Lecture 8

Continuation of codim 1 bifurcations of limit cycles in ODEs

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March 26, 2014

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1. Linear nonautonomous equations

- Consider two **adjoint** linear ODEs

$$\dot{v} = A(t)v, \ \dot{w} = -A^*(t)w, \quad v, w \in \mathbb{R}^n,$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous and * denotes transpose.

- The **fundamental matrix solutions**:

$$\dot{M} = A(t)M, \quad M(0) = I_n,$$

and

$$\dot{N} = -A^*(t)N, \quad N(0) = I_n.$$

One has: $N(t) = [M^{-1}(t)]^*$.

- The solution of the linear ODE

$$\dot{v} = A(t)v + b(t), \quad v \in \mathbb{R}^n,$$

with continuous $b : \mathbb{R} \to \mathbb{R}^n$ is given by

$$\begin{aligned} v(t) &= M(t)v(0) + \int_0^t M(t)M^{-1}(\tau)b(\tau)\, d\tau \\ &= M(t)v(0) + \int_0^t M(t)N^*(\tau)b(\tau)\, d\tau \\ &= M(t)\left[v(0) + \int_0^t N^*(\tau)b(\tau)\, d\tau\right]. \end{aligned}$$

2. Monodromy matrices of limit cycles

- Consider a smooth ODE system

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}.$$

A **cycle** with period T at a parameter value α corresponds to a solution $x \in \mathcal{C}^1([0,1], \mathbb{R}^n)$ of the BVP

$$\begin{cases} \dot{x} - Tf(x, \alpha) &= 0, \\ x(1) - x(0) &= 0. \end{cases}$$

- **Monodromy matrix**:

$$\dot{\Phi}(t) - Tf_x(x(t), \alpha)\Phi(t) = 0, \quad \Phi(0) = I_n.$$

The eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n-1}$, and $\mu_n = 1$ of $\Phi(1)$ are the **multipliers** of the cycle.

- **Adjoint monodromy matrix**:

$$\dot{\Psi}(t) + Tf_x^*(x(t), \alpha)\Psi(t) = 0, \quad \Psi(0) = I_n,$$

where * denotes transpose.

- One has: $\Psi(t) = [\Phi^{-1}(t)]^*$ and

$$(\Phi(1) - I_n)q_0 = (\Psi(1) - I_n)p_0 = 0,$$
$$(\Phi(1) - I_n)^* p_0 = (\Psi(1) - I_n)^* q_0 = 0,$$

with $q_0^* q_0 = p_0^* p_0 = 1$. Notice that $q_0 = c_0 \dot{x}(0)$ with $c_0 \in \mathbb{R}$, $c_0 \neq 0$.

3. Regular cycles and simple bifurcations

Def. 1 *A cycle is called* **regular** *if $\mu_n = 1$ has geometric multiplicity* 1.

Def. 2 *A cycle bifurcation is called* **simple** *if only the following critical multipliers are present:*

- **LPC**: $\mu_1 = \mu_n = 1$ *with algebraic multiplicity* 2 *and geometric multiplicity* 1

  $$(\Phi(1) - I_n)q_1 = q_0, \quad (\Psi(1) - I_n)p_1 = p_0.$$

- **PD**: $\mu_1 = -1$ *and* $\mu_n = 1$ *both with algebraic multiplicity* 1

  $$(\Phi(1) + I_n)q_2 = 0, \quad (\Psi(1) + I_n)p_2 = 0.$$

- **NS**: $\mu_{1,2} = e^{\pm i\theta_0}$ *and* $\mu_n = 1$ *all with algebraic multiplicity* 1

  $$(\Phi(1) - \mathrm{e}^{i\theta_0} I_n)(q_3 + iq_4) = 0,$$
  $$(\Psi(1) - \mathrm{e}^{i\theta_0} I_n)(p_3 + ip_4) = 0.$$

  We have $(I_n - 2\kappa\Phi(1) + \Phi^2(1))q_{3,4} = 0$ *where* $\kappa = \cos\theta_0$.

4. Defining systems for codim 1 bifurcations

- **LPC** and **PD**: $(x, T, \alpha) \in \mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}$

$$\left\{ \begin{array}{rcl} \dot{x}(\tau) - Tf(x(\tau), \alpha) & = & 0, \quad \tau \in [0,1], \\ x(0) - x(1) & = & 0, \\ \int_0^1 \langle \dot{x}_0(\tau), x(\tau) \rangle \, d\tau & = & 0, \\ G[x, T, \alpha] & = & 0. \end{array} \right.$$

- **NS**: $(x, T, \alpha, \kappa) \in \mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$\left\{ \begin{array}{rcl} \dot{x}(\tau) - Tf(x(\tau), \alpha) & = & 0, \quad \tau \in [0,1], \\ x(0) - x(1) & = & 0, \\ \int_0^1 \langle \dot{x}_0(\tau), x(\tau) \rangle \, d\tau & = & 0, \\ G_{11}[u, T, \alpha, \kappa] & = & 0, \\ G_{22}[u, T, \alpha, \kappa] & = & 0. \end{array} \right.$$

When the linearized system is nonsingular at a solution, this solution can be continued w.r.t. another parameter, thus obtaining a **bifurcation curve** in the (α_1, α_2)-plane.

LPC-computation

- There exist $v_{01}, w_{01} \in \mathcal{C}^0([0,1],\mathbb{R}^n), w_{02} \in \mathbb{R}^n$, and $v_{02}, w_{03} \in \mathbb{R}$ such that

  $N_1 : \mathcal{C}^1([0,1],\mathbb{R}^n)\times\mathbb{R}^2 \to \mathcal{C}^0([0,1],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^2$,

  $$N_1 = \begin{bmatrix} D - Tf_x(x,\alpha) & -f(x,\alpha) & w_{01} \\ \delta_0 - \delta_1 & 0 & w_{02} \\ \mathrm{Int}_{f(x,\alpha)} & 0 & w_{03} \\ \mathrm{Int}_{v_{01}} & v_{02} & 0 \end{bmatrix},$$

  is one-to-one and onto near a simple LPC bifurcation point.

- Define G by solving

  $$N_1 \begin{pmatrix} v \\ S \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

- "Classical" form:

  $$\begin{cases} \dot{v}(\tau) - Tf_x(x(\tau),\alpha)v(\tau) & \\ \qquad -Sf(u(\tau),\alpha) + Gw_{01}(\tau) & = 0, \\ v(0) - v(1) + Gw_{02} & = 0, \\ \int_0^1 \langle f(x(\tau),\alpha), v(\tau)\rangle d\tau + Gw_{03} & = 0, \\ \int_0^1 \langle v_{01}(\tau), v(\tau)\rangle d\tau + Sv_{02} & = 1. \end{cases}$$

Lemma 1 *If $(x(\cdot), T, \alpha)$ corresponds to a regular cycle then the operator*

$$M_1 = \begin{bmatrix} D - Tf_x(x,\alpha) & -f(x,\alpha) \\ \delta_1 - \delta_0 & 0 \\ \mathrm{Int}_{f(x,\alpha)} & 0 \end{bmatrix}$$

from $\mathcal{C}^1([0,1],\mathbb{R}^n) \times \mathbb{R}$ into $\mathcal{C}^0([0,1],\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$ is one-to-one if the multiplier 1 has algebraic multiplicity 1. If the multiplier 1 has algebraic multiplicity 2, i.e. at a simple LPC, then M_1 has a one-dimensional kernel, spanned by the vector

$$\begin{pmatrix} v \\ 1 \end{pmatrix} \in \mathcal{C}^1([0,1],\mathbb{R}^n) \times \mathbb{R},$$

where

$$v(t) = \frac{c_0}{T}\Phi(t)(c_2 q_0 - (q_1 - tq_0))$$

and c_2 is determined by the condition that

$$q_0^* \int_0^1 \Phi^*(\tau)\Phi(\tau)[c_2 q_0 - (q_1 - \tau q_0)]\ d\tau = 0,$$

while c_0 is such that $\dot{x}(0) = c_0 q_0$.

Proof: Consider the homogeneous equation

$$\left[\begin{array}{cc} D - Tf_x(x,\alpha) & -f(x,\alpha) \\ \delta_1 - \delta_0 & 0 \\ \text{Int}_{f(x,\alpha)} & 0 \end{array}\right] \begin{pmatrix} v \\ S \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From the first row we have

$$\dot{v} - Tf_x(x(t),\alpha)v = Sf(x(t),\alpha)$$

implying

$$\begin{aligned} v(t) &= \Phi(t)\left[v(0) + S\int_0^t \Psi^*(\tau)f(x(\tau),\alpha)\ d\tau\right] \\ &= \Phi(t)\left[v(0) + \frac{S}{T}\int_0^t \Psi^*(\tau)\dot{x}(\tau)d\tau\right] \\ &= \Phi(t)\left[v(0) + \frac{S}{T}\int_0^t \Psi^*(\tau)\Phi(\tau)d\tau\ \dot{x}(0)\right] \\ &= \Phi(t)\left[v(0) + \frac{St}{T}\dot{x}(0)\right], \end{aligned}$$

since $\Psi^*(\tau)\Phi(\tau) = I_n$ and $\dot{x}(t) = \Phi(t)\dot{x}(0)$.

From the second row we have

$$0 = v(1) - v(0) = (\Phi(1) - I)v(0) + \frac{S}{T}\dot{x}(0)$$

or

$$(\Phi(1) - I)v(0) = -\frac{S}{T}\dot{x}(0).$$

Because $\dot{x}(0) = c_0 q_0$ for some $c_0 \in \mathbb{R}$, $c_0 \neq 0$, we must solve

$$(\Phi(1) - I)v(0) = -c_0 \frac{S}{T} q_0, \tag{1}$$

where q_0 spans the kernel of $\Phi(1) - I$.

If the multiplier 1 has algebraic multiplicity 1 then we must have $S = 0$, $v(0) = c_1 q_0$, and hence $v(t) = c_1 \Phi(t) q_0$. From the third row,

$$\begin{aligned} 0 &= \int_0^1 f^*(x(\tau), \alpha) v(\tau)\ d\tau \\ &= \frac{1}{T} \int_0^1 \dot{x}^*(\tau) v(\tau)\ d\tau \\ &= \frac{1}{T} \int_0^1 [\Phi(\tau)\dot{x}(0)]^* c_1 \Phi(\tau) q_0\ d\tau \end{aligned}$$

or

$$c_0 c_1\ q_0^* \left(\int_0^1 \Phi^*(\tau)\Phi(\tau)\ d\tau \right) q_0 = 0,$$

from which it follows that $c_1 = 0$. Thus $v(t) \equiv 0$. It follows that the operator M_1 is one-to-one.

At a simple fold the multiplier 1 has algebraic multiplicity 2. In this case (1) is also solvable if S is nonzero, namely

$$v(0) = -c_0 \frac{S}{T} q_1 + c_2 q_0,$$

where $c_2 \in \mathbb{R}$ is arbitrary. The third row then implies

$$\begin{aligned}
0 &= \int_0^1 \dot{x}^*(\tau) v(\tau)\ d\tau \\
&= \int_0^1 \dot{x}^*(\tau) \Phi(\tau) \left[v(0) + \frac{S\tau}{T} \dot{x}(0) \right] d\tau \\
&= \int_0^1 [\Phi(\tau)\dot{x}(0)]^* \Phi(\tau) \left[-c_0 \frac{S}{T} q_1 + c_2 q_0 + \frac{S\tau}{T} c_0 q_0 \right] d\tau \\
&= c_0 q_0^* \int_0^1 \Phi^*(\tau) \Phi(\tau) \left[-c_0 \frac{S}{T} q_1 + c_2 q_0 + \frac{S\tau}{T} c_0 q_0 \right] d\tau,
\end{aligned}$$

from which it follows that

$$c_2 = \frac{c_0 S q_0^* \int_0^1 \Phi^*(\tau)\Phi(\tau)[q_1 - \tau q_0]\ d\tau}{T\ q_0^* \int_0^1 \Phi^*(\tau)\Phi(\tau)\ d\tau\ q_0}.$$

Take now $S = 1$ and $v(t) = \Phi(t) v(0)$ to get a null-vector of M_1.

Lemma 2 *Let* $(x(\cdot), T, \alpha)$ *corresponds to a regular cycle and consider the operator*

$$M_1 = \left[\begin{array}{cc} D - Tf_x(x,\alpha) & -f(x,\alpha) \\ \delta_1 - \delta_0 & 0 \\ \mathrm{Int}_{f(x,\alpha)} & 0 \end{array} \right] \qquad (2)$$

from $\mathcal{C}^1([0,1],\mathbb{R}^n)\times\mathbb{R}$ *into* $\mathcal{C}^0([0,1],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}$. *If the multiplier* 1 *has algebraic multiplicity* 1, *then* M_1 *is onto.*

If it has algebraic multiplicity 2, *i.e., at a simple LPC, then the range of* M_1 *has codimension* 1 *and the vector*

$$\left(\begin{array}{c} \Psi p_0 \\ -p_0 \\ 0 \end{array} \right) \in \mathcal{C}^0([0,1],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R} \qquad (3)$$

is complementary to the range space.

Proof: Consider a vector

$$\begin{pmatrix} \xi \\ \eta \\ \omega \end{pmatrix} \in \mathcal{C}^0([0,1],\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}.$$

It is in the range of M_1 if and only if there exists

$$\begin{pmatrix} v \\ S \end{pmatrix} \in \mathcal{C}^1([0,1],\mathbb{R}^n) \times \mathbb{R}$$

such that

$$\begin{bmatrix} D - Tf_x(x,\alpha) & -f(x,\alpha) \\ \delta_1 - \delta_0 & 0 \\ \mathrm{Int}_{f(x,\alpha)} & 0 \end{bmatrix} \begin{pmatrix} v \\ S \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \\ \omega \end{pmatrix}.$$

The first row implies that

$$v(t) = \Phi(t)\left[v(0) + \int_0^t \Psi^*(\tau)(\xi(\tau) + Sf(x(\tau),\alpha))d\tau\right].$$

The second row then implies

$$\begin{aligned}
\eta &= v(1) - v(0) \\
&= (\Phi(1) - I)v(0) + \\
&\quad \Phi(1)\int_0^1 \Psi^*(\tau)(\xi(\tau) + Sf(x(\tau),\alpha))d\tau \\
&= (\Phi(1) - I)v(0) + \\
&\quad \Phi(1)\int_0^1 \Psi^*(\tau)\xi(\tau)d\tau + \frac{Sc_0}{T}q_0.
\end{aligned}$$

Thus

$$\eta = (\Phi(1) - I)v(0) + \frac{Sc_0}{T} q_0 + \Phi(1) \int_0^1 \Psi^*(\tau)\xi(\tau)\, d\tau. \tag{4}$$

If 1 is an algebraically simple eigenvalue of $\Phi(1)$ then q_0 is not in the range of $(\Phi(1) - I)$. For given ξ and η, equation (4) can be solved for $v(0)$ and S. Moreover, the solution is unique up to the addition of a scalar multiple of q_0 to $v(0)$. Since

$$\begin{aligned} \int_0^1 (\dot{x}(\tau))^* \Phi(\tau) q_0 \, d\tau &= c_0 \int_0^1 (\Phi(\tau) q_0)^* \Phi(\tau) q_0 \, d\tau \\ &\neq 0, \end{aligned}$$

the scalar is determined uniquely by the third row of the main system.

If 1 is an algebraically double eigenvalue of $\Phi(1)$, i.e., at a simple LPC point, then (4) is solvable if and only if

$$p_0^* \eta = p_0^* \int_0^1 \Psi^*(\tau)\xi(\tau)\, d\tau.$$

If so, the third row of the main system again determines the solution uniquely.

Lemma 3 *If* $(x(\cdot), T, \alpha)$ *corresponds to a regular cycle then the operator*

$$M_2 = \begin{bmatrix} D + T f_x^*(x,\alpha) & -f(x,\alpha) \\ \delta_1 - \delta_0 & 0 \\ \mathrm{Int}_{f(x,\alpha)} & 0 \end{bmatrix}$$

from $\mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R} \to \mathcal{C}^0([0,1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$ *is one-to-one and onto if the multiplier* 1 *has algebraic multiplicity* 1.

If the multiplier 1 *has algebraic multiplicity* 2*, i.e., at a simple LPC, then* M_2 *has a one-dimensional kernel, spanned by*

$$\begin{pmatrix} \Psi^* p_0 \\ 0 \end{pmatrix} \in \mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R},$$

while its range has codimension 1*, and the vector*

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathcal{C}^0([0,1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$$

is complementary to the range space.

Th. 1 *Let* $(x(\cdot), T, \alpha)$ *corresponds to a simple LPC, i.e.,* $\Phi(1)$ *has eigenvalue* 1 *with algebraic multiplicity* 2. *Then there exist* $v_{01}, w_{01}, v_{11}, w_{11} \in \mathcal{C}^0([0,1], \mathbb{R}^n)$, $w_{02}, v_{12} \in \mathbb{R}^n$, $w_{03}, v_{02}, v_{13}, w_{12} \in \mathbb{R}$ *such that operator*

$$N_1 = \begin{bmatrix} D - Tf_x(x,\alpha) & -f(x,\alpha) & w_{01} \\ \delta_1 - \delta_0 & 0 & w_{02} \\ \mathrm{Int}_{f(x,\alpha)} & 0 & w_{03} \\ \mathrm{Int}_{v_{01}} & v_{02} & 0 \end{bmatrix}$$

from $\mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$ *to* $\mathcal{C}^0([0,1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ *is one-to-one and onto.*

Proof: We choose

$$\begin{pmatrix} v_{01}(t) \\ v_{02} \end{pmatrix} = \begin{pmatrix} v(t) \\ 1 \end{pmatrix},$$

where v is given in the statement of Lemma 1. Further we set

$$\begin{pmatrix} w_{01}(t) \\ w_{02} \\ w_{03} \end{pmatrix} = \begin{pmatrix} \Psi^*(t)p_0 \\ 0 \\ 0 \end{pmatrix}.$$

By Lemmas 1 and 3, N_1 is one-to-one and onto.

PD-computation

- There exist $v_{01}, w_{01} \in \mathcal{C}^0([0,1],\mathbb{R}^n)$, and $w_{02} \in \mathbb{R}^n$, such that

  $N_2 : \mathcal{C}^1([0,1],\mathbb{R}^n)\times\mathbb{R} \to \mathcal{C}^0([0,1],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}$,

  $$N_2 = \begin{bmatrix} D - Tf_x(x,\alpha) & w_{01} \\ \delta_0 + \delta_1 & w_{02} \\ \mathrm{Int}_{v_{01}} & 0 \end{bmatrix},$$

  is one-to-one and onto near a simple PD bifurcation point.

- Define G by solving

  $$N_2 \begin{pmatrix} v \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- The BVP for (v,G) can be written in the "classical form"

  $$\left\{ \begin{array}{rcl} \dot{v}(\tau) - Tf_x(x(\tau),\alpha)v(\tau) + Gw_{01}(\tau) & = & 0, \\ v(0) + v(1) + Gw_{02} & = & 0, \\ \int_0^1 \langle v_{01}(\tau), v(\tau)\rangle d\tau & = & 1. \end{array} \right.$$

NS-computation

- There exist $v_{01}, v_{02}, w_{11}, w_{12} \in \mathcal{C}^0([0,2], \mathbb{R}^n)$, and $w_{21}, w_{22} \in \mathbb{R}^n$, such that

  $N_3 : \mathcal{C}^1([0,2], \mathbb{R}^n) \times \mathbb{R}^2 \to \mathcal{C}^0([0,2], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2$,

  $$N_3 = \begin{bmatrix} D - Tf_x(x,\alpha) & w_{11} & w_{12} \\ \delta_0 - 2\kappa\delta_1 + \delta_2 & w_{21} & w_{22} \\ \mathrm{Int}_{v_{01}} & 0 & 0 \\ \mathrm{Int}_{v_{02}} & 0 & 0 \end{bmatrix},$$

  is one-to-one and onto near a simple NS bifurcation point.

- Define G_{jk} by solving the **3-point BVP**

  $$N_3 \begin{pmatrix} r & s \\ G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- At the NS-cycle: $\kappa = \cos\theta$.

5. Remarks

- After discretization via orthogonal collocation, all linear BVPs for G's have sparsity structure that is identical to that of the linearization of the BVP for limit cycles.

- For each defining system holds: *Simplicity of the bifurcation* + *Transversality* $\Rightarrow$ *Regularity of the defining BVP*, allowing for the two-parameter continuation with $\alpha \in \mathbb{R}^2$.

- Jacobian matrix of each (discretized) defining BVP can be efficiently computed using adjoint linear BVP.

- Border adaptation using solutions of the adjoint linear BVPs.

- Actually implemented in MATCONT.

- Maximally augmented BVPCPs (AUTO)

LPC:

$$\left\{\begin{array}{rcl} \dot{x}(\tau) - Tf(x(\tau),\alpha) & = & 0,\\ x(1) - x(0) & = & 0,\\ \int_0^1 \langle \dot{x}_0(\tau), x(\tau)\rangle \, d\tau & = & 0,\\ \dot{v}(\tau) - Tf_x(x(\tau),\alpha)v(\tau) - Sf(x(\tau),\alpha) & = & 0,\\ v(1) - v(0) & = & 0,\\ \int_0^1 \langle \dot{x}_0(\tau), v(\tau)\rangle \, d\tau & = & 0,\\ \int_0^1 \langle v_0(\tau), v(\tau)\rangle \, d\tau + S^2 - 1 & = & 0. \end{array}\right.$$

PD:

$$\left\{\begin{array}{rcl} \dot{x}(\tau) - Tf(x(\tau),\alpha) & = & 0,\\ x(1) - x(0) & = & 0,\\ \int_0^1 \langle \dot{x}_0(\tau), x(\tau)\rangle \, d\tau & = & 0,\\ \dot{v}(\tau) - Tf_x(x(\tau),\alpha)v(\tau) & = & 0,\\ v(1) + v(0) & = & 0,\\ \int_0^1 \langle v_0(\tau), v(\tau)\rangle \, d\tau - 1 & = & 0. \end{array}\right.$$

NS:

$$\left\{\begin{array}{rcl} \dot{x}(\tau) - Tf(x(\tau),\alpha) & = & 0,\\ x(1) - x(0) & = & 0,\\ \int_0^1 \langle \dot{x}_0(\tau), x(\tau)\rangle \, d\tau & = & 0,\\ \dot{w}(\tau) - Tf_x(x(\tau),\alpha)w(\tau) & = & 0,\\ w(1) - e^{i\theta}w(0) & = & 0,\\ \int_0^1 \langle w_0(\tau), w(\tau)\rangle_{\mathbb{C}^n} \, d\tau - 1 & = & 0. \end{array}\right.$$