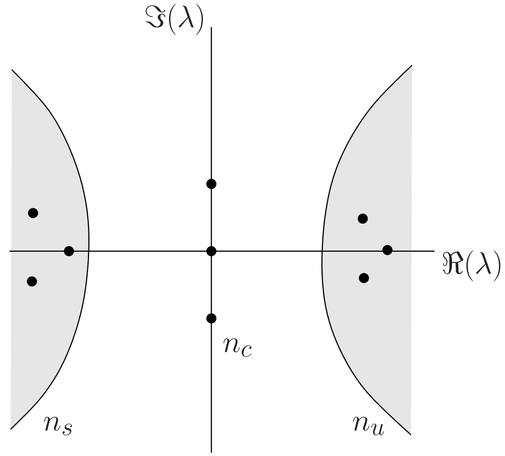
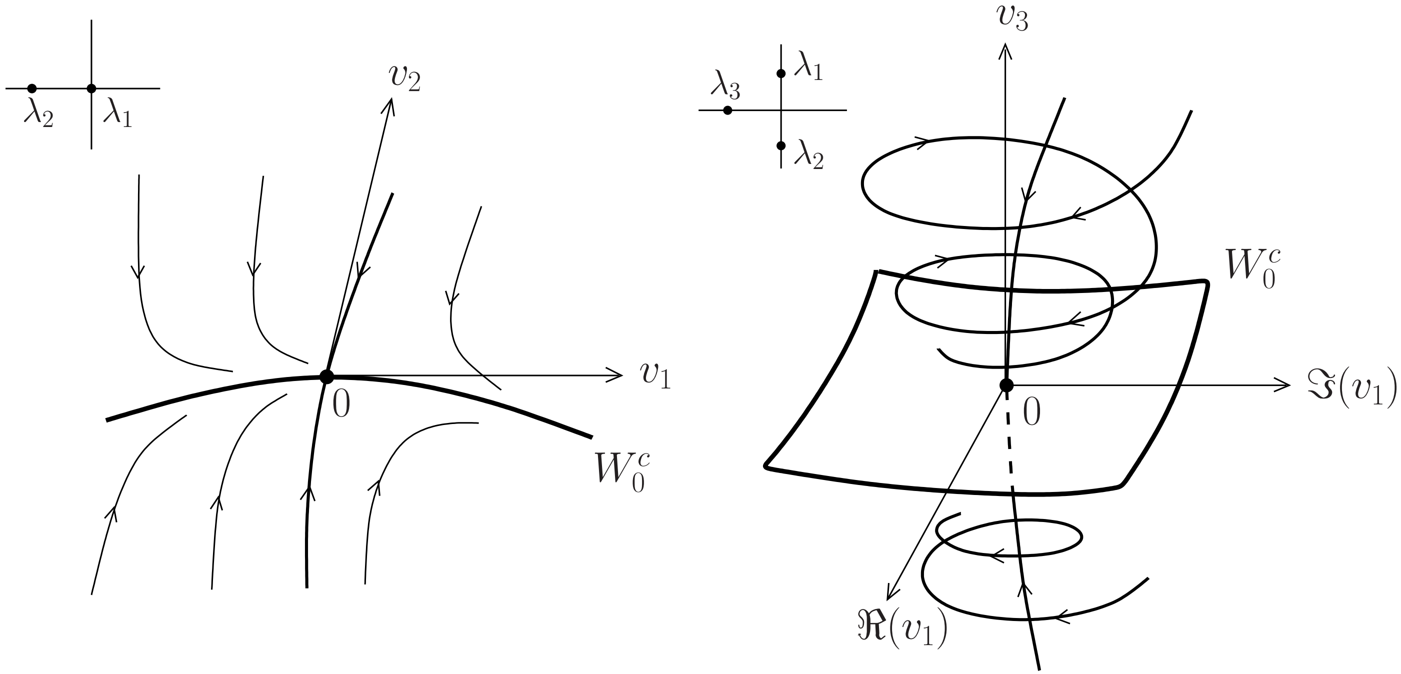
2. Bifurcations of n-dimensional ODEs $\dot{u} = f(u, \alpha)$

- **Local (equilibrium) bifurcations**

  Center manifold reduction: Let $u_0 = 0$ at $\alpha = 0$ be non-hyperbolic with stable, usntable, and critical eigenvalues:



Th. 2 *For all sufficiently small $\|\alpha\|$, there exists a local invariant* **center manifold** W^c_α *of dimension n_c that is locally attracting if $n_u = 0$, repelling if $n_s = 0$, and of saddle type if $n_s n_u > 0$. Moreover W^c_0 is tangent to the critical eigenspace of $A = f_u(0, 0)$.*



Remark: W_0^c is **not unique**; however, all W_0^c have the same Taylor expansion.

Th. 3 *If $\dot{\xi} = g(\xi, \alpha)$ is the restriction of $\dot{u} = f(u, \alpha)$ to W_α^c, then this system is locally topologically equivalent to*

$$\begin{cases} \dot{\xi} &= g(\xi, \alpha), \quad \xi \in \mathbb{R}^{n_c}, \alpha \in \mathbb{R}^m, \\ \dot{x} &= -x, \quad x \in \mathbb{R}^{n_s}, \\ \dot{y} &= +y, \quad y \in \mathbb{R}^{n_u}. \end{cases}$$

Codimension 1 bifurcations of equilibria

- Consider a smooth ODE system

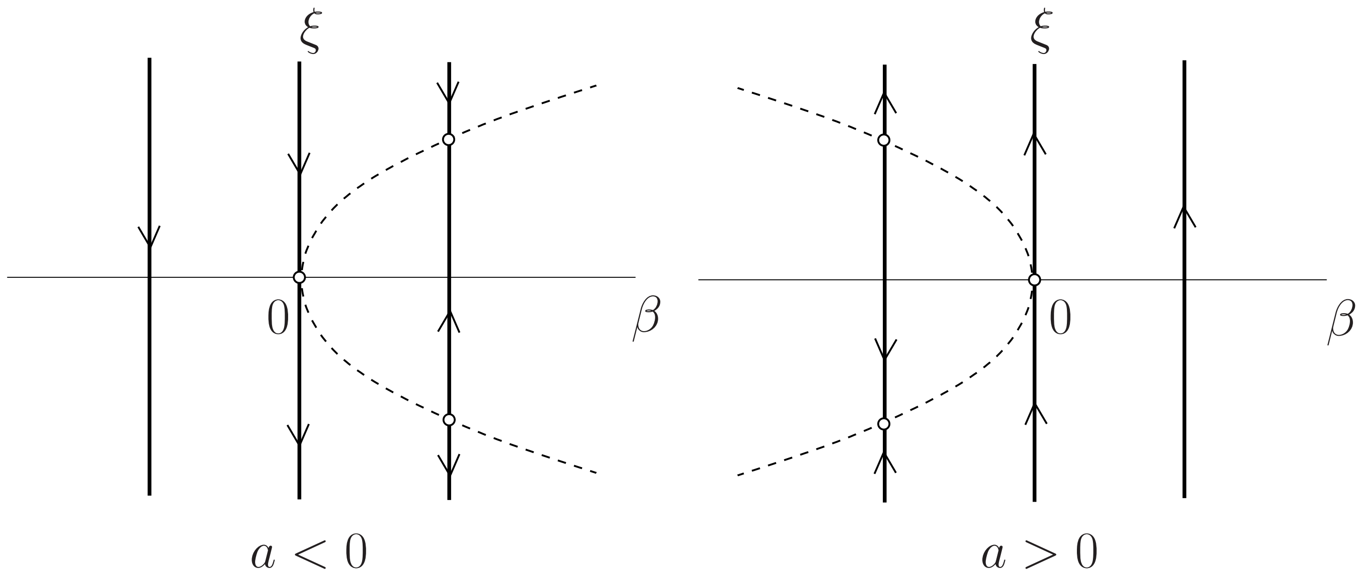
$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}.$$

- Critical cases:

  - **Fold (limit point, LP)**: $\lambda_1 = 0$;
  - **Andronov-Hopf (H)**: $\lambda_{1,2} = \pm i\omega_0, \quad \omega_0 > 0$.

LP smooth normal form on $W^c_{\beta(\alpha)}$

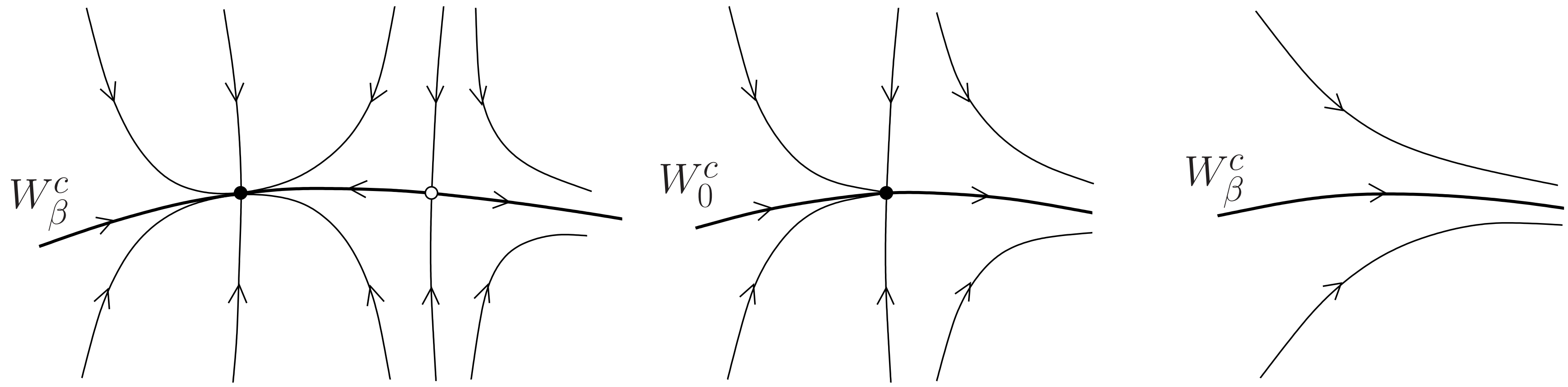
- $\dot{\xi} = \beta(\alpha) + a(\alpha)\xi^2 + O(|\xi|^3), \quad a(0) \neq 0.$



- Approximation of equilibria:

$$\beta + a\xi^2 = 0 \Rightarrow \xi_{1,2} = \pm\sqrt{-\frac{\beta}{a}}$$

Generic LP bifurcation: $\lambda_1 = 0 \quad (a > 0)$

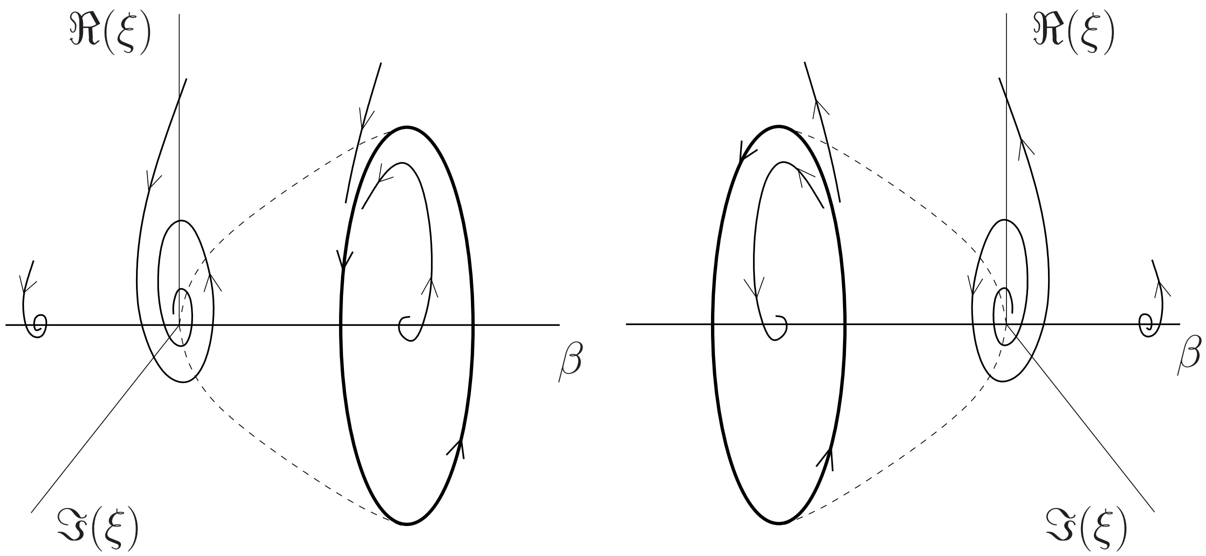


$\beta < 0$ $\beta = 0$ $\beta > 0$

Collision of two equilibria.

Hopf smooth normal form on $W^c_{\beta(\alpha)}$

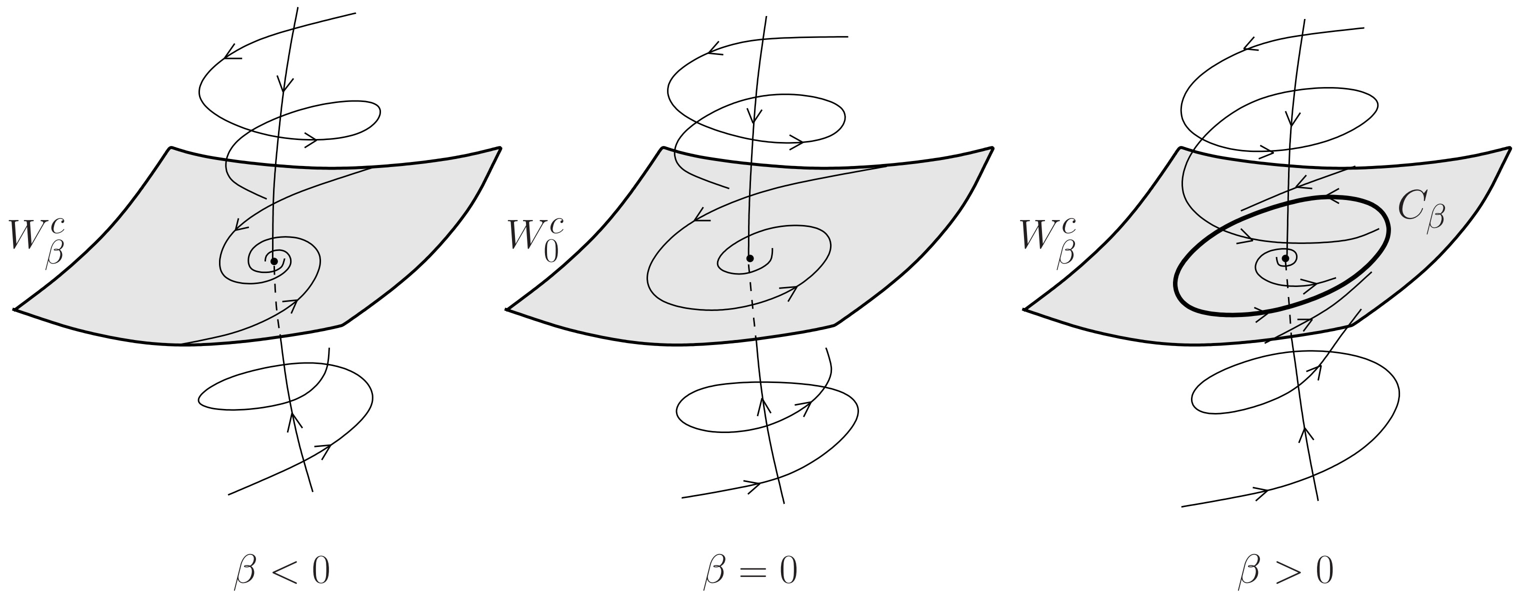
- $\dot{\xi} = (\beta(\alpha) + i\omega(\alpha))\xi + c_1(\alpha)\xi|\xi|^2 + O(|\xi|^4), \quad \omega(0) = \omega_0, l_1 \neq 0$
- **First Lyapunov coeffeicient**: $l_1 = \frac{1}{\omega_0}\Re(c_1(0))$



$l_1 < 0$ $\qquad$ $l_1 > 0$

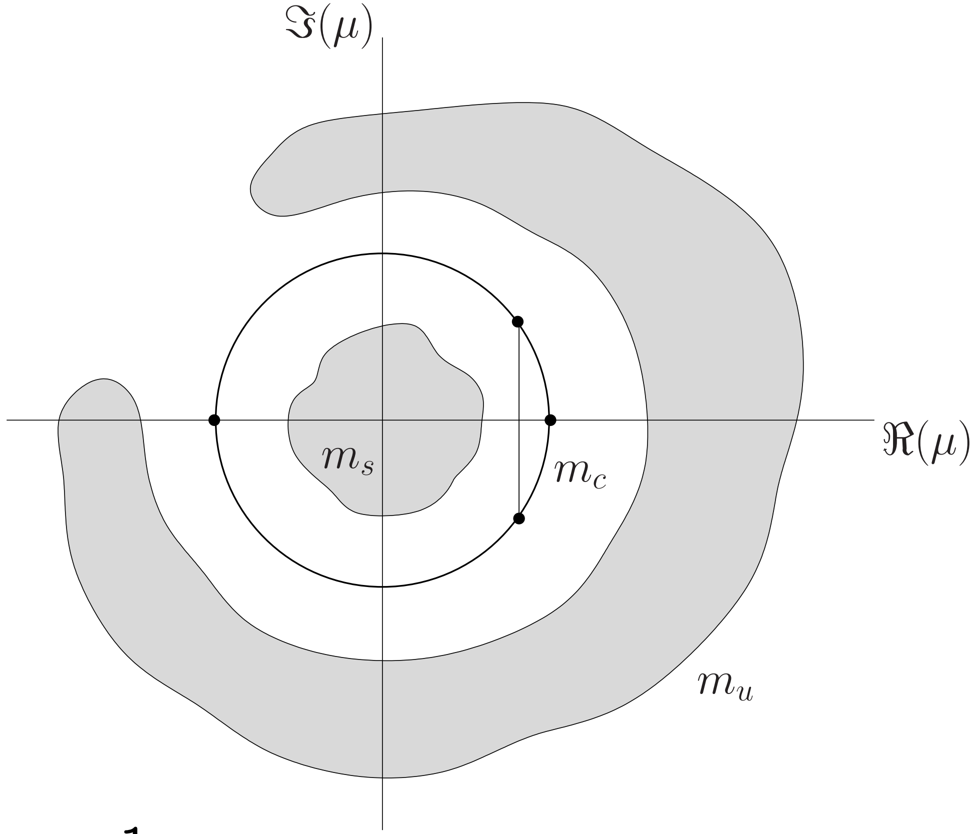
- Approximate cycle: $\begin{cases} \dot{\rho} = \rho(\beta + \Re(c_1)\rho^2), \\ \dot{\varphi} = \omega + \Im(c_1)\rho^2, \end{cases} \Rightarrow \rho_0 = \sqrt{-\frac{\beta}{\Re(c_1)}}$

Generic Hopf bifurcation: $\lambda_{1,2} = \pm i\omega_0$



Birth of a limit cycle.

Codimension 1 local bifurcations of cycles



$$\dim W^c_\alpha = m_c$$

Critical cases:

- **cyclic fold (LPC)**: $\mu_1 = 1$
- **period-doubling (PD)**: $\mu_1 = -1$
- **Neimark-Sacker (NS):** $\mu_{1,2} = e^{\pm i\theta_0},\ 0 < \theta_0 < \pi,\ \theta_0 \neq \frac{\pi}{2}, \frac{2\pi}{3}$

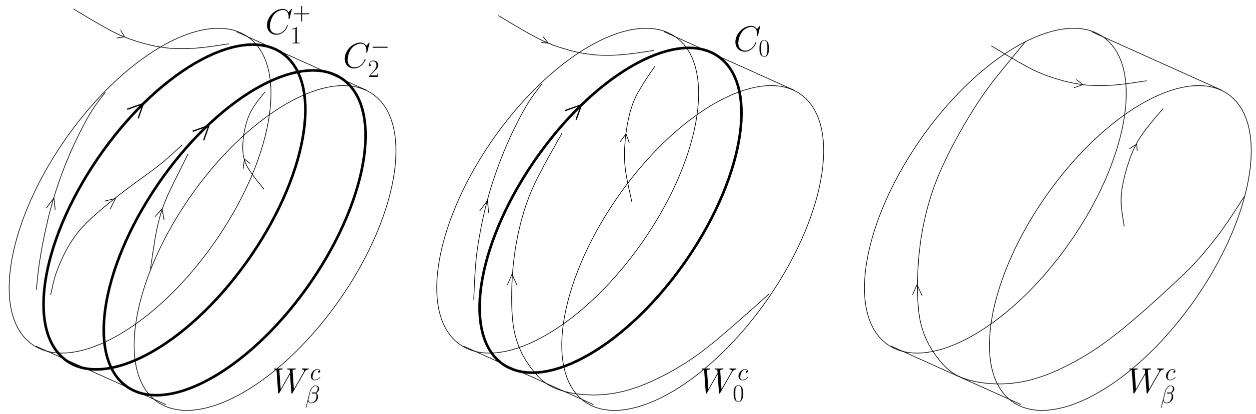
Generic LPC bifurcation

- Periodic parameter-dependent smooth normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\beta) - \xi + a(\beta)\xi^2 + \mathcal{O}(\xi^3), \\ \dfrac{d\xi}{dt} = \beta + b(\beta)\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

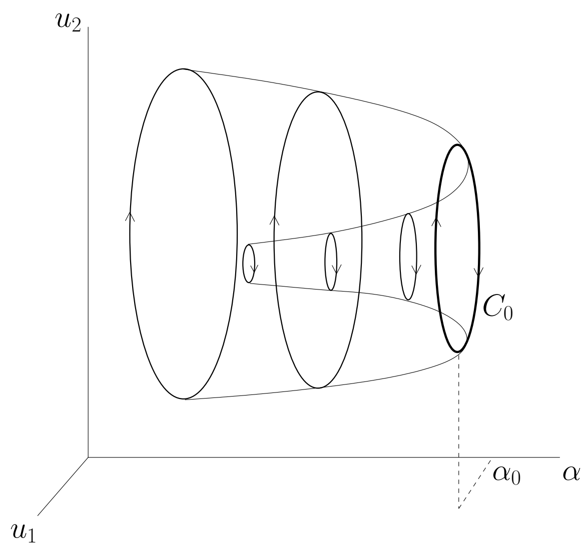
  where $a, b \in \mathbb{R}$ and the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

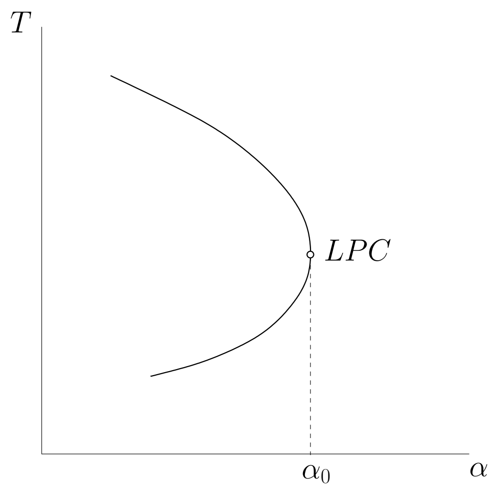
- Collision and disappearance of two limit cycles ($b(0) > 0$):



$\beta < 0$ $\beta = 0$ $\beta > 0$

Cycle manifold near LPC





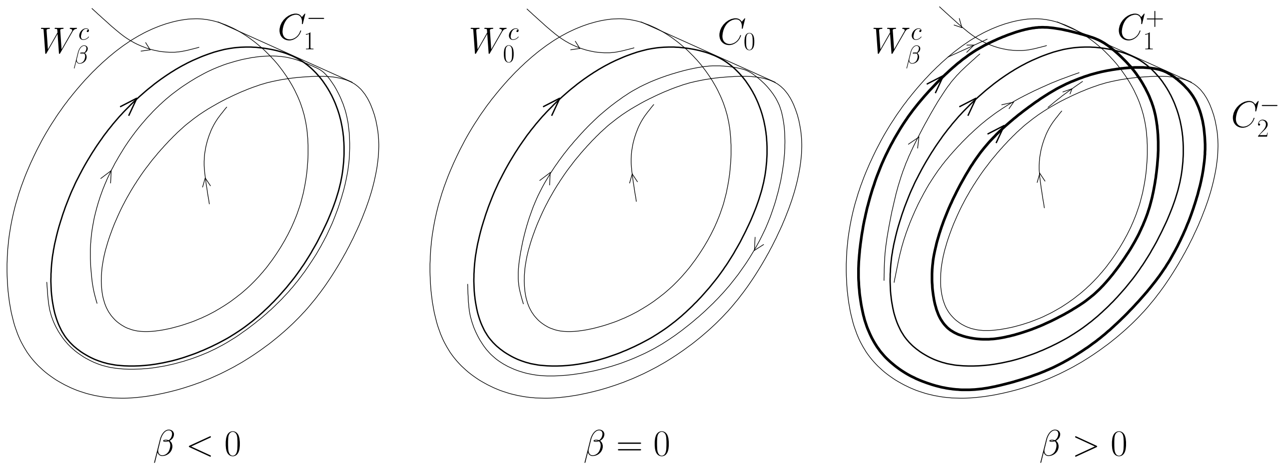
Generic PD bifurcation

- Periodic parameter-dependent smooth normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1+\nu(\beta)+a(\beta)\xi^2+\mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} &= \beta\xi+c(\beta)\xi^3+\mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$ and the $\mathcal{O}(\xi^3)$-terms are $2T_0$-periodic in τ.

- Period-doubling ($c(0) < 0$):



Generic NS bifurcation

- Periodic parameter-dependent smooth normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1 + \nu(\beta) + a(\beta)|\xi|^2 + \mathcal{O}(|\xi|^4), \\ \dfrac{d\xi}{dt} &= \left(\beta + \dfrac{i\theta(\beta)}{T(\beta)}\right)\xi + d(\beta)\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{cases}$$

where $a \in \mathbb{R}, d \in \mathbb{C}$ and the $\mathcal{O}(\|\xi\|^4)$-terms are T_0-periodic in τ

- Torus generation ($\Re(d(0)) < 0$):