Lecture 5: Location and continuation of limit cycles

5.1 Limit cycles of autonomous ODEs

Assume, the ODE system

$$\dot{u} = f(u), \quad u \in \mathbb{R}^n, \tag{5.1}$$

has an isolated periodic orbit (**limit cycle**). Let $u_0(t + T_0) = u_0(t)$ be the corresponding periodic solution with minimal period $T_0 > 0$. The **multipliers** of the cycle are the eigenvalues

$$\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{C}$$

of the $n \times n$ **monodromy matrix** $M(T_0)$, where $M(t)$ satisfies

$$\begin{cases} \dot{M}(t) &= f_u(u_0(t))M(t), \\ M(0) &= I_n. \end{cases}$$

There is always a trivial multiplier $\mu_n = 1$. If $|\mu| < 1$ for each multiplier except $\mu_n = 1$, the cycle is (orbitally) stable. If $|\mu| > 1$ for at least one multiplier, it is unstable.

Consider a **periodic boundary-value problem** on the unit interval:

$$\begin{cases} \dot{x}(t) - Tf(x(t)) &= 0, \quad t \in [0,1], \\ x(0) - x(1) &= 0. \end{cases}$$

Clearly, $x(t) = u_0(T_0 t + \sigma)$ is a solution to this BVP for $T = T_0$ and any phase shift σ.

Let $w(t)$ be a smooth period-1 function. To fix σ, impose the **integral phase condition**:

$$\Psi[x] = \int_0^1 \langle \dot{w}(\tau), x(\tau) \rangle d\tau = 0$$

Lemma 11 *The condition*

$$\int_0^1 \langle \dot{w}(\tau), x(\tau) \rangle d\tau = 0$$

is a necessary condition for the L_2**-distance**

$$\rho(\sigma) = \int_0^1 \| x(\tau + \sigma) - w(\tau) \|^2 d\tau$$

between 1*-periodic smooth functions* x *and* w *to achieve a local minimum with respect to possible shifts* σ *at* $\sigma = 0$*.*

Since $\|x\|^2 = \langle x, x \rangle$,

$$\begin{aligned} \frac{1}{2}\dot{\rho}(0) &= \left. \int_0^1 \langle x(\tau+\sigma) - w(\tau), \dot{x}(\tau+\sigma) \rangle d\tau \right|_{\sigma=0} \\ &= \int_0^1 \langle x(\tau) - w(\tau), \dot{x}(\tau) \rangle d\tau \\ &= \int_0^1 \langle x(\tau), \dot{x}(\tau) \rangle d\tau - \int_0^1 \langle w(\tau), \dot{x}(\tau) \rangle d\tau \\ &= \frac{1}{2} \int_0^1 d\|x(\tau)\|^2 - \int_0^1 \langle w(\tau), \dot{x}(\tau) \rangle d\tau \\ &= \int_0^1 \langle \dot{w}(\tau), x(\tau) \rangle d\tau \ . \end{aligned}$$

□

5.2 BVP for periodic solutions

A periodic solution to (5.1) can thus be computed by solving the BVP

$$\begin{cases} \dot{x}(t) - Tf(x(t)) &=& 0, \quad t \in [0,1], \\ x(0) - x(1) &=& 0, \\ \int_0^1 \langle \dot{w}(\tau), x(\tau) \rangle \, d\tau &=& 0, \end{cases} \tag{5.2}$$

where $w \in \mathcal{C}^1([0,1], \mathbb{R}^n)$ is a reference period-1 function. If $(x_0(\cdot), T_0) \in \mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R}$ satisfies this BVP, then

$$u(t) = x_0\left(\frac{t}{T_0}\right)$$

gives the T_0-periodic solution of (5.1) with $u(0) = x_0(0)$.

For the monodromy matrix of the corresponding cycle we have $M(T) = \Phi(1)$, where

$$\dot{\Phi}(t) - Tf_x(x(t))\Phi(t) = 0, \quad \Phi(0) = I_n.$$

The eigenvalues of $\Psi(1)$ coincide with the above introduced multipliers of the cycle.

Introduce also the **adjoint monodromy matrix** $\Psi(1)$ using the solution of

$$\dot{\Psi}(t) + Tf_x^{\mathrm{T}}(x(t))\Psi(t) = 0, \quad \Psi(0) = I_n.$$

One has

$$\Psi(t) = [\Phi^{-1}(t)]^{\mathrm{T}}.$$

Any solution v of an inhomogeneous linear system

$$\dot{v} - Tf_x(x(t))v = b(t),$$

where $b \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^n)$, can be written as

$$v(t) = \Phi(t)\left[v(0) + \int_0^t \Phi^{-1}(\tau)b(\tau)d\tau\right] = \Phi(t)\left[v(0) + \int_0^t \Psi^{\mathrm{T}}(\tau)b(\tau)d\tau\right].$$

A cycle is called **simple** if $\mu_n = 1$ has algebraic multiplicity 1. Let $q_0, p_0 \in \mathbb{R}^n$ denote the left and right eigenvectors of the monodromy matrix corresponding to the trivial multiplier,

$$\begin{array}{rcccl} (\Phi(1) - I_n)q_0 &=& (\Psi(1) - I_n)p_0 &=& 0, \\ (\Phi(1) - I_n)^{\mathrm{T}}p_0 &=& (\Psi(1) - I_n)^{\mathrm{T}}q_0 &=& 0, \end{array}$$

such that $p_0^{\mathrm{T}}p_0 = q_0^{\mathrm{T}}q_0 = 1$. One can take $q_0 = c_0 f(x(0))$ with $c_0 \in \mathbb{R}$, $c_0 \neq 0$.

5.3 Regularity of the defining system

One Newton iterate for problem (5.2) is

$$(x, T) \mapsto (x + v, T + S),$$

where $(v(\cdot), S) \in \mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R}$ is the solution of the linearized inhomogeneous BVP

$$\left\{ \begin{array}{rcl} \dot{v}(t) - Tf_x(x(t))v - Sf(x(t)) & = & -\dot{x}(t) + Tf(x(t)), \quad t \in [0,1], \\ v(0) - v(1) & = & -x(1) + x(0), \\ \int_0^1 \langle \dot{w}(\tau), v(\tau) \rangle \, d\tau & = & -\int_0^1 \langle \dot{w}(\tau), x(\tau) \rangle \, d\tau. \end{array} \right. \tag{5.3}$$

The LHS of (5.3) can be re-written in the matrix form

$$\left[\begin{array}{cc} D - Tf_x(x) & -f(x) \\ \delta_0 - \delta_1 & 0 \\ \mathrm{Int}_{\dot{w}} & 0 \end{array} \right] \left(\begin{array}{c} v \\ S \end{array} \right),$$

where D is the differentiation operator, δ_a is the evaluation at $t = a$, i.e. $\delta_a v = v(a)$, and

$$\mathrm{Int}_{\dot{w}}\ v = \int_0^1 \langle \dot{w}(\tau), v(\tau) \rangle \, d\tau.$$

First we assume that $w = x$ so that $\dot{w} = \dot{x} = Tf(x)$ on solutions of (5.2). The next lemma is also valid for all w close to such x, i.e. when w is some reference period-1 function, e.g. the solution at the previous continuation step. We can also replace $Tf(x)$ in the integral operator by $f(x)$ without affecting essential properties of the operator.

Theorem 7 *If $(x(\cdot), T)$ corresponds to a simple cycle then the operator*

$$L = \left[\begin{array}{cc} D - Tf_x(x) & -f(x) \\ \delta_1 - \delta_0 & 0 \\ \mathrm{Int}_{f(x)} & 0 \end{array} \right]$$

from $\mathcal{C}^1([0,1], \mathbb{R}^n) \times \mathbb{R}$ into $\mathcal{C}^0([0,1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$ is one-to-one and onto.

Proof:

(1) Consider the homogeneous system

$$\left[\begin{array}{cc} D - Tf_x(x) & -f(x) \\ \delta_1 - \delta_0 & 0 \\ \mathrm{Int}_{f(x)} & 0 \end{array} \right] \left(\begin{array}{c} v \\ S \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right). \tag{5.4}$$

From the first row of (5.4) we have

$$\dot{v} - Tf_x(x(t))v = Sf(x(t))$$

implying

$$\begin{aligned} v(t) &= \Phi(t)\left[v(0) + S\int_0^t \Psi^{\mathrm{T}}(\tau)f(x(\tau))\,d\tau\right] \\ &= \Phi(t)\left[v(0) + \frac{S}{T}\int_0^t \Psi^{\mathrm{T}}(\tau)\dot{x}(\tau)d\tau\right] \\ &= \Phi(t)\left[v(0) + \frac{S}{T}\int_0^t \Psi^{\mathrm{T}}(\tau)\Phi(\tau)d\tau\ \dot{x}(0)\right] \\ &= \Phi(t)\left[v(0) + \frac{St}{T}\dot{x}(0)\right], \end{aligned}$$

since $\Psi^{\mathrm{T}}(\tau)\Phi(\tau) = I_n$ and $\dot{x}(t) = \Phi(t)\dot{x}(0)$.

From the second row of (5.4) we now have

$$0 = v(1) - v(0) = (\Phi(1) - I)v(0) + \frac{S}{T}\dot{x}(0)$$

or

$$(\Phi(1) - I)v(0) = -\frac{S}{T}\dot{x}(0).$$

Because $\dot{x}(0) = c_0 q_0$ for some $c_0 \in \mathbb{R}$, $c_0 \neq 0$, we must solve

$$(\Phi(1) - I)v(0) = -c_0\frac{S}{T}q_0, \tag{5.5}$$

where q_0 spans the kernel of $\Phi(1) - I$.

Since the trivial multiplier 1 has algebraic multiplicity 1, we must have $S = 0$, $v(0) = c_1 q_0$, and hence $v(t) = c_1\Phi(t)q_0$.

From the third row of (5.4) it follows that

$$0 = \int_0^1 f^{\mathrm{T}}(x(\tau))v(\tau)\,d\tau = \frac{1}{T}\int_0^1 \dot{x}^{\mathrm{T}}(\tau)v(\tau)\,d\tau = \frac{1}{T}\int_0^1 [\Phi(\tau)\dot{x}(0)]^{\mathrm{T}} c_1\Phi(\tau)q_0\,d\tau$$

or

$$c_0 c_1\ q_0^{\mathrm{T}}\left(\int_0^1 \Phi^{\mathrm{T}}(\tau)\Phi(\tau)\,d\tau\right)q_0 = 0,$$

from which it follows that $c_1 = 0$. Thus $v(t) \equiv 0$, so that the operator L is one-to-one.

(2) Consider a vector

$$\begin{pmatrix} \xi \\ \eta \\ \omega \end{pmatrix} \in \mathcal{C}^0([0,1],\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}.$$

It is in the range of L if and only if there exists

$$\begin{pmatrix} v \\ S \end{pmatrix} \in \mathcal{C}^1([0,1],\mathbb{R}^n) \times \mathbb{R}$$

such that

$$\begin{bmatrix} D - Tf_x(x) & -f(x) \\ \delta_1 - \delta_0 & 0 \\ \mathrm{Int}_{f(x)} & 0 \end{bmatrix} \begin{pmatrix} v \\ S \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \\ \omega \end{pmatrix}. \qquad (5.6)$$

The first row implies that

$$v(t) = \Phi(t)\left[v(0) + \int_0^t \Psi^{\mathrm{T}}(\tau)(\xi(\tau) + Sf(x(\tau)))d\tau \right].$$

The second row of (5.6) then implies

$$\begin{aligned} \eta &= v(1) - v(0) \\ &= (\Phi(1) - I)v(0) + \Phi(1)\int_0^1 \Psi^{\mathrm{T}}(\tau)(\xi(\tau) + Sf(x(\tau)))d\tau \\ &= (\Phi(1) - I)v(0) + \Phi(1)\int_0^1 \Psi^{\mathrm{T}}(\tau)\xi(\tau)d\tau + \frac{Sc_0}{T}q_0, \end{aligned}$$

since

$$\int_0^1 \Psi^{\mathrm{T}}(\tau)f(x(\tau))d\tau = \frac{1}{T}\int_0^1 \Psi^{\mathrm{T}}(\tau)\dot{x}(\tau)d\tau = \frac{1}{T}\int_0^1 \Psi^{\mathrm{T}}(\tau)c_0\Phi(\tau)q_0 d\tau = \frac{c_0}{T}q_0.$$

Thus

$$\eta = (\Phi(1) - I)v(0) + \frac{Sc_0}{T}q_0 + \Phi(1)\int_0^1 \Psi^{\mathrm{T}}(\tau)\xi(\tau)\, d\tau. \qquad (5.7)$$

Since 1 is an algebraically simple eigenvalue of $\Phi(1)$, q_0 is not in the range of $(\Phi(1) - I)$. For given ξ and η, equation (5.7) can be solved for $v(0)$ and S. Moreover, the solution is unique up to the addition of a scalar multiple of q_0 to $v(0)$. Since

$$\begin{aligned} \int_0^1 (\dot{x}(\tau))^{\mathrm{T}}\Phi(\tau)q_0\, d\tau &= c_0\int_0^1 (\Phi(\tau)q_0)^{\mathrm{T}}\Phi(\tau)q_0\, d\tau \\ &= c_0 q_0^{\mathrm{T}}\left(\int_0^1 \Phi^{\mathrm{T}}(\tau)\Phi(\tau)d\tau\right) q_0 \\ &\neq 0, \end{aligned}$$

the scalar is determined uniquely by the third row of system (5.6). It follows that the operator L is onto. □

The established regularity of the linearization of the defining BVP at the periodic solution ensures that this solution can be found by Newton's iterations.

To continue a **limit cycle branch** in

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}. \tag{5.8}$$

w.r.t. parameter $\alpha \in \mathbb{R}$, the following BVP can be used:

$$\left\{ \begin{array}{rcl} \dot{x}(\tau) - Tf(x(\tau), \alpha) & = & 0, \ \tau \in [0,1], \\ x(0) - x(1) & = & 0, \\ \int_0^1 \langle x(\tau), \dot{x}_0(\tau) \rangle \, d\tau & = & 0. \end{array} \right. \tag{5.9}$$

Theorem 7 together with an appropriate implicit function arguments implies that a simple cycle has a locally unique continuation w.r.t. α. Moreover the derivative operator of (5.9) with respect to (x, T, α):

$$\begin{bmatrix} D - Tf_x(x, \alpha) & -f(x, \alpha) & -Tf_\alpha(x, \alpha) \\ \delta_0 - \delta_1 & 0 & 0 \\ \mathrm{Int}_{\dot{x}_0} & 0 & 0 \end{bmatrix} \tag{5.10}$$

has the one-dimensional null-space at a simple cycle.

5.4 BVCPs and their discretization

A **Boundary Value Continuation Problem** (**BVCP**) consists of finding a **branch** of solutions $(u(\cdot), \beta)$ to the boundary-value problem with integral constraints

$$\left\{ \begin{array}{rcl} \dot{u}(\tau) - H(u(\tau), \beta) & = & 0, \quad \tau \in [0,1], \\ B(u(0), u(1), \beta) & = & 0, \\ \displaystyle\int_0^1 C(u(\tau), \beta) \, d\tau & = & 0, \end{array} \right. \tag{5.11}$$

starting from a given solution $(u_0(\cdot), \beta_0)$. Here $u \in \mathbb{R}^{n_u}, \beta \in \mathbb{R}^{n_\beta}$ and

$$\begin{aligned} H &: \mathbb{R}^{n_u} \times \mathbb{R}^{n_\beta} \to \mathbb{R}^{n_u}, \\ B &: \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_\beta} \to \mathbb{R}^{n_b}, \\ C &: \mathbb{R}^{n_u} \times \mathbb{R}^{n_\beta} \to \mathbb{R}^{n_c} \end{aligned}$$

are smooth functions.

BVCP (5.11) is (formally) **well posed** if

$$n_\beta = n_b + n_c - n_u + 1. \tag{5.12}$$

The most widely used discretization of BVCP (5.11) is based on **orthogonal collocation**. Introduce the primary **mesh points**

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = 1$$

and the **basis points**

$$\tau_{i,j} = \tau_i + \frac{j}{m}(\tau_{i+1} - \tau_i),$$

where $i = 0, 1, \ldots, N-1,\ j = 0, 1, \ldots, m$. Then approximate the solution u by

$$u^{(i)}(\tau) = \sum_{j=0}^{m} u^{i,j} l_{i,j}(\tau), \quad \tau \in [\tau_i, \tau_{i+1}],$$

where $l_{i,j}(\tau)$ are the **Lagrange basis polynomials**

$$l_{i,j}(\tau) = \prod_{k=0,k\neq j}^{m} \frac{\tau - \tau_{i,k}}{\tau_{i,j} - \tau_{i,k}}$$

and $u^{i,m} = u^{i+1,0}$.

Finally, apply the **orthogonal collocation**, i.e. require that

$$F : \left\{ \begin{array}{r} \left(\sum_{j=0}^{m} u^{i,j} l'_{i,j}(\zeta_{i,k})\right) - H(\sum_{j=0}^{m} u^{i,j} l_{i,j}(\zeta_{i,k}), \beta) = 0, \\ B(u^{0,0}, u^{N-1,m}, \beta) = 0, \\ \sum_{i=0}^{N-1} \sum_{j=0}^{m} \omega_{i,j} C(u^{i,j}, \beta) = 0, \end{array} \right. \tag{5.13}$$

where $\zeta_{i,k},\ k = 1, 2, \ldots, m$, are the **Gauss points** (roots of the Legendre polynomials relative to the interval $[\tau_i, \tau_{i+1}]$), and $\omega_{i,j}$ are the **Lagrange quadrature coefficients**.

If $h = \max_{i=1,2,\ldots,N} |\tau_i - \tau_{i-1}|$ then the **approximation error**, i.e. the deviation from the exact solution,

- in the basis points: $\|u(\tau_{i,j}) - u^{i,j}\| = O(h^m)$;
- in the mesh points: $\|u(\tau_i) - u^{i,0}\| = O(h^{2m})$.

Moreover, the deviation from the exact parameter values is also $O(h^{2m})$ (super-convergence).

5.5 Discretized BVCP for limit cycles

The problem (5.9) is a particular instance of a BVPC with

$$\begin{cases} u &= x, \\ \beta &= (T, \alpha), \end{cases}$$

and

$$H(u,\beta) = Tf(x,\alpha),\ B(u(0),u(1),\beta) = u(0)-u(1),\ C(u,\beta) = \langle x, \dot{x}_0 \rangle,$$

so that $n_u = n_b = n, n_c = 1, n_\beta = 2$, and (5.12) holds. The corresponding to (5.9) discretized via the orthogonal collocation system (5.13) is a *huge* ALCP:

$$F(X) = 0, \quad X = (\{x^{j,k}\}, T, \alpha) \in \mathbb{R}^{mnN+n+2}$$

where $j = 0, 1, \ldots, N-1,\ k = 0, 1, \ldots, m$. Its Jacobian matrix F_X

$$\left(\begin{array}{cccccccccccccccc}
x_1^{0,0} & & x_1^{0,1} & & x_1^{1,0} & & x_1^{1,1} & & x_1^{2,0} & & x_1^{2,1} & & x_1^{3,0} & & T_1 & \alpha_1 \\
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\end{array}\right)$$

is sparse, corresponds to the linear operator (5.10), and has the one-dimensional null-space at generic points satisfying $F(X) = 0$.

Suppose that Newton-like corrections of the continuation algorithm converged to a point

$$X_0 = (\{x_0^{j,k}\}, T_0, \alpha_0)$$

on the cycle branch, which therefore approximates a solution to (5.9). The matrix $F_X(X_0)$ can be transformed by Gauss elimination to the form

$$
\left(\begin{array}{cccccccccccccccc}
x_1^{0,0} & & x_1^{0,1} & & x_1^{1,0} & & x_1^{1,1} & & x_1^{2,0} & & x_1^{2,1} & & x_1^{3,0} & & T_1 & \alpha_1 \\
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\end{array}\right)
$$

where $\circ$'s denote eliminated entries. Let P_0 be the matrix block marked by $\ast$'s and P_1 the matrix block marked by $\star$'s. Applying this matrix to the vector

$$(\{v^{j,k}\}, 0, 0)$$

and using $v^{0,0} = v(0), v^{N,0} = v(1)$, we see that

$$P_0 v(0) + P_1 v(1) = 0 \;\Rightarrow\; M(T_0) \approx -P_1^{-1} P_0.$$

Thus, the eigenvalues of $(-P_1^{-1} P_0)$ approximate the multipliers of the cycle. Note that the matrices above correspond to the case $N = 3, m = n = 2$.