

TUTORIAL III:
One-parameter bifurcation analysis of limit cycles with
MATCONT

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This session was tested on MATCONT7.3 with MATLAB2020b, R9.9. It is devoted to the numerical initialization and continuation of limit cycles in systems of autonomous ODEs depending on one parameter

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R},$$

and detection of their bifurcations. We will also show how to start the continuation of a limit cycle from the Hopf bifurcation and a doubled cycle from the period-doubling bifurcation.

1 Cycle initialization from a converging orbit

We consider again the *Rössler system* that was introduced in TUTORIAL I:

$$\begin{cases} \dot{x} &= -y - z \\ \dot{y} &= x + Ay \\ \dot{z} &= Bx - Cz + xz, \end{cases} \quad (1)$$

where (x, y, z) are the phase variables, and (A, B, C) are the parameters.

Input system (1) in MATCONT, generating the derivatives of order 1, 2 and 3 symbolically. We start with computing by time integration an orbit, using the **Starter** and **Integrator** windows presented in Figure 1.

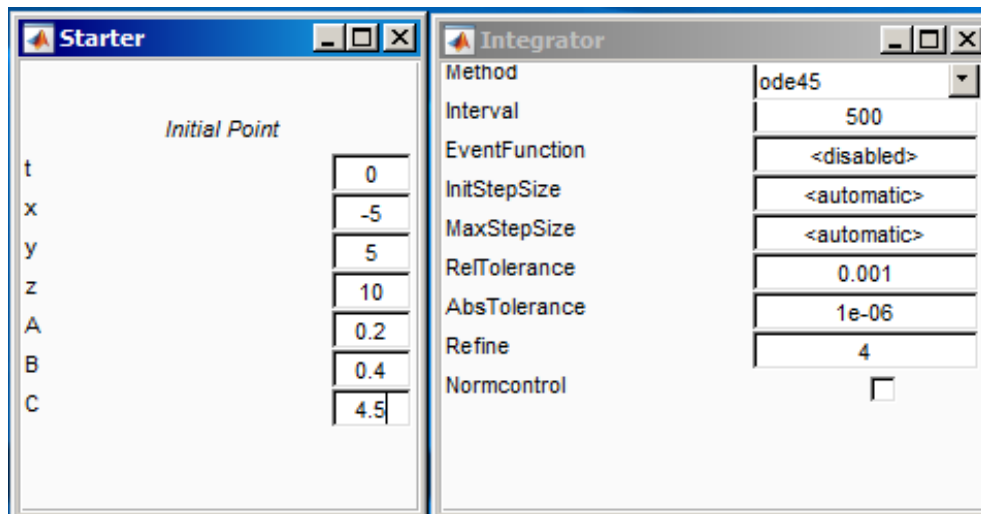


Figure 1: The **Starter** and **Integrator** windows for the initial time integration.

The computed orbit is presented in the 3D-plot in Figure 2 in (x, y, z) - space within the region

$$-9 \leq x \leq 9, \quad -9 \leq y \leq 9, \quad -2 \leq z \leq 11.$$

It is visually clear that the orbit has converged to a (presumably stable) limit cycle. We select the last point of the orbit using **Select|Initial Point|Last Point|Select Point** and clear the **Plot3D** window.

It is readily seen that time integration from the selected point over a time interval 5 does not close the orbit, see the **Starter** and **Integrator** windows in Figure 3, and the orbit displayed in Figure 4.

Now perform a time integration over an interval 8 and check that it closes the limit cycle. This implies that 8 is larger than the period of the limit cycle but smaller than twice that period, which is the condition for starting the continuation of limit cycles from an orbit that is computed along the limit cycle. That is why time integrations over an interval 5 or 500 are of no use.

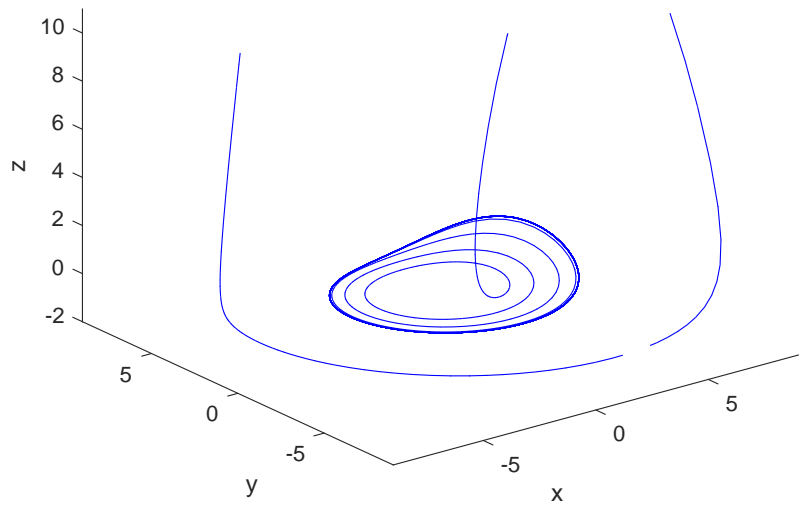


Figure 2: 3D-plot of the computed orbit.

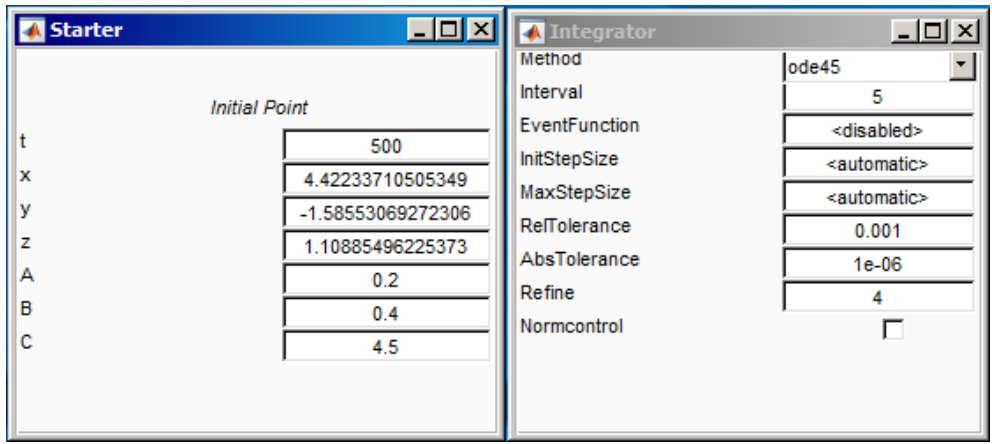


Figure 3: **Starter** and **Integrator** windows for time integration from a selected point on the orbit.

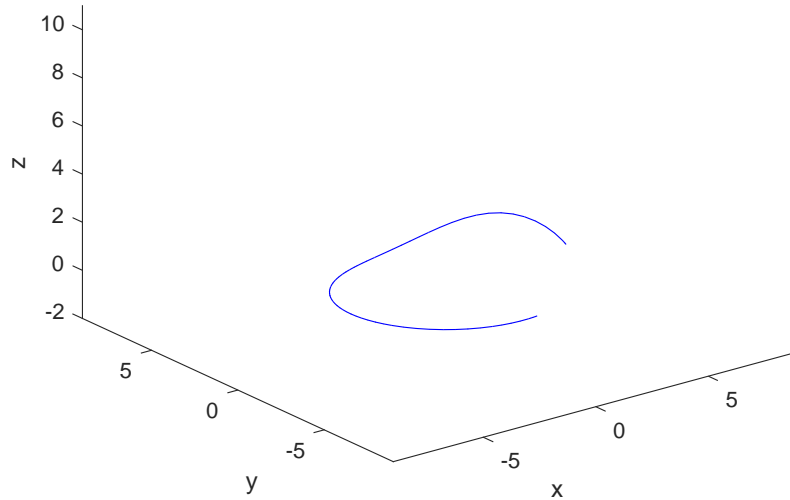


Figure 4: Computed orbit over an interval 5.

Now press **Control|View Result** and double-click at **Select Cycle** (we note that a single-click at **Select Cycle** followed by clicking **Select Point** also works). This opens a small **Choose tolerance ...** input window with two input fields. The top one allows to change the tolerance for detecting a limit cycle along the orbit. The bottom one allows to change the number of mesh intervals that will be used to discretize the limit cycle. Make sure that it is set to 40; the window should now look like Figure 5.

Now press **OK**. The main MATCONT window is then prepared for the continuation of a limit cycle, see Figure 6(left). In the corresponding **Starter** window activate the parameter **A** and the **Period**. Also, check all singularity monitoring boxes as in Figure 6(right). We note that the initial value of the **Period** is estimated by the initializer and equal to 6.27..., indeed a value between 5 and 8.

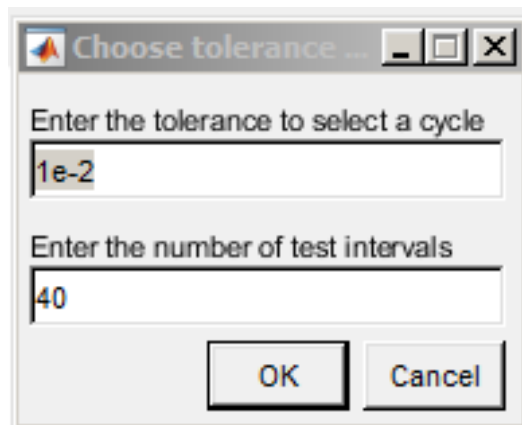


Figure 5: Input window opened by pressing the **Select Cycle** button.

In the corresponding **Continuer** window only default values are to be used. Open a **Numeric** window and check **Multipliers** in its **Layout** menu for the multipliers of the periodic orbits to be displayed in the **Numeric** window during continuation.

Press **Compute|Forward** (depending on details of the previous computations it may be neces-

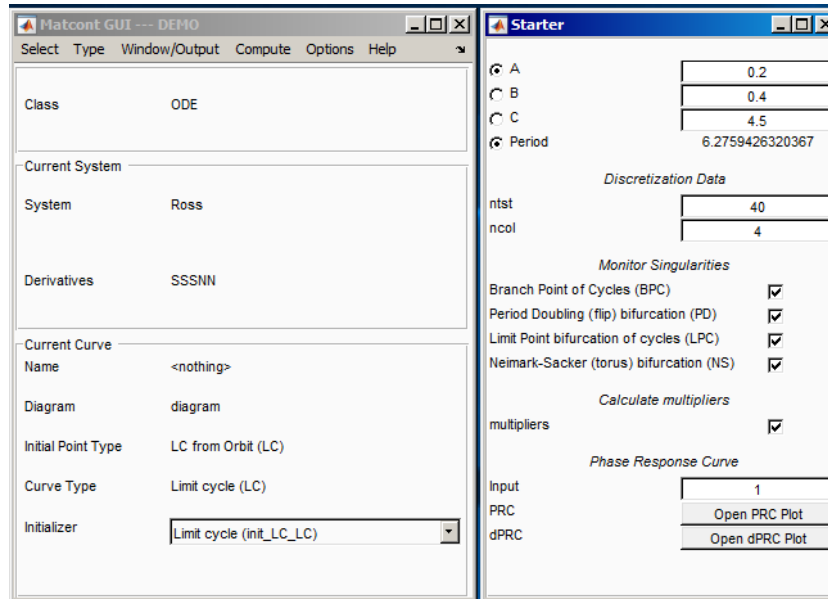


Figure 6: The main MATCONT window when prepared to start the continuation of limit cycles.

sary to press **Compute** | **Backward** instead). The continuation of limit cycles now starts. Observe that there is period-doubling bifurcation for $A = 0.27191773$. The **Plot3D** window and the **Nu-meric** window are displayed in Figures 7. Observe that at the PD-point there is a multiplier equal to -1 (i.e., absolute value 1 and argument equal to 180°).

In the MATLAB command line the output

```
Period Doubling (period = 6.148967e+00, parameter = 2.719177e-01)
Normal form coefficient = -4.283828e-03
```

is generated, including the normal form coefficient of the PD bifurcation.

Stop the computation and end the session by closing all output windows.

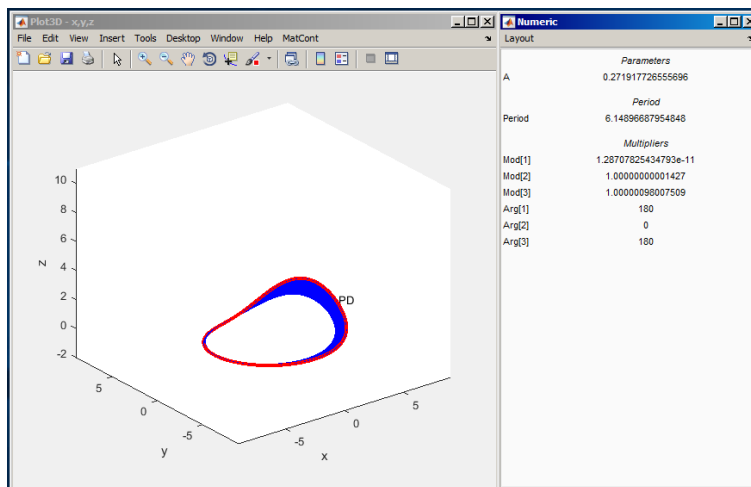


Figure 7: The **Plot3D** and the **Numeric** windows at the period-doubling bifurcation.

2 Fold and Neimark-Sacker bifurcations of cycles in a chemical model

Consider the following chemical model by *Steinmetz and Larter*:

$$\begin{cases} \dot{A} &= -k_1ABX - k_3ABY + k_7 - k_{-7}A, \\ \dot{B} &= -k_1ABX - k_3ABY + k_8, \\ \dot{X} &= k_1ABX - 2k_2X^2 + 2k_3ABY - k_4X + k_6, \\ \dot{Y} &= -k_3ABY + 2k_2X^2 - k_5Y. \end{cases} \quad (2)$$

We will fix all parameters but k_7 and study periodic solutions (limit cycles) of (2) when this parameter varies.

2.1 System specification

Specify a new ODE system – say **StLar** – in **MATCONT**

```
A'=-k1*A*B*X-k3*A*B*Y+k7-km7*A
B'=-k1*A*B*X-k3*A*B*Y+k8
X'=k1*A*B*X-2*k2*X^2+2*k3*A*B*Y-k4*X+k6
Y'=-k3*A*B*Y+2*k2*X^2-k5*Y
```

where (A, B, X, Y) are the coordinates and $(k_1, k_2, k_3, k_4, k_5, k_6, k_7, km_7, k_8)$ are the parameters. Use (default) **t** for time and generate symbolical derivatives of order 1, 2 and 3.

2.2 Fold bifurcation of limit cycles

2.2.1 Continuation of an equilibrium

We will continue an equilibrium of (2) and detect its Hopf bifurcation, from which the continuation of a limit cycle can be started.

Input **Type|Initial point|Equilibrium** and **Type|Curve|Equilibrium** in the main **MATCONT** window.

Input the following numerical data in the appeared **Starter** window:

A	31.78997
B	1.45468
X	0.01524586
Y	0.1776113
k1	0.1631021
k2	1250
k3	0.046875
k4	20
k5	1.104
k6	0.001
k7	4.235322
km7	0.1175
k8	0.5

These values correspond to an unstable equilibrium (A, B, X, Y) in the system. Activate the parameter k_7 .

Open **Window|Output|Numeric** and change its appearance via the **Layout** menu. Namely, check **Eigenvalues** to be also shown in the window.

Use **Window|Output|Graphic|2D plot** to open the corresponding window and select the coordinates (A, B) as abscissa and ordinate, respectively, with the visibility limits:

```

Abscissa:          32          38
Ordinate:          0           4

```

respectively.

Start **Compute|Forward**. The equilibrium curve will be continued and you get a *Hopf bifurcation point* labeled by H. The message in the MATLAB Command Window

```

label = H , x = ( 34.808899 1.328517 0.015246 0.177611 4.590046 )
First Lyapunov coefficient = 1.527549e-02

```

at $k_7 = 4.590046\dots$ indicates a *subcritical* Hopf bifurcation. Indeed, there are two eigenvalues of the equilibrium with $\text{Re } \lambda_{1,2} \approx 0$ at this parameter value visible in the **Numeric** window. The critical frequency $\text{Im } \lambda_1 \neq 0$, while the first Lyapunov coefficient is positive. Thus, there should exist an unstable limit cycle, bifurcating from the equilibrium. Resume computations and terminate them when the curve leaves the graphic window.

For later use, rename the computed curve via **Select|Curve|Rename** into **Equilibrium(+)**.

2.2.2 Cycle continuation

Click **Select|Initial point|Hopf|Select Point** to select the H: Hopf point in the **Equilibrium(+)** curve as initial point. The **Starter** and **Continuer** windows for the continuation of the limit cycle from the Hopf point will appear.

Ensure that all boxes in the Monitor **Singularities** field of the **Starter** window are checked. Increase the **MaxStepsize** to 1.0 in the **Continuer** window to allow larger steps along the curve and set **MaxNumPoints** to 50. The **Starter** and **Continuer** windows should look like in Figure 8.

Change the layout of the **Numeric** window via **Numeric|Layout** by selecting all **Multipliers** to be shown. The absolute values (*modulae*) and arguments in *angular grads* will be displayed.

Compute|Forward will produce a family of cycles with a *cycle limit point* labeled by LPC at $k_7 = 4.74838\dots$ (The initial Hopf point may also be detected as an LPC right after starting the continuation; in that case ignore it and resume the computation). The **Numeric** window corresponding to LPC is shown in Figure 9. Recall that exact numerical values could deviate slightly.

In the MATLAB Command Window, the following message appears:

```

Limit point cycle (period = 1.036108e+01, parameter = 4.748384e+00)
Normal form coefficient = -2.714838e-01

```

The critical cycle has (approximately) a double multiplier $\mu = 1$ and the normal form coefficient is nonzero. Thus, the limit cycle manifold has a fold here. Resume the computations. The continuation algorithm will automatically follow the second (stable) cycle branch after the LPC point. Verify this by looking at the nontrivial multipliers in the **Numeric** window ($|\mu| < 1$ for all such multipliers). The computations produce Figure 10.

To visualize LPC in another way, open another **Plot2D** window and select parameter **k7** and **Period** of the cycle as abscissa and ordinate, respectively. Use the visibility limits:

```

Abscissa:          4.5          4.8
Ordinate:          7           12

```

After **MatCont|Redraw Diagram** in the **Plot2D** window, you will get a curve with a limit point, clearly indicating the presence of two limit cycles with different periods for $k_7 < 4.74838$ near LPC (see Figure 11). Close both **Plot2D** windows.

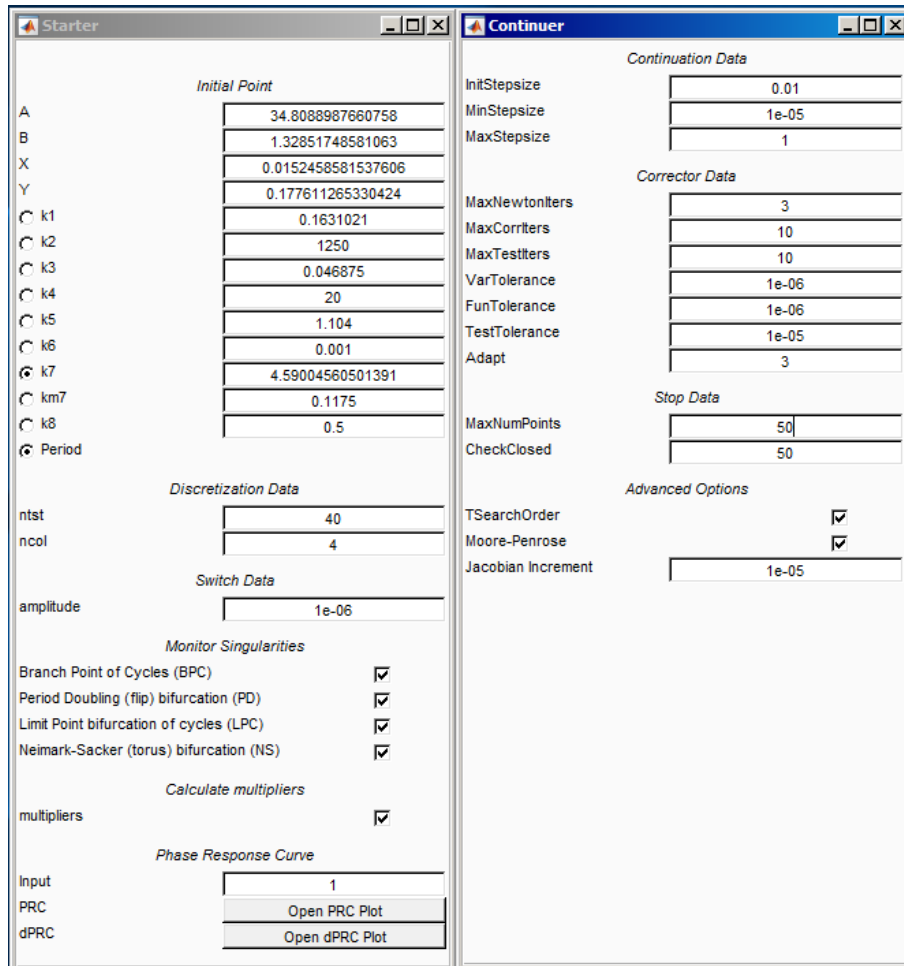


Figure 8: The **Starter** and **Continuer** windows for the cycle continuation.

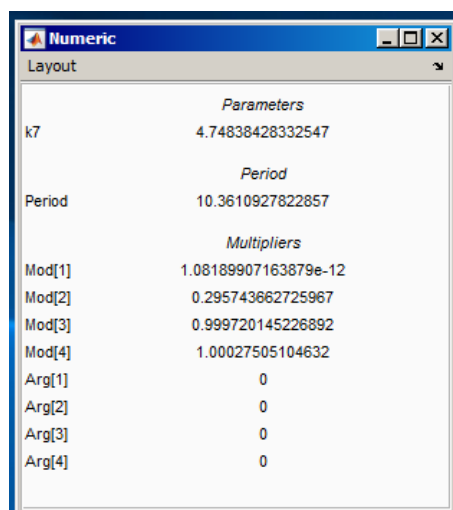


Figure 9: The **Numeric** window at the LPC-point.

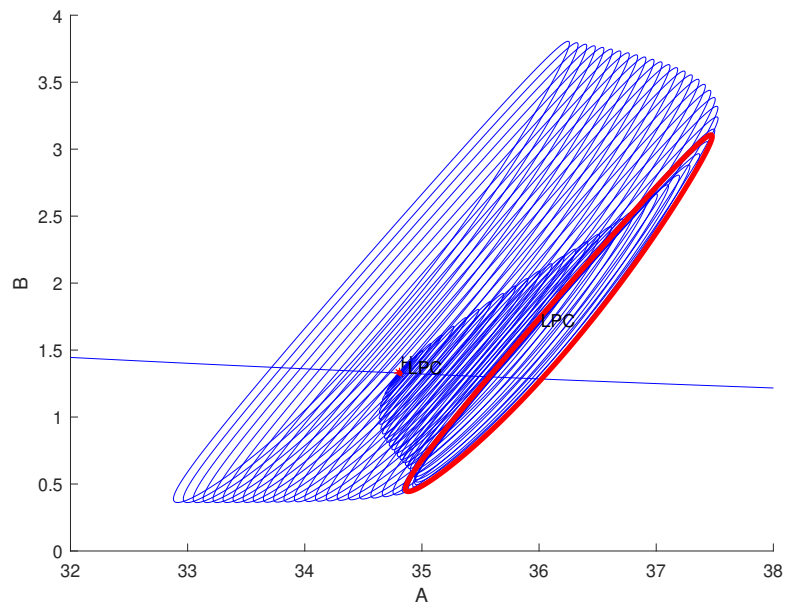


Figure 10: The family of limit cycles bifurcating from the Hopf point H : LPC is a fold bifurcation of the cycle.

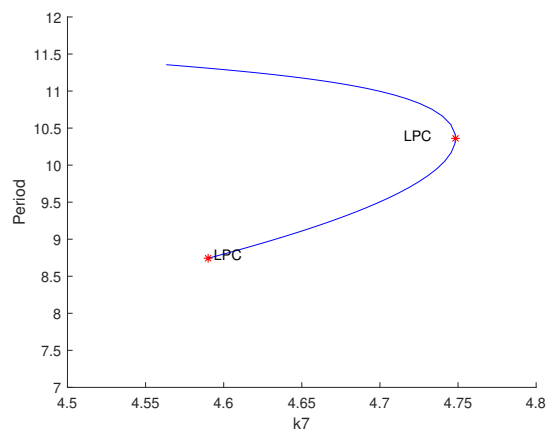


Figure 11: Period of the cycle versus k_7 .

2.3 Neimark-Sacker bifurcation

2.3.1 Backward continuation of the equilibrium

Load the first computed equilibrium curve `Equilibrium(+)` in the **Data Browser** window appearing after the `Select|Curve` command. Be sure to select `Type|Curve|Equilibrium` for continuation.

Open a new **Plot2D** window but select now `B` and `X` as abscissa and ordinate, respectively, with the visibility limits:

```
Abscissa:          25          27
Ordinate:           0          0.03
```

`Compute|Backward` and `Extend` several times until you get another *Hopf bifurcation* at $k_7 = 0.712475\dots$ with the message

```
label = H , x = ( 1.808301 25.573303 0.015246 0.177611 0.712475 )
First Lyapunov coefficient = -2.371880e-02
```

in the MATLAB Command Window. The first Lyapunov coefficient is negative now. This means that a *stable* limit cycle bifurcates from the equilibrium, when it loses stability. Resume computations and terminate them when the equilibrium curve leaves the graphic window. Rename the computed curve into `Equilibrium(-)`.

2.3.2 Cycle continuation

Select the Hopf point in the curve `Equilibrium(-)` as initial. `MATCONT` will prepare to continue a limit cycle curve from the Hopf point (curve type `H-LC`). Check all boxes in the **Monitor Singularities** section of the **Starter** window. Activate `k7` and the **Period** in the **Starter** window. Set

```
MaxStepsize          1
MaxNumPoints         25
```

in the **Continuer** window. Check **Multipliers** to be shown in the **Numeric** window.

Click `Compute|Forward` to start the continuation of the limit cycle. At $k_7 = 0.716434\dots$ the message **Neimark-Sacker** indicates a *torus* bifurcation. (The initial Hopf point may also be detected as an LPC right after starting the continuation; in that case ignore it and resume the computation). In the MATLAB Command Window, the following message appears:

```
Neimark-Sacker (period = 1.091213e+01, parameter = 7.164336e-01)
Normal form coefficient = -4.912065e-08.
```

Indeed, there are two complex multipliers with (approximately) $|\mu| = 1$ and one trivial at $\mu = 1$. This can be seen in the **Numeric** window that stays open, see Figure 12. The normal form coefficient is negative and small but nonzero, indicating that a stable two-dimensional *invariant torus* bifurcates from the limit cycle.

Resume computations further to see that after the **NS**-point the cycle becomes unstable (with two multipliers satisfying $|\mu| > 1$). You should get a family of limit cycles bifurcating from the Hopf point as in Figure 13.

Close the **Numeric** and all output windows.

2.3.3 Dynamics on a stable torus

Click `Select|Initial point...` and select the **NS: Neimark-Sacker** point in the computed limit cycle curve. The critical parameter values will be read in from the archive. Immediately after that select `Type|Initial point|Point` and `Type|Curve|Orbit`. A point on the critical cycle will thus be selected as the initial for orbit integration.

Increase slightly the value of the parameter k_7 and perturb the initial point by altering the value of `B`, namely set:

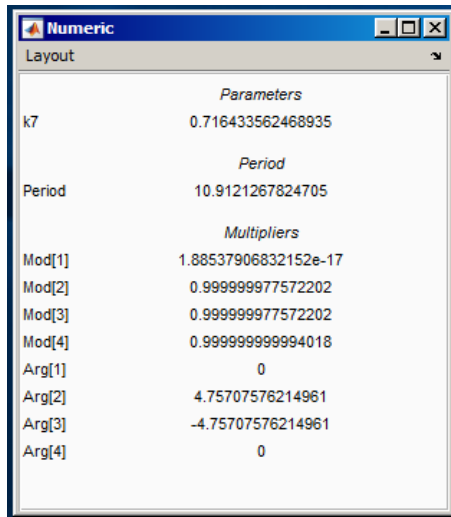


Figure 12: The **Numeric** window at the NS-point.

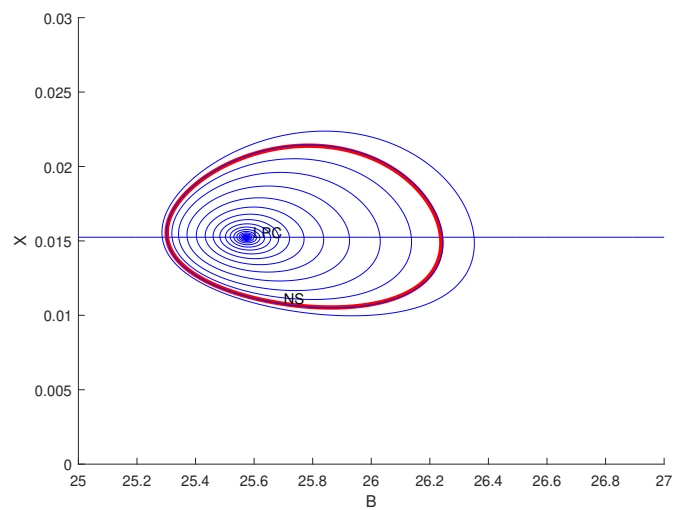


Figure 13: Family of limit cycles bifurcating from the second Hopf point.

k7 0.7165
 B 26.2

Change **Method** of integration to `ode23s` and alter

Interval 1000
 Rel.Tolerance 1e-6
 Abs.Tolerance 1e-9

in the **Integrator** window.

To speed up visualization, click **Options|Output Interval** and set it to 1000, see Figure 14

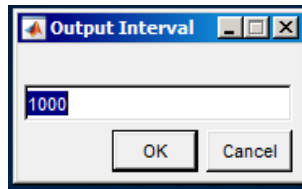


Figure 14: **Output Interval** window.

Clear the graphic window and **Compute|Forward** followed by **Compute|Extend**. The integration will take some time. After a transient, the orbit will exhibit modulated oscillations with two frequencies near the limit cycle. This is a motion on a stable two-dimensional torus born via the Neimark-Sacker bifurcation, see Figure 15. Open another **2Dplot** window with time t and

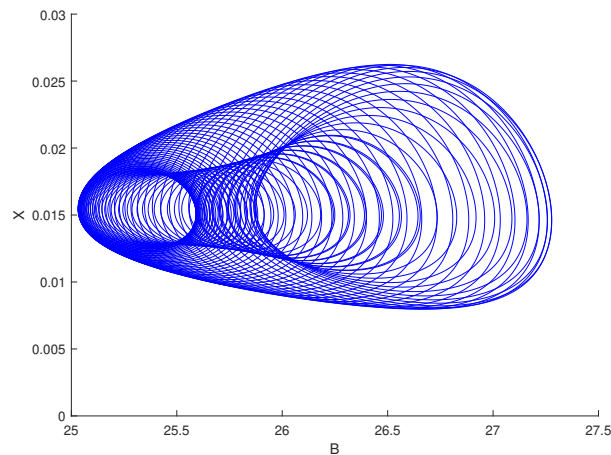


Figure 15: Dynamics on a stable 2-torus.

coordinate B as abscissa and ordinate, with the visibility limits:

Abscissa: 1000 2000
 Ordinate: 25.0 26.5

After **MatCont|Redraw Curve**, you will see the high-frequency oscillations with low-frequency modulation as in Figure 16.

Restore the original setting (i.e., 1) of **Options|Output Interval** via the main **MATCONT** window.

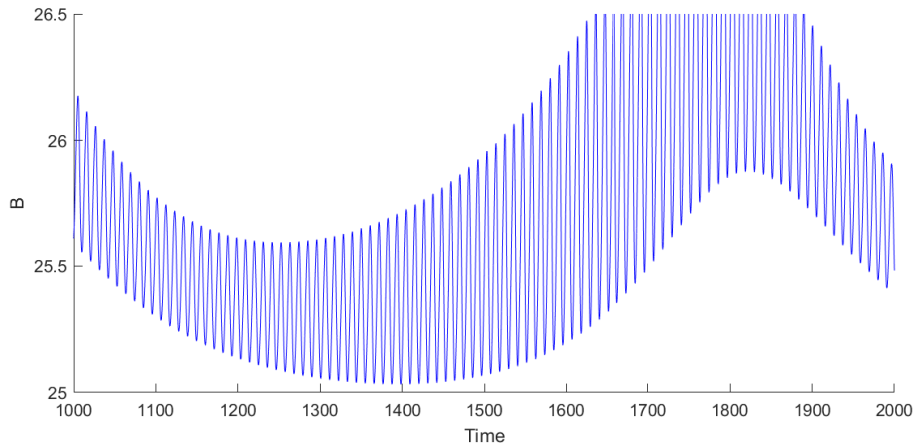


Figure 16: Modulated chemical oscillations.

3 Period-doubling bifurcation in an adaptive control model

Consider the following *adaptive control system of Lur'e type*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -\alpha z - \beta y - x + x^2. \end{cases} \quad (3)$$

We will fix $\alpha = 1$ and continue limit cycles of (3) using β as a bifurcation parameter.

3.1 System specification

As usual, define a new system in MATCONT with the coordinates x, y, z , parameters α, β , and select derivatives of order 1, 2, and 3 to be generated symbolically.

3.2 Equilibrium continuation

Select **Type|Initial point|Equilibrium** and **Type|Curve|Equilibrium**. In the **Starter** window, set α equal to 1 and activate β .

Open a **Plot3D** window and select β, x and y as variables along the coordinate axes with the visibility limits:

Abscissa:	0.4	1.2
Ordinate:	-0.5	1.2
Applicate:	-0.6	0.7

respectively.

Compute|Forward results in an equilibrium curve with a Hopf bifurcation (labeled **H**) detected with the following message in the MATLAB Command Window:

```
label = H , x = ( 0.000000 0.000000 0.000000 1.000000 )
First Lyapunov coefficient = -3.000000e-01
```

The Hopf bifurcation occurs at $\beta = 1$ and is supercritical, thus generating a stable limit cycle. Resume the continuation and eventually stop it, when the curve leaves the plotting region.

3.3 Cycle continuation starting from the H-point

Click **Control|View Result** and double-click at the H: Hopf point. The **Starter** and **Continuer** windows appear, corresponding to the limit cycle curve H.LC. Make sure that **beta** and **Period** are activated.

In the **Starter** window, use the number of mesh points **ntst** equal to 40 and check all boxes in the **Monitor Singularities** field. In the **Continuer** window, set **MaxStepsize** to 0.7 and **MaxNumPoints** to 80.

With **Window/Output|Numeric**, open a **Numeric** window and make the cycle **Multipliers** visible by checking them in the **Numeric|Layout** menu.

Compute|Forward will produce a branch of limit cycles of increasing amplitude with two *period-doubling bifurcations* (labeled PD). The **Numeric** windows at both PD bifurcations are shown in Figures 17 and 18.

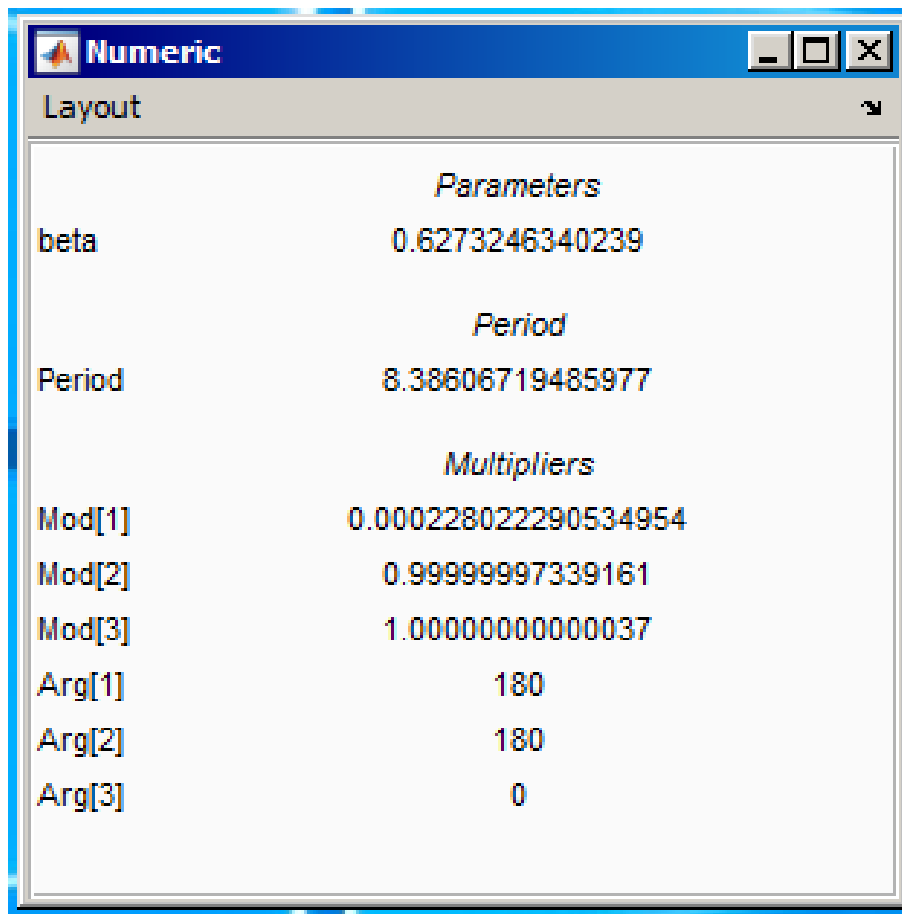


Figure 17: The **Numeric** window at the first PD bifurcation. The bifurcation parameter value is $\beta = 0.6273246\dots$

Resume the computations at both PD points. The MATLAB Command Window will contain the messages

```
Period Doubling (period = 8.386068e+00, parameter = 6.273245e-01)
Normal form coefficient = -3.405330e-03
Period Doubling (period = 9.864971e+00, parameter = 5.417461e-01)
Normal form coefficient = -7.655596e-04
```

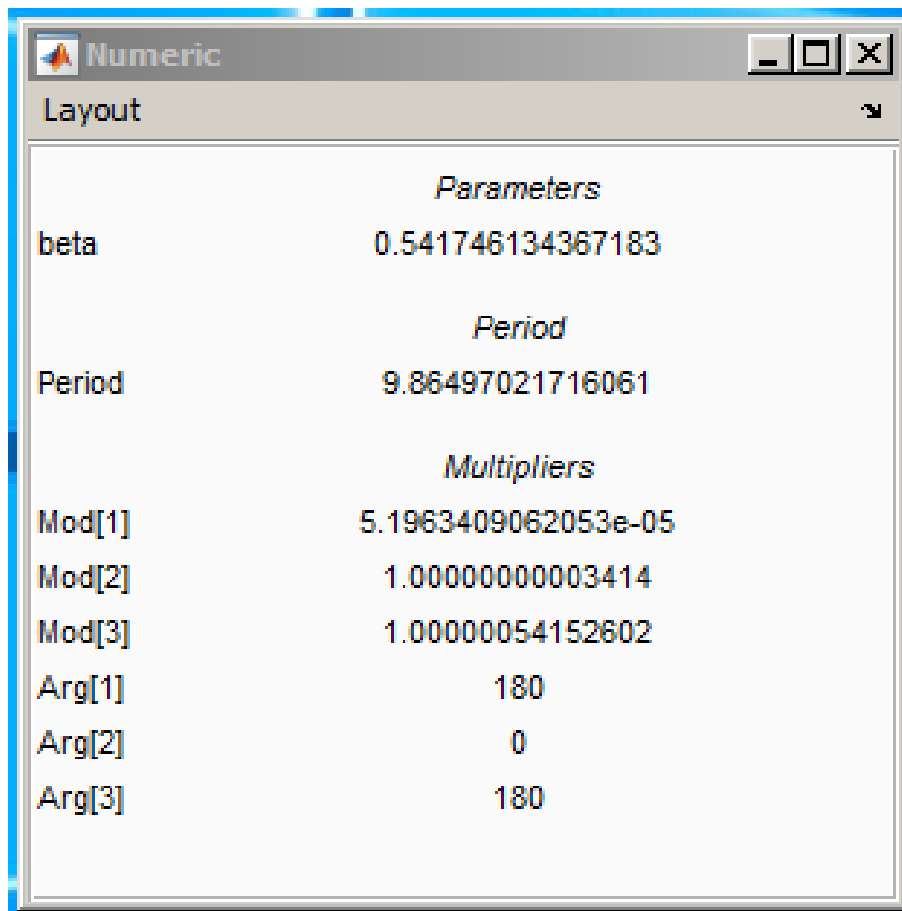


Figure 18: The **Numeric** window at the second PD bifurcation. The bifurcation parameter value is $\beta = 0.5417461\dots$

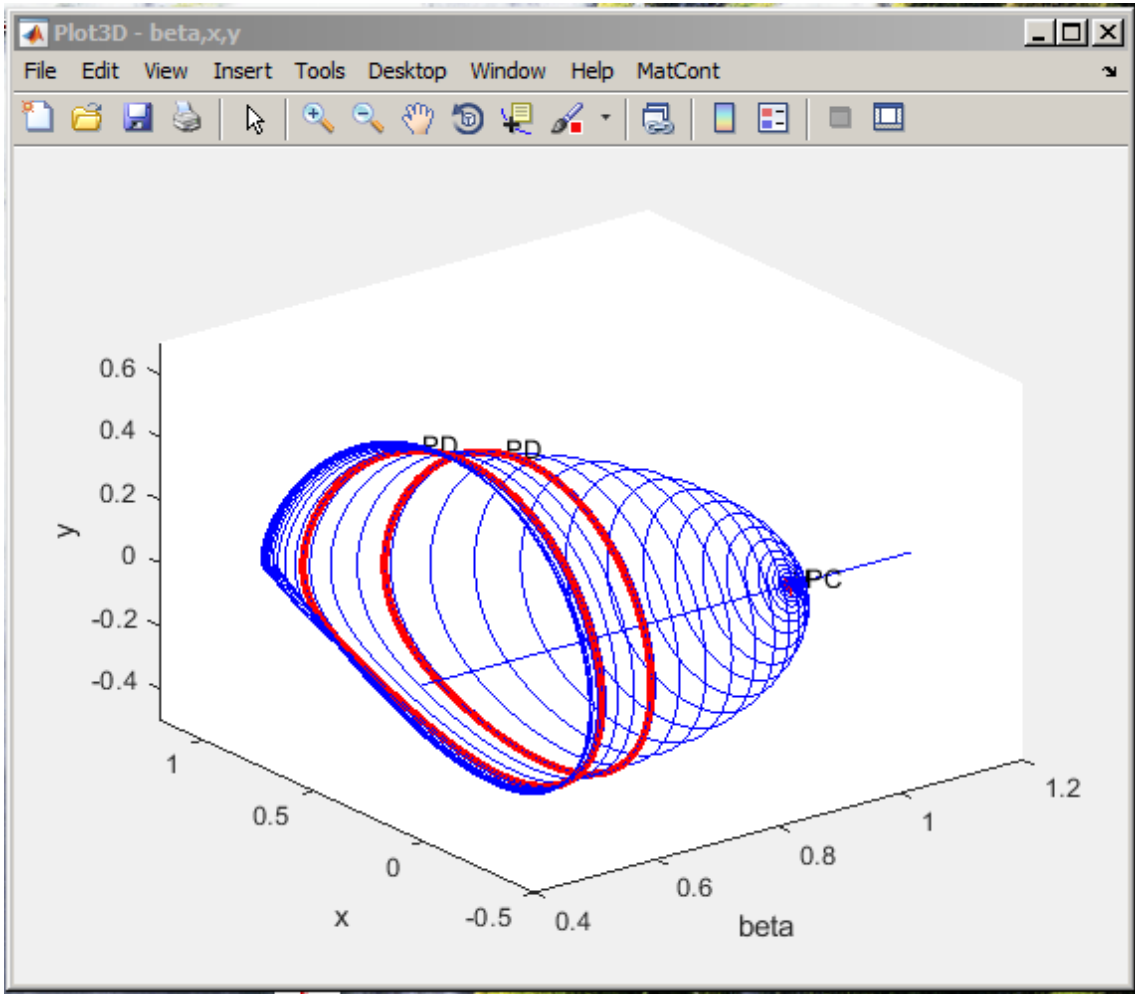


Figure 19: The cycle branch in system (3) rooted at the Hopf point. Two critical cycles undergoing the period-doubling bifurcations are labeled by PD.

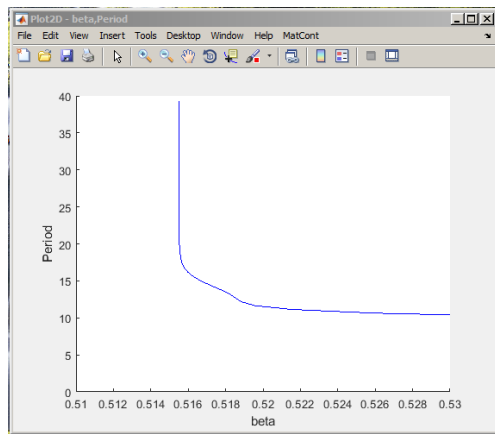


Figure 20: The cycle period near the homoclinic bifurcation.

indicating that in both cases the normal form coefficients are negative, so that stable double-period cycles are involved.

You get a cycle manifold shown in Figure 19. The period of the cycle rapidly increases when β approaches $\beta_{\text{hom}} = 0.515489\dots$. To see this, open a **Plot2D** window (see Figure 20) with the axes **beta** and **Period** and the plotting region

Abscissa: 0.51 0.53
 Ordinate: 0.0 40.0

and apply **MatCont|Redraw Curve**.

Actually, the limit cycle approaches a *homoclinic orbit* to the second equilibrium $(x, y, z) = (1, 0, 0)$ of (3), which is a global bifurcation¹. Below β_{hom} no cycle exists. Near the homoclinic bifurcation, the computation of the cycle and its multipliers becomes inaccurate. The loss of accuracy can be seen in the **Numeric** window, where no multiplier close to 1 is present at the last computed point.

3.4 Continuation of the double cycle from the PD-point

Using **Select|Initial point**, take the first PD: Period Doubling point in the computed above curve as initial.

Adjust the **Plot3D** window by setting the visibility limits for **beta** as follows:

Abscissa: 0.5 0.63

¹At $\beta = \beta_{\text{hom}}$, this equilibrium is a saddle-focus with the *saddle quantity* $\sigma < 0$. Thus, according to Shilnikov's Theorem, from this homoclinic orbit only one limit cycle bifurcates - the one we computed.

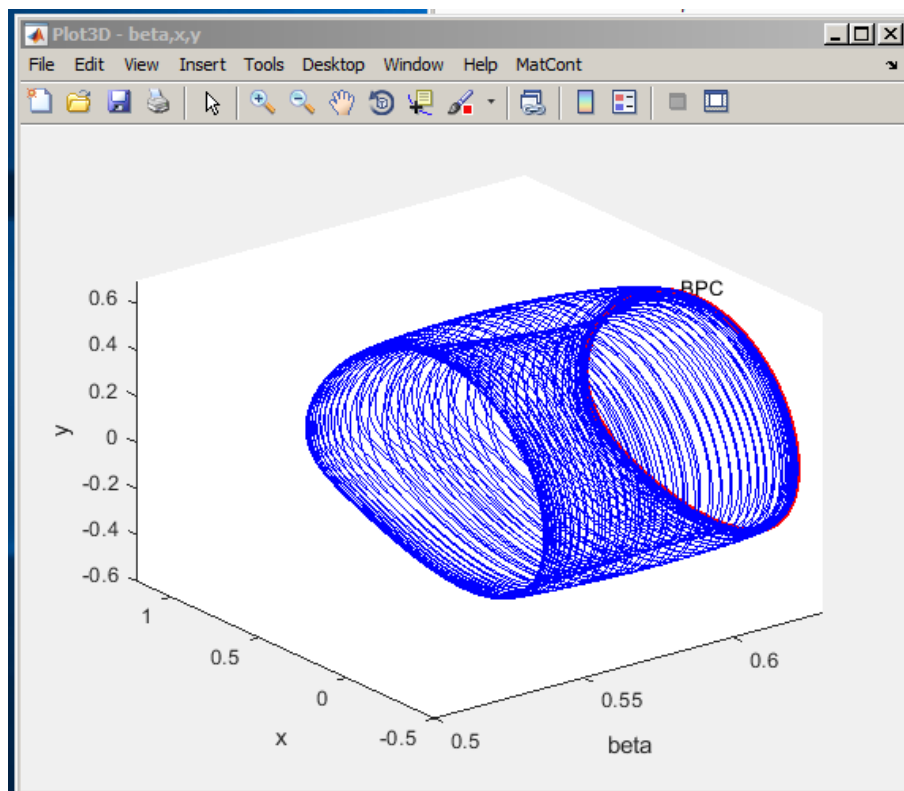


Figure 21: The branch of stable doubled cycles connects two PD points.

In the **Starter** window, set the number of mesh points `ntst` to 80, increase the `amplitude` to 0.005, and check only **Branch Point of Cycles (BPC)** in the **Monitor Singularities** section.

In the **Continuer** window, increase `MaxTestIters` to 30, and make sure that `MaxStepsize` remains equal to 0.7.

Click **Compute|Forward** and continue the doubled cycle bifurcating from the first PD-point at $\beta = \beta_{PD,1}$. This cycle remains stable until at (approximately) $\beta = \beta_{PD,2}$ a *branching point cycle* (BPC) is detected. This is not surprising, since the PD-bifurcation for the original limit cycles corresponds to a branching point for the doubled cycles. Stop computations at BPC, clear **Plot3D** window and make **MatCont|Redraw Curve** to get Figure 21.

Exit `MATCONT`.

4 Additional Problems

A. Study numerically limit cycles in the system

$$\begin{cases} \dot{x} &= \mu x - y - xz, \\ \dot{y} &= x + \mu y, \\ \dot{z} &= -z + y^2 + x^2 z, \end{cases}$$

that is another model of a feedback control system.

B. Consider the following *predator-prey* model

$$\begin{cases} \dot{x} &= x - \frac{xy}{1 + \alpha x}, \\ \dot{y} &= -y + \frac{xy}{1 + \alpha x} - \delta y^2. \end{cases}$$

1. Prove that the following polynomial system

$$\begin{cases} \dot{x} &= x(1 + \alpha x) - xy, \\ \dot{y} &= -(y + \delta y^2)(1 + \alpha x) + xy, \end{cases} \quad (4)$$

has the same orbits in the positive quadrant as the original model. Derive an explicit condition on (α, δ) for system (4) to have a positive equilibrium with one zero eigenvalue. Derive a similar condition for (4) to have a positive equilibrium with a pair of purely imaginary eigenvalues.

2. Introduce new variables

$$\begin{cases} \xi &= \ln x, \\ \eta &= \ln y, \end{cases}$$

in which the original system becomes:

$$\begin{cases} \dot{\xi} &= 1 - \frac{\exp(\eta)}{1 + \alpha \exp(\xi)}, \\ \dot{\eta} &= -1 + \frac{\exp(\xi)}{1 + \alpha \exp(\xi)} - \delta \exp(\eta). \end{cases} \quad (5)$$

Fix $\alpha = 0.3$ and study numerically limit cycles of (5), when the parameter δ varies.

Hint: Begin with finding an equilibrium at $\delta = 0.35$ by integration.

3. Relate your analytical and numerical results.

C. Consider the following *predator-double prey* system:

$$\begin{cases} \dot{x} &= x(2.4 - x - 6y - 4z), \\ \dot{y} &= y(\beta - x - y - 10z), \\ \dot{z} &= -z(1 - 0.25x - 4y + z), \end{cases}$$

modelling dynamics of two prey populations affected by a predator.

1. Starting from $\beta = 1.77$, find its *positive* equilibrium by orbit integration.
2. Continue the equilibrium with respect to β until it exhibits a Hopf bifurcation.
3. Continue a limit cycle from the Hopf point and monitor the dependence of its period upon β . *Hint*: Set `ntst` to a high value.
4. Plot the cycle at different period values and try to understand its asymptotic shape. *Hint*: Compute and analyze the equilibria located on the coordinate planes.

D. Consider the famous *Lorenz system*

$$\begin{cases} \dot{x} &= \sigma(-x + y), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{cases}$$

with the standard parameter values $\sigma = 10, b = \frac{8}{3}$ and $r > 0$. Use `MATCONT` to analyse its simplest limit cycles numerically.

1. Compute a critical parameter value r_{BP} , at which the trivial equilibrium $(0, 0, 0)$ undergoes a *symmetric* pitchfork bifurcation.
2. Compute a critical parameter value r_{H} , corresponding to a Hopf bifurcation of *nontrivial* equilibria.
3. Continue the limit cycle born at the Hopf bifurcation with respect to r and find out what happens to it. Plot the cycle period as function of r , and monitor stability.
4. Demonstrate by numerical integration that for $r = 400$ the system has a stable symmetric limit cycle and determine its period approximately. Continue this cycle with respect to r and find the critical parameter value r_{BPC} corresponding to a *branch point of cycles (symmetry breaking)*. *Hint*: Use the procedure to start LC continuation from a simulation as done in Section 1.

Derive explicit formulas for r_{BP} and r_{H} as functions of (σ, b) and verify the numerical values obtained with `MATCONT`.