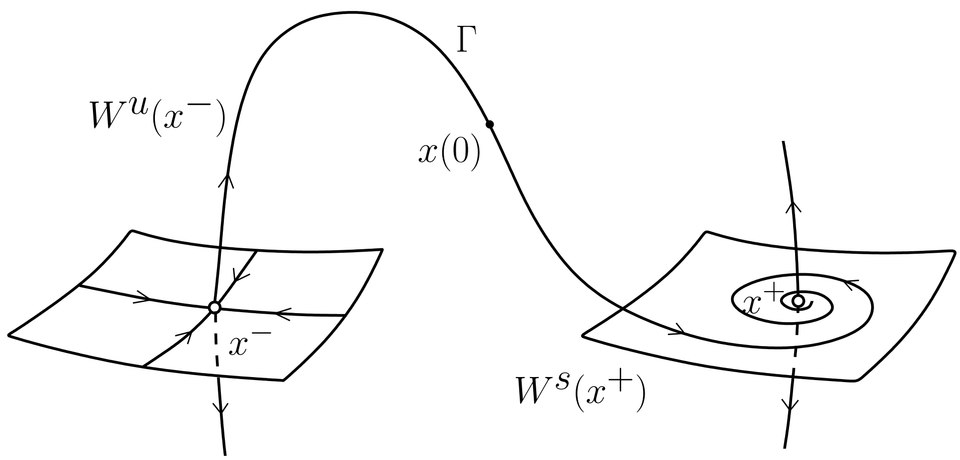
- Consider a **family of ODEs**

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R},$$

having equilibria x^- and x^+, $f(x^\pm, \alpha) = 0$.

Def. 3 *An orbit* $\Gamma = \{x = x(t) : t \in \mathbb{R}\}$, *where* $x(t)$ *is a solution to the ODE system at some* α, *is called* **heteroclinic** *between* x^- *and* x^+ *if*

$$\lim_{t \to \pm\infty} x(t) = x^\pm.$$



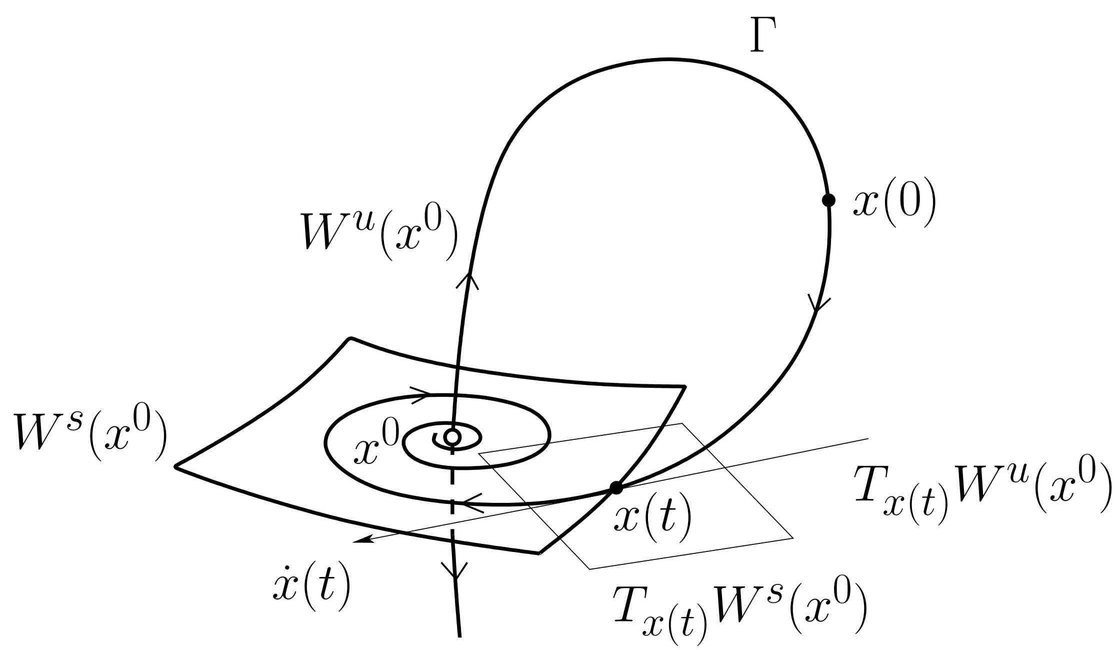
If $x^\pm = x^0$, *it is called* **homoclinic** *to* x^0.

- Introduce **unstable** and **stable invariant sets**

$$W^u(x^-) = \{x(0) \in \mathbb{R}^n : \lim_{t \to -\infty} x(t) = x^-\},$$
$$W^s(x^+) = \{x(0) \in \mathbb{R}^n : \lim_{t \to +\infty} x(t) = x^+\}.$$

Then $\Gamma \subset W^u(x^-) \cap W^s(x^+)$.

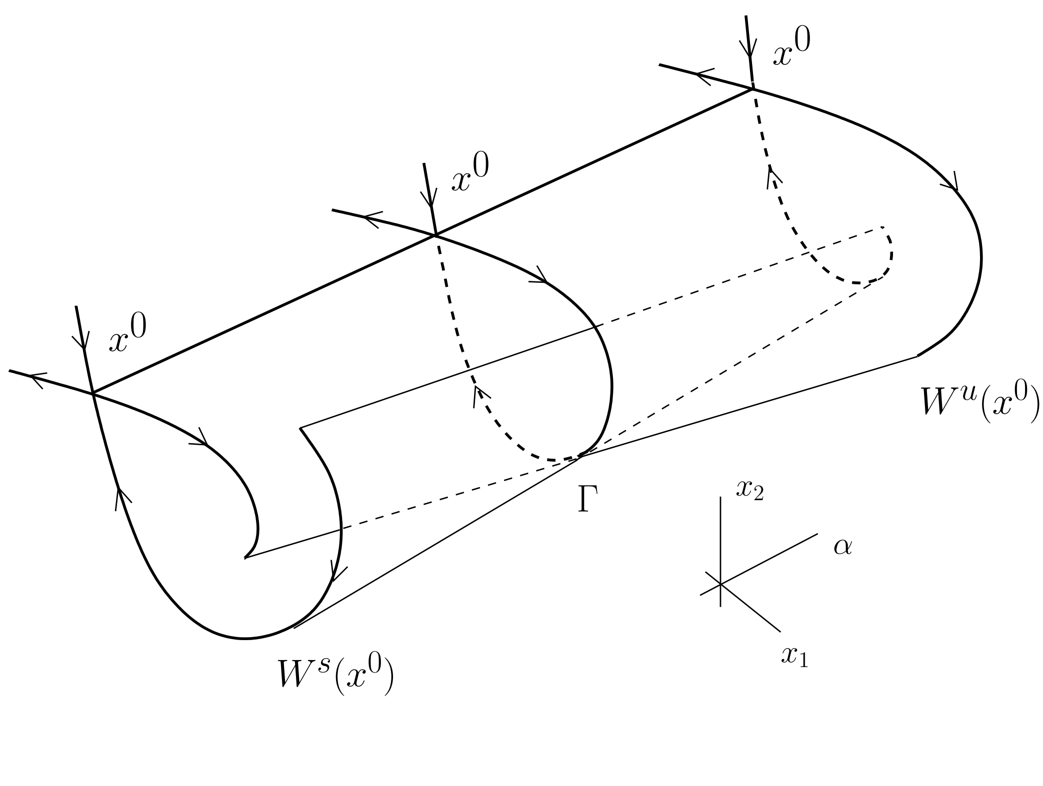
- The intersection of $W^u(x^0)$ and $W^s(x^0)$ cannot be transversal along a homoclinic orbit Γ, since $\dot{x}(t) \in T_{x(t)}W^u(x^0) \cap T_{x(t)}W^s(x^0)$.



- Homoclinic orbts exist in generic ODE families only at isolated parameter values.

Def. 4 *A homoclinic orbit* Γ *is called* **regular** *if*

- $f_x(x^0)$ *has no eigenvalues with* $\Re(\lambda) = 0$;
- $\dim(T_{x(t)}W^u(x^0) \cap T_{x(t)}W^s(x^0)) = 1$;
- *The intersection of the* **traces** *of* $W^u(x^0)$ *and* $W^s(x^0)$ *along* Γ *is transversal in the* (x, α)*-space.*



2. Continuation of homoclinic orbits of maps

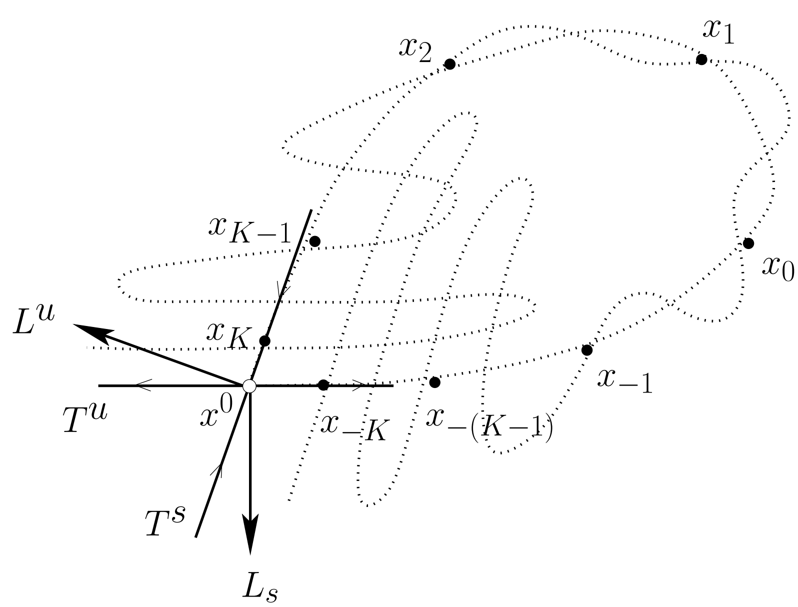
- **Homoclinic problem**

$$\begin{cases} f(x^0,\alpha) - x^0 &= 0, \\ x_{k+1} - f(x_k,\alpha) &= 0, \ k \in \mathbb{Z}, \\ \lim\limits_{k\to\pm\infty} x_k - x^0 &= 0. \end{cases}$$

- **Truncate** with the **projection boundary conditions**:

$$\begin{cases} f(x^0,\alpha) - x^0 &= 0, \\ x_{k+1} - f(x_k,\alpha) &= 0, \ k \in [-K, K-1], \\ L_s^{\top}(x^0,\alpha)(x_{-K} - x^0) &= 0, \\ L_u^{\top}(x^0,\alpha)(x_{+K} - x^0) &= 0, \end{cases}$$

where the columns of L_s and L_u span the orthogonal complements to $T^u = T_{x^0}W^u(x^0)$ and $T^s = T_{x^0}W^s(x^0)$, resp.



- Assume the eigenvalues of $A = f_x(x^0, \alpha)$ are arranged as follows:

$$|\mu_{n_s}| \leq \cdots \leq |\mu_1| < 1 < |\lambda_1| \leq \cdots \leq |\lambda_{n_u}|$$

  If $V^* = \left\{v_1^*, \ldots, v_{n_s}^*\right\}$ and $W^* = \left\{w_1^*, \ldots, w_{n_u}^*\right\}$ span the stable and unstable eigenspaces of A^T, then **Fredholm's Alternative** implies: $L_s = [V^*]$ and $L_u = [W^*]$.

- Let (μ, λ) satisfy $|\mu_1| < \mu < 1 < \lambda < |\lambda_1|$ and $\nu = \max(\mu, \lambda^{-1})$.

  Th. 1 (Beyn–Kleinkauf) *There is a locally unique solution to the truncated problem for a regular homoclinic orbit with an error that is $O(\nu^{2K})$.*

- The truncated system is an ALCP in $\mathbb{R}^{2nK+2n+1}$ to which the standard continuation methods are applicable.

3. Continuation of homoclinic orbits of ODEs

- **Homoclinic problem**

$$\begin{cases} f(x^0,\alpha) &=& 0, \\ \dot{x}(t) - f(x(t),\alpha) &=& 0, \\ \lim\limits_{t\to\pm\infty} x(t) - x^0 &=& 0, \ t \in \mathbb{R}, \\ \int_{-\infty}^{\infty} \langle \dot{y}(t), x(t) - y(t) \rangle dt &=& 0, \end{cases}$$

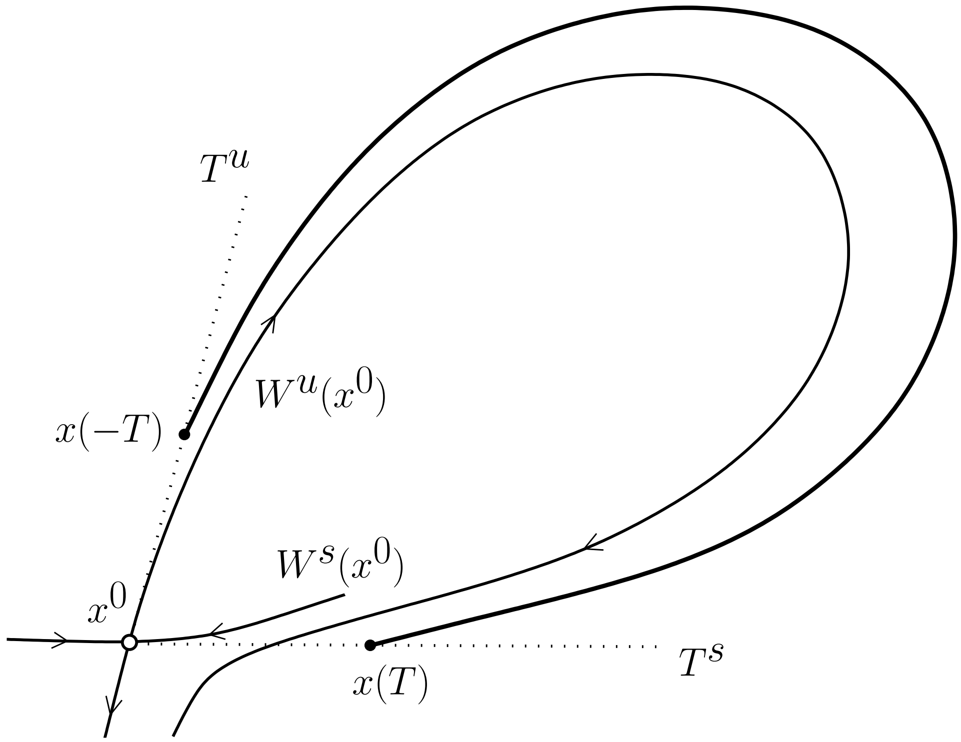
  where y is a reference homoclinic solution.

- **Truncate** with the **projection boundary conditions**:

$$\begin{cases} f(x^0,\alpha) &=& 0, \\ \dot{x}(t) - f(x(t),\alpha) &=& 0, \ t \in [-T,T] \\ L_s^\top(x^0,\alpha)(x(-T) - x^0) &=& 0, \\ L_u^\top(x^0,\alpha)(x(+T) - x^0) &=& 0, \\ \int_{-T}^{T} \langle \dot{y}(t), x(t) - y(t) \rangle dt &=& 0, \end{cases}$$

  where the columns of L_s and L_u span the orthogonal complements to $T^u = T_{x^0} W^u(x^0)$ and $T^s = T_{x^0} W^s(x^0)$, resp.

- The truncated system is a BVCP to which the standard discretization and continuation methods are applicable when $\alpha \in \mathbb{R}^2$.



- Assume the eigenvalues of $A = f_x(x^0, \alpha)$ are arranged as follows:

  $$\Re(\mu_{n_s}) \le \cdots \le \Re(\mu_1) < 0 < \Re(\lambda_1) \le \cdots \le \Re(\lambda_{n_u})$$

  If $V^* = \left\{v_1^*, \ldots, v_{n_s}^*\right\}$ and $W^* = \left\{w_1^*, \ldots, w_{n_u}^*\right\}$ span the stable and unstable eigenspaces of A^T, then $L_s = [V^*]$ and $L_u = [W^*]$.

- Let (μ, λ) satisfy $\Re(\mu_1) < \mu < 0 < \lambda < \Re(\lambda_1)$ and $\omega = \min(|\mu|, \lambda)$.

  Th. 2 (Beyn) *There is a locally unique solution to the truncated problem for a regular homoclinic orbit with the $(x(\cdot), \alpha)$-error that is $O(e^{-2\omega T})$.*

Remarks:

1. If W^u is **one-dimensional**, one can use the explicit boundary conditions

$$\begin{aligned} x(-T) - (x^0 + \varepsilon w_1) &= 0, \\ \langle w_1^*, x(T) - x^0 \rangle &= 0, \end{aligned}$$

   where $Aw_1 = \lambda_1 w_1$ and $A^\mathsf{T} w_1^* = \lambda_1 w_1^*$, without the integral phase condition.

2. Under implementation in MATCONT with possibilities to start

   (i) from a large period cycle;

   (ii) by homotopy.

   (iii) from a codim 2 bifurcations of equilibria, i.e. `BT` and `ZH`;

3. **Th. 3 (L.P. Shilnikov)** *There is always at least one* **limit cycle** *arbitrary close to* Γ *near the bifurcation. There are* **infinitely many cycles** *nearby when* μ_1 *and* λ_1 *are both complex, or when one of them is complex and has the smallest absolute value of the real part.*

4. Continuation of invariant subspaces

Th. 4 (Smooth Schur Block Factorization)
Any paramter-dependent matrix $A(s) \in \mathbb{R}^{n\times n}$ with nontrivial stable and unstable eigenspaces can be written as

$$A(s) = Q(s) \begin{bmatrix} R_{11}(s) & R_{12}(s) \\ 0 & R_{22}(s) \end{bmatrix} Q^{\mathsf{T}}(s),$$

where $Q(s) = [Q_1(s) \;\; Q_2(s)]$ such that

- *$Q(s)$ is orthogonal, i.e. $Q^{\mathsf{T}}(s)Q(s) = I_n$;*
- *the eigenvalues of $R_{11}(s) \in \mathbb{R}^{m\times m}$ are the unstable eigenvalues of $A(s)$, while the eigenvalues of $R_{22}(s) \in \mathbb{R}^{(n-m)\times(n-m)}$ are the remaning $(n-m)$ eigenvalues of $A(s)$;*
- *the columns of $Q_1(s) \in \mathbb{R}^{n\times m}$ span the eigenspace $\mathcal{E}(s)$ of $A(s)$ corresponding to its m unstable eigenvalues;*
- *the columns of $Q_2(s) \in \mathbb{R}^{n\times(n-m)}$ span the orthogonal complement $\mathcal{E}^{\perp}(s)$.*
- *$Q_i(s)$ and $R_{ij}(s)$ have the same smoothness as $A(s)$.*

Then holds the **invariant subspace relation**:

$$Q_2^{\mathsf{T}}(s)A(s)Q_1(s) = 0.$$

CIS-algorithm [Dieci & Friedman]

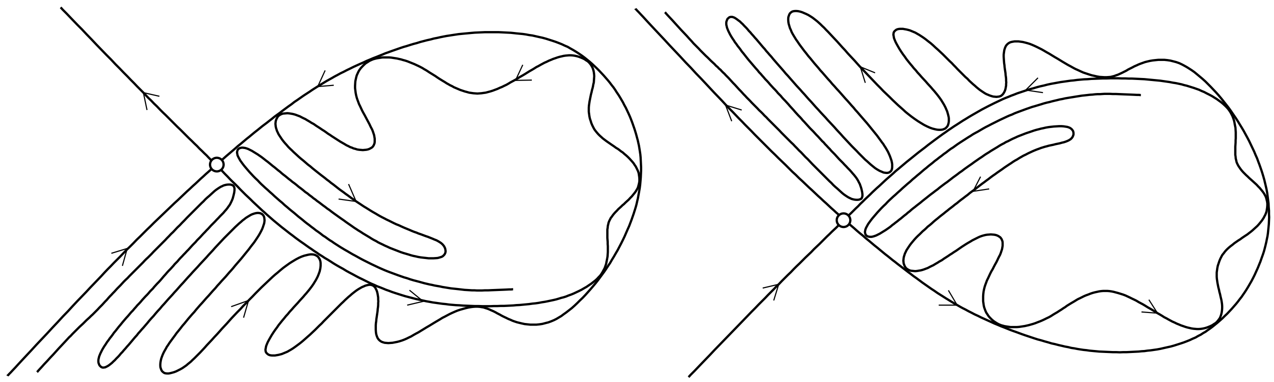
- Define
$$\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} = Q^{\top}(0)A(s)Q(0)$$
for small $|s|$, where $T_{11}(s) \in \mathbb{R}^{m\times m}$.

- Compute by Newton's method $Y \in \mathbb{R}^{(n-m)\times m}$ satisfying the **Riccati matrix equation**
$$YT_{11}(s) - T_{22}(s)Y + YT_{12}(s)Y = T_{21}(s).$$

- Then $Q(s) = Q(0)U(s)$ where
$$U(s) = [U_1(s) \quad U_2(s)]$$
with
$$\begin{aligned} U_1(s) &= \begin{pmatrix} I_m \\ Y \end{pmatrix} (I_{n-m} + Y^{\top}Y)^{-\frac{1}{2}}, \\ U_2(s) &= \begin{pmatrix} -Y^{\top} \\ I_{n-m} \end{pmatrix} (I_{n-m} + YY^{\top})^{-\frac{1}{2}}, \end{aligned}$$
so that columns of $Q_1(s) = Q(0)U_1(s)$ and $Q_2(s) = Q(0)U_2(s)$ form orthogonal bases in $\mathcal{E}(s)$ and $\mathcal{E}^{\perp}(s)$.

5. Detection of higher-order singularities

- For homoclinic orbits to fixed points, x^0 can exhibit one of codim 1 bifucations. `LP`'s of the (truncated) ALCP correspond to **homoclinic tangencies**:



- For homoclinic orbits to equilibria, there are many codim 2 cases:

  1. fold or Hopf bifurcations of x^0;

  2. special eigenvalue configurations (e.g. $\sigma = \Re(\mu_1) + \Re(\lambda_1) = 0$ or $\mu_1 - \mu_2 = 0$);

  3. change of global topology of W^s and W^n (orbit and inclination flips);

  4. higher nontransversality.

6. Cycle-to-cycle connections in 3D ODEs

- Consider

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^p.$$

- Let O^- be a limit cycle with only one (trivial) multiplier satisfying $|\mu| = 1$ and having $\dim W_-^u = m_u^-$.

- Let O^+ be a limit cycle with only one (trivial) multiplier satisfying $|\mu| = 1$ and having $\dim W_+^s = m_s^+$.

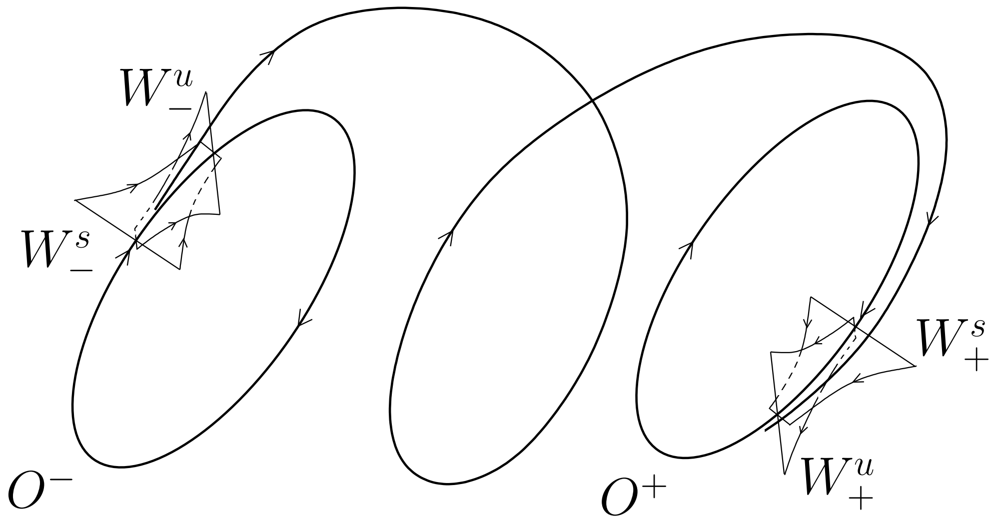
- Let $x^\pm(t)$ be periodic solutions (with minimal periods $T^\pm$) corresponding to $O^\pm$ and $M^\pm$ the corresponding **monodromy matrices**, i.e. $M(T^\pm)$ where

$$\dot{M} = f_x(x^\pm(t), \alpha)M, \quad M(0) = I_n.$$

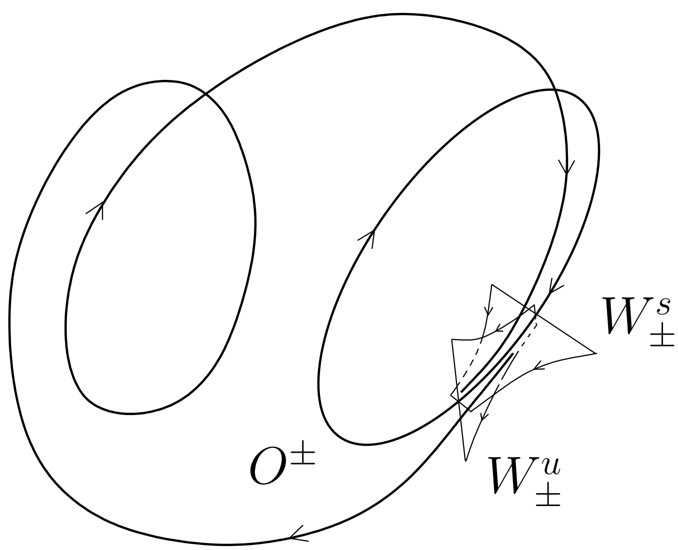
- Then $m_s^+ = n_s^+ + 1$ and $m_u^- = n_u^- + 1$, where n_s^+ and n_u^- are the numbers of eigenvalues of $M^\pm$ satisfying $|\mu| < 1$ and $|\mu| > 1$, resp.

Isolated families of connecting orbits

- **Beyn's equality**: $p = n - m_s^+ - m_u^- + 2$.
- The cycle-to-cycle connections in $\mathbb{R}^3$:



heteroclinic orbit



homoclinic orbit ⇒ infinite number of cycles

Truncated BVCP

- The connecting solution $u(t)$ is **truncated** to an interval $[\tau_-, \tau_+]$.

- The points $u(\tau_+)$ and $u(\tau_-)$ are required to belong to the linear subspaces that are tangent to the stable and unstable invariant manifolds of O^+ and O^-, respectively:

$$\begin{cases} L_+^\top(u(\tau_+) - x^+(0)) &= 0, \\ L_-^\top(u(\tau_-) - x^-(0)) &= 0. \end{cases}$$

- Generically, the truncated BVP composed of the ODE, the above projection BC's, and a phase condition on u, has a unique solution family $(\hat{u}, \hat{\alpha})$, provided that the ODE has a connecting solution family satisfying the pa-hase condition and Beyn's equality.

Th. 5 (Pampel–Dieci–Rebaza) *If u is a generic connecting solution to the ODE at parameter value α, then the following estimate holds:*

$$\|(u|_{[\tau_-,\tau_+]},\alpha)-(\widehat{u},\widehat{\alpha})\| \le C\mathrm{e}^{-2\min(\mu_-|\tau_-|,\mu_+|\tau_+|)},$$

where

- $\|\cdot\|$ *is an appropriate norm in the space* $C^1([\tau_-,\tau_+],\mathbb{R}^n)\times\mathbb{R}^p$,
- $u|_{[\tau_-,\tau_+]}$ *is the restriction of u to the truncation interval,*
- $\mu_\pm$ *are determined by the eigenvalues of the monodromy matrices $M^\pm$.*

Adjoint variational eqiation:

$$\dot{w} = -f_x^\mathsf{T}(x^\pm(t),\alpha)w, \quad w \in \mathbb{R}^n.$$

Let $N(t)$ be the solution to

$$\dot{N} = -f_x^\mathsf{T}(x^\pm(t),\alpha)N, \quad N(0) = I_n.$$

Then $N(T^\pm) = [M^{-1}(T^\pm)]^\mathsf{T}$.

The defining BVCP in 3D

Cycle-related equations:

- Periodic solutions:

$$\begin{cases} \dot{x}^{\pm} - f(x^{\pm}, \alpha) &= 0 \, , \\ x^{\pm}(0) - x^{\pm}(T^{\pm}) &= 0 \, . \end{cases}$$

- Adjoint eigenfunctions: $\mu^{+} = \frac{1}{\mu_u^{+}}$, $\mu^{-} = \frac{1}{\mu_s^{-}}$.

$$\begin{cases} \dot{w}^{\pm} + f_u^{\mathsf{T}}(x^{\pm}, \alpha) w^{\pm} &= 0 \, , \\ w^{\pm}(T^{\pm}) - \mu^{\pm} w^{\pm}(0) &= 0 \, , \\ \langle w^{\pm}(0), w^{\pm}(0) \rangle - 1 &= 0 \, , \end{cases}$$

  or equivalently

$$\begin{cases} \dot{w}^{\pm} + f_u^{\mathsf{T}}(x^{\pm}, \alpha) w^{\pm} + \lambda^{\pm} w^{\pm} &= 0 \, , \\ w^{\pm}(T^{\pm}) - s^{\pm} w^{\pm}(0) &= 0 \, , \\ \langle w^{\pm}(0), w^{\pm}(0) \rangle - 1 &= 0 \, , \end{cases}$$

  where $\lambda^{\pm} = \ln |\mu^{\pm}|$, $s^{\pm} = \mathrm{sign}(\mu^{\pm})$.

- Projection BC: $\langle w^{\pm}(0), u(\tau_{\pm}) - x^{\pm}(0) \rangle = 0$.

Connection-related equations:

- The equation for the connection:

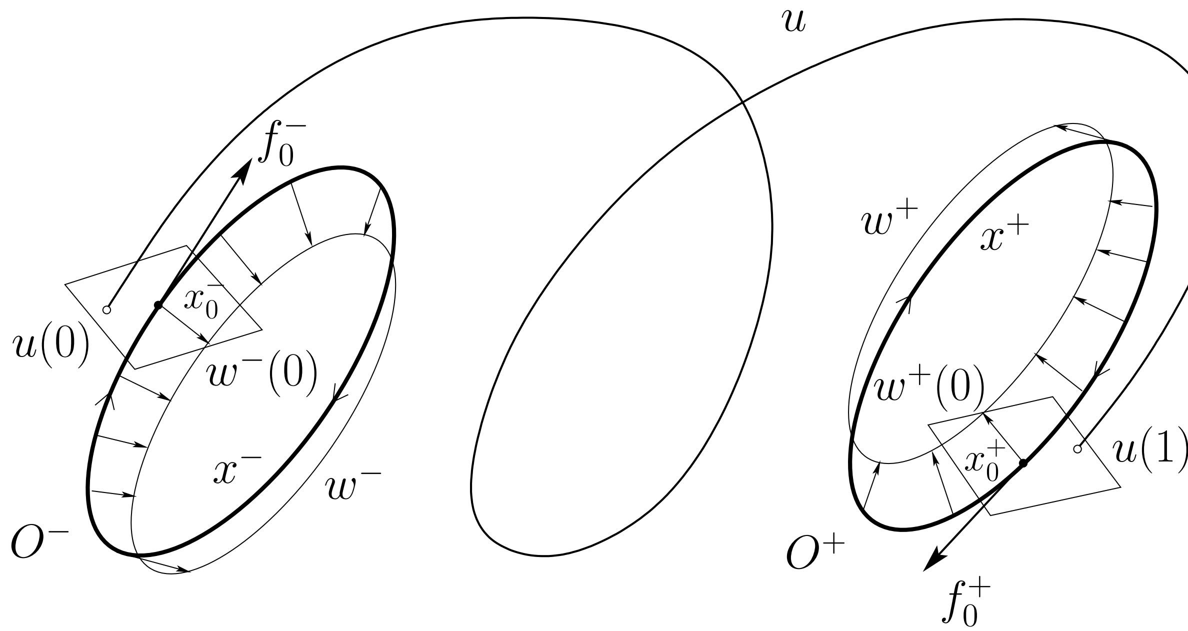
$$\dot{u} - f(u, \alpha) = 0 \ .$$

- We need the base points $x^{\pm}(0)$ to move freely and independently upon each other along the corresponding cycles $O^{\pm}$.

- We require the end-point of the connection to belong to a plane orthogonal to the vector $f(x^{+}(0), \alpha)$, and the starting point of the connection to belong to a plane orthogonal to the vector $f(x^{-}(0), \alpha)$:

$$\langle f(x^{\pm}(0), \alpha), u(\tau_{\pm}) - x^{\pm}(0) \rangle = 0 \ .$$

The defining BVCP in 3D:



$$
\left\{
\begin{array}{rcl}
\dot{x}^{\pm} - T^{\pm} f(x^{\pm}, \alpha) & = & 0, \\
x^{\pm}(0) - x^{\pm}(1) & = & 0, \\
\dot{w}^{\pm} + T^{\pm} f_u^{\mathsf{T}}(x^{\pm}, \alpha) w^{\pm} + \lambda^{\pm} w^{\pm} & = & 0, \\
w^{\pm}(1) - s^{\pm} w^{\pm}(0) & = & 0, \\
\langle w^{\pm}(0), w^{\pm}(0) \rangle - 1 & = & 0, \\
\dot{u} - T f(u, \alpha) & = & 0, \\
\langle f(x^{+}(0), \alpha), u(1) - x^{+}(0) \rangle & = & 0, \\
\langle f(x^{-}(0), \alpha), u(0) - x^{-}(0) \rangle & = & 0, \\
\langle w^{+}(0), u(1) - x^{+}(0) \rangle & = & 0, \\
\langle w^{-}(0), u(0) - x^{-}(0) \rangle & = & 0, \\
\| u(0) - x^{-}(0) \|^2 - \varepsilon^2 & = & 0.
\end{array}
\right.
$$

There is an efficient **homotopy method** to find a starting solution.