Lecture 1: Location and analysis of equilibria

1.1 Multivariate Taylor expansions

Let $F : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto F(x)$, be a smooth map. Its **multivariate Taylor expansion** at $x_0 \in \mathbb{R}^n$ for $h \in \mathbb{R}^n$ with small $\|h\|$ can be written as

$$F(x_0 + h) = F(x_0) + A(x_0)h + \frac{1}{2}B(x_0; h, h) + O(\|h\|^3),$$

where $\|h\| = \sqrt{\langle h, h\rangle}$ with

$$\langle x, y\rangle := x^{\mathrm{T}} y, \quad x, y \in \mathbb{R}^n,$$

and $A(x_0) = DF(x_0)$, while $B(x_0; r, s) = D^2F(x_0)(r, s)$. In components:

$$\begin{aligned} (A(x_0)h)_i &= \sum_{j=1}^{n} \frac{\partial F_i(x_0)}{\partial x_j} h_j, \\ B_i(x_0; r, s) &= \sum_{j,k=1}^{n} \frac{\partial^2 F_i(x_0)}{\partial x_j \partial x_k} r_j s_k, \end{aligned}$$

for $r, s \in \mathbb{R}^n$ and $i = 1, 2, \ldots, n$. Given x_0, the mapping $h \mapsto A(x_0)h$ is linear, while $(r, s) \mapsto B(x_0; r, s)$ is a bilinear form.

Lemma 1 *Let $x, x_0 \in \mathbb{R}^n$ and $h := x - x_0$. Then for all x sufficiently close to x_0, i.e. $\|h\|$ small enough, hold*

$$F(x) = F(x_0) + \int_0^1 A(x_0 + th)h \, dt, \tag{1.1}$$

$$F(x) = F(x_0) + A(x_0)h + \int_0^1 (1 - t)B(x_0 + th; h, h) \, dt. \tag{1.2}$$

Proof:

Introduce $f(t) := F(x_0 + th)$. Then $\dot{f}(t) = A(x_0 + th)h$ and $\ddot{f}(t) = B(x_0 + th; h, h)$. This gives

$$\int_0^1 A(x_0 + th)h \; dt = \int_0^1 \dot{f}(t) \; dt = f(1) - f(0) = F(x_0 + h) - F(x_0),$$

from which (1.1) follows. Similarly,

$$\begin{aligned}\int_0^1 (1-t)B(x_0 + th; h, h) \; dt &= \int_0^1 (1-t)\ddot{f}(t)dt \\ &= \dot{f}(t)(1-t)\Big|_0^1 + \int_0^1 \dot{f}(t)dt \\ &= -\dot{f}(0) + f(1) - f(0) \\ &= -A(x_0)h + F(x_0 + h) - F(x_0),\end{aligned}$$

implying (1.2). □

Define

$$\|A\|(x) := \sup_{h \neq 0} \frac{\|A(x)h\|}{\|h\|} \quad \text{and} \quad \|B\|(x) := \sup_{h \neq 0} \frac{\|B(x; h, h)\|}{\|h\|^2}.$$

Then

$$\|A(x)h\| \leq \|A\|(x)\|h\| \quad \text{and} \quad \|B(x; h, h)\| \leq \|B\|(x)\|h\|^2$$

for all $x, h \in \mathbb{R}^n$.

1.2 Newton's method

Consider an "algebraic" equation

$$F(x) = 0, \tag{1.3}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map.

Theorem 1 *Let x_* be a solution to (1.3), i.e. $F(x_*) = 0$, and suppose that $A(x_*)$ is invertible. Then there exist $\varepsilon, C > 0$ such that for any x_0 with $\|x_0 - x_*\| \leq \varepsilon$ the* **Newton iterations**

$$x_{k+1} = x_k - A^{-1}(x_k)F(x_k), \quad k = 0, 1, 2, \ldots, \tag{1.4}$$

are well defined, satisfy

$$\begin{aligned}\|x_k - x_*\| &\leq \varepsilon, &(1.5)\\ \|x_{k+1} - x_*\| &\leq C\|x_k - x_*\|^2, &(1.6)\end{aligned}$$

for all $k \geq 0$, implying that the sequence $\{x_k\}$ converges to x_* quadratically.

Proof:

Provided $A^{-1}(x_k)$ exists, it follows from (1.4) that

$$x_{k+1} - x_* = A^{-1}(x_k)(A(x_k)(x_k - x_*) - F(x_k) + F(x_*)).$$

Using (1.2) and making a substitution in the integral, we see that

$$A(x_k)(x_k - x_*) - F(x_k) + F(x_*) = \int_0^1 tB(x_* + th; h, h)\ dt,$$

where $h = x_k - x_*$. Thus

$$x_{k+1} - x_* = A^{-1}(x_k) \int_0^1 tB(x_* + th; h, h)\ dt. \tag{1.7}$$

Since F is smooth and $A(x_*)$ is invertible, there exist $\delta, M > 0$ such that for all x with $\|x - x_*\| \le \delta$ the matrix $A(x)$ is also invertible and

$$\|A^{-1}(x)\| \le \frac{1}{M}.$$

Let

$$L = \sup_{\|x - x_*\| \le \delta} \|B\|(x).$$

From (1.7) it follows that

$$\|x_{k+1} - x_*\| \le \frac{L}{2M}\|h\|^2 = \frac{L}{2M}\|x_k - x_*\|^2 \tag{1.8}$$

for $\|x_k - x_*\| \le \delta$. Therefore, if $\{x_k\}$ is well defined and (1.5) holds for some $0 < \varepsilon < \delta$, then (1.6) also holds with

$$C = \frac{L}{2M}.$$

Now take ε satisfying

$$0 < \varepsilon < \min\left(\delta, \frac{2M}{L}\right).$$

Then, if x_k satisfies $\|x_k - x_*\| \le \varepsilon$, it is also true that $\|x_k - x_*\| \le \delta$, so that $A^{-1}(x_k)$ exists and x_{k+1} is well defined. Moreover, (1.8) gives

$$\|x_{k+1} - x_*\| \le \frac{L}{2M}\|x_k - x_*\|\|x_k - x_*\| < \frac{L}{2M}\frac{2M}{L}\varepsilon = \varepsilon.$$

We have $\varepsilon > 0$ such that for any x_0 with $\|x_0 - x_*\| \le \varepsilon$ holds $\|x_k - x_*\| \le \varepsilon$ for all $k \ge 1$ (by induction). Thus, both estimates (1.5) and (1.6) are established.

Introducing

$$\xi_k = \frac{L}{2M}\|x_k - x_*\|,$$

we get from (1.8) that $\xi_{k+1} \le \xi_k^2$ for $k \ge 0$, which implies

$$\xi_k \le \xi_0^{2^k}.$$

Here $\xi_0 < 1$, since

$$\frac{L}{2M}\|x_0 - x_*\| \le \frac{L}{2M}\varepsilon < \frac{L}{2M}\frac{2M}{L} = 1.$$

Thus $\xi_k \to 0$ as $k \to \infty$, meaning that $\{x_k\}$ converges to x_* quadratically. □

1.3 Approximation of 1D invariant manifolds

Let $x_* = 0$ be a hyperbolic equilibrium of a smooth ODE system

$$\dot{x} = Ax + \frac{1}{2}B(x,x) + O(\|x\|^3), \quad x \in \mathbb{R}^n. \tag{1.9}$$

Suppose that A has one simple eigenvalue $\lambda > 0$ and $(n-1)$ eigenvalues with $\Re(\lambda) < 0$. Then

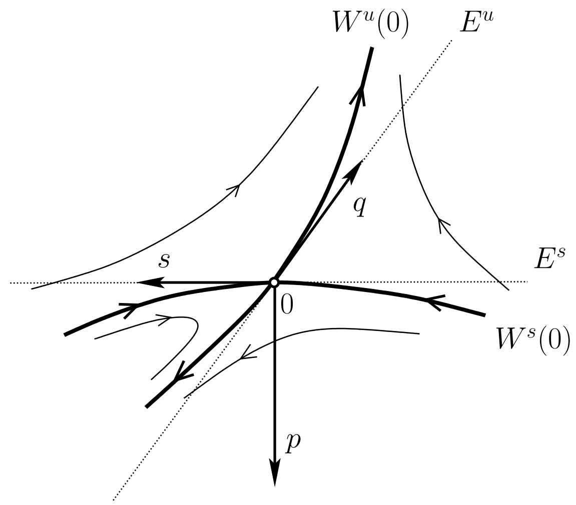

Figure 1.1: One-dimensional invariant manifolds of a saddle in the plane.

there exists a one-dimensional invaraint manifold $W^u(0)$ that is tangent at the equilibrium 0 to the one-dimensional unstable eigenspace E^u of A spanned by the eigenvector $q \in \mathbb{R}^n$:

$$Aq = \lambda q, \quad \langle q, q \rangle = 1.$$

The manifold $W^u(0)$ can be parametrized near the origin by

$$x = \xi q + \frac{1}{2}\xi^2 s + O(|\xi|^3), \tag{1.10}$$

with $\xi \in \mathbb{R}$ and some fixed vector $s \in \mathbb{R}^n$ satisfying

$$\langle p, s \rangle = 0, \tag{1.11}$$

where $p \in \mathbb{R}^n$ is the adjoint eigenvector:

$$A^{\mathrm{T}} p = \lambda p, \quad \langle p, q \rangle = 1.$$

The condition (1.11) means that s belongs the $(n-1)$-dimensional stable eigenspace E^s of A, so that the quadratic term in (1.10) does not contain any component in the q-direction.

The restriction of (1.9) to its $W^u(0)$ can be written as

$$\dot{\xi} = \lambda \xi + a\xi^2 + O(|\xi|^3), \tag{1.12}$$

where $a \in \mathbb{R}$. Notice that both a and s are unknown at this stage.

Using the invariancy of $W^u(0)$, we obtain

$$\begin{aligned}\dot{x} &= \dot{\xi}q + \xi\dot{\xi}s + \dots \\ &= (\lambda\xi + a\xi^2 + \dots)q + \xi(\lambda\xi + a\xi^2 + \dots)s + \dots \\ &= \lambda\xi q + \xi^2(aq + \lambda s) + \dots,\end{aligned}$$

as well as

$$\begin{aligned}\dot{x} &= A(\xi q + \frac{1}{2}\xi^2 s + \dots) + \frac{1}{2}B(\xi q + \dots, \xi q + \dots) + \dots \\ &= \xi Aq + \frac{1}{2}\xi^2 As + \frac{1}{2}\xi^2 B(q,q) + \dots \\ &= \lambda\xi q + \frac{1}{2}\xi^2(As + B(q,q)) + \dots,\end{aligned}$$

where "..." denote the $O(|\xi|^3)$-terms. Collecting the ξ^2 terms, we obtain the following nonsingular linear system

$$(A - 2\lambda I_n)s = 2aq - B(q,q). \tag{1.13}$$

Since

$$\begin{aligned}\langle p, (A - 2\lambda I_n)s\rangle &= \langle (A - 2\lambda I_n)^{\mathrm{T}}p, s\rangle = \langle A^{\mathrm{T}}p, s\rangle - 2\lambda\langle p, s\rangle \\ &= \lambda\langle p, s\rangle - 2\lambda\langle p, s\rangle = -\lambda\langle p, s\rangle \\ &= 0,\end{aligned}$$

due to (1.11), we must have $\langle p, 2aq - B(q,q)\rangle = 0$. This implies

$$a = \frac{1}{2}\langle p, B(q,q)\rangle$$

and (1.13) finally gives

$$s = (A - 2\lambda I_n)^{-1}(\langle p, B(q,q)\rangle q - B(q,q)).$$