TUTORIAL M-III:
Computation of invariant manifolds and connecting orbits of saddle fixed points in MatContM

May 11, 2018

This tutorial is devoted to the numerical construction of 1D stable and unstable invariant manifolds of saddle fixed points of (orientation-preserving) maps

$$u \mapsto f(u, \alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}^m,$$

at fixed parameter values, and to the continuation of their intersections (i.e. orbits connecting saddles) with respect to one or two parameters[1].

Notice that in this case a "point" on a continuation curve represents an entire orbit. We must first obtain an initial connecting orbit. Such an orbit lies in the intersection of the unstable manifold of a fixed point u_1 and the stable manifold of a fixed point u_2. We note that there can be multiple connecting orbits between the same saddle points. In the case of a homoclinic orbit, u_1 and u_2 are the same point u_0. MatContM can compute 1D stable and unstable manifolds and in the case of 2D maps find intersection points between these manifolds. The intersection points then lie on connecting orbits.

The user can also provide MatContM with an orbit that has been obtained through other means, if the orbit is stored as a matrix in the MATLAB workspace.

As an example, we will use a map called the *generalized Hénon map* (GHM):

$$\text{GeneralizedHenon} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \alpha - \beta x - y^2 + Rxy + Sy^3 \end{pmatrix} \tag{1}$$

This system may be already implemented under the name `GeneralizedHenon`. If not, you can re-enter it in MatContM as in Figure 1.
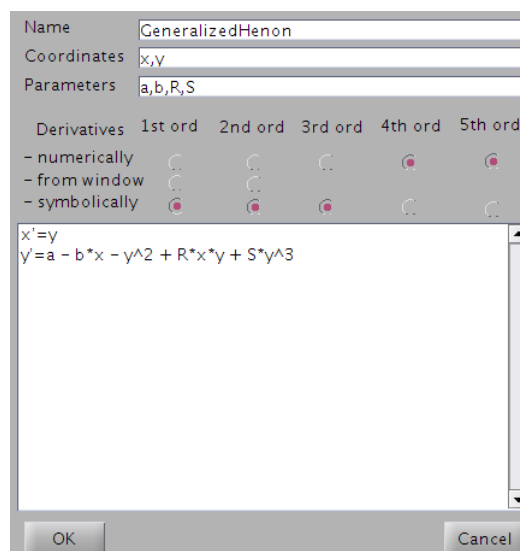

Figure 1: Using the systembrowser, add the following system into MatContM.

1 Computation of 1D invariant manifolds of a saddle

Once the system is loaded we can start by computing a connecting orbit for continuation. We do this by changing the type of the initial point. We select in the main menu: **Type|Initial Point|Connecting Orbit** (Figure 2). Once the type of the initial point has been set to an orbit, you will be able to select or compute a connecting orbit. You can choose between a homoclinic orbit (`HO-curve`) and a heteroclinic orbit (`HE-curve`) (See Figure 3). Make sure the homoclinic orbit option is selected.

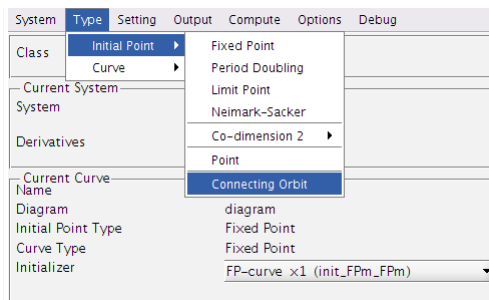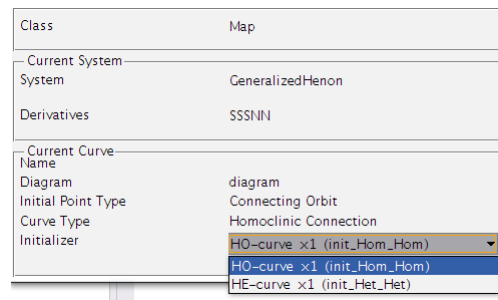Figure 2: Set the type of the initial point to "Connecting Orbit"

Figure 3: Select the homoclinic connection continuation (HO-Curve)

We select the Starter window. The "Initial Point" subpanel of the Starter window contains either zeros or a point that was selected on a curve of fixed points. This information is only used as a dummy fixed point for computing manifolds and plays no further role in the continuation of connecting orbits. Notice that a special panel has appeared on the Starter window when changing the type of the initial point to connecting orbits. This panel will allow to compute manifolds, find intersections and select connecting orbits.

Before we start computing manifolds, we first have to enter the values of the parameters in the Starter window. Set $a = -0.4$, $b = 1.03$, $R = -0.1$ and $S = 0$ (Figure 4).
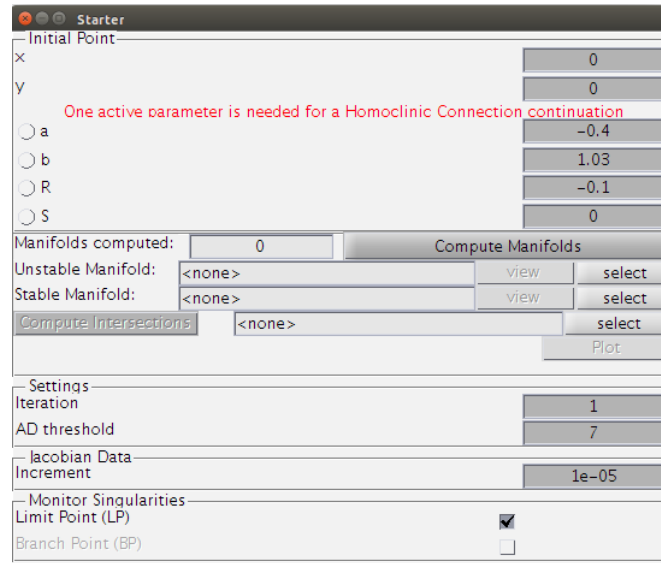
Figure 4: The Starter window after entering the parameter values.

Start computation of a stable and an unstable manifold by pressing the **Compute Manifolds** button in the Starter window. A Compute Manifolds window for computing manifolds will appear (Figure 5).

First we need to enter a fixed point. Enter the point $(-1.621146385, -1.621146385)$. Proceed by pressing the **Select fixed point and proceed to configure** button. We can now configure the algorithm used to compute the manifold. The most important setting defines whether we want to compute a stable or unstable manifold. We will first compute the unstable manifold: next to `function`, we select `Unstable Manifold`. We will adjust some other settings as well:

- set `nmax` to `4000` (maximum number of points in the manifold)

- set `distanceInit` to `-1e-08` (initial displacement from the fixed point)

- set `deltaMax` to `0.01`

It is important to set `distanceInit` *negative*, since it determines which of two branches of the manifold will be computed.
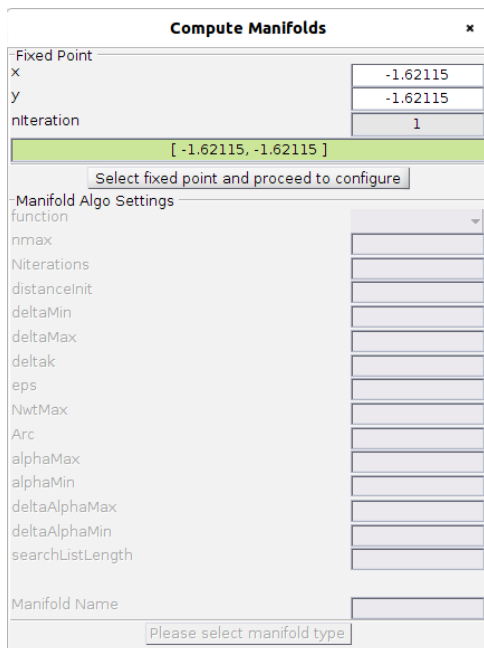
4

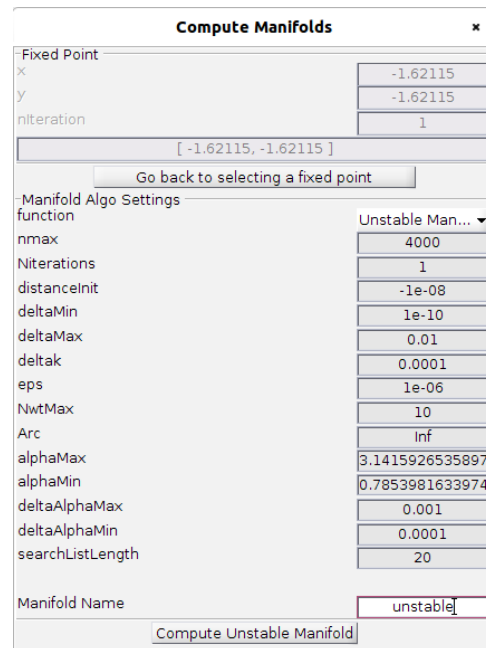Figure 5: Manifold computing window after entering a fixed point

Figure 6: Manifold computing window configured for the computation of an unstable manifold

The Compute Manifolds window should now look like Figure 6. You can give the manifold a name, e.g. unstable (Figure 6, see Manifold Name). If no name is given, a default name will be chosen.

Press **Compute Unstable Manifold**. A window will appear that displays the output of the algorithm, close that window when the computing stops and go back to the Compute Manifolds window (Figure 6).

We will now compute the stable manifold, go to function and select Stable Manifold. We give this manifold the name stable. Press **Compute Stable Manifold**.

We have computed an unstable and a stable manifold of the same fixed point at given parameter values. Close the Compute Manifolds window and return to the Starter window.

We have computed two manifolds, a stable and an unstable manifold. In the Starter window, go to the Unstable Manifold section and press the **select** button. A browser window pops up and lists the available unstable manifolds, we only have one available[2]. We select the manifold by double-clicking on the name in the list. Repeat the same process for Stable Manifold in the Starter window.

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[2]Here you can plot the computed manifold by pressing the corresponding **Plot xxx** button.

2 Loading the computed manifolds into the workspace

Often the computed manifolds are used only to find connections and will not be permanently stored. However, it is possible to load them in the MATLAB workspace to keep them for further usage. The following MATLAB session demonstrates how this can be done[3].

```
>> manstable=manifolds.stable

manstable =

  Manifold with properties:

    points: [24000 double]
    arclen: 25.5306
      optM: [11 struct]
    man_ds: [11 struct]
      name: 'stable'

>> manunstable=manifolds.unstable

manunstable =

  Manifold with properties:

    points: [24000 double]
    arclen: 25.4053
      optM: [11 struct]
    man_ds: [11 struct]
      name: 'unstable'

>> manall=manifolds.all

manall =

  12 struct array with fields:

    name
    manifold

>> manall.name
```

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[3]You can skip this section, if not interested in such low-level functionality.

```
ans =

    'unstable'


ans =

    'stable'

>> manall.manifold

ans =

  Manifold with properties:

    points: [24000 double]
    arclen: 25.4053
      optM: [11 struct]
    man_ds: [11 struct]
      name: 'unstable'


ans =

  Manifold with properties:

    points: [24000 double]
    arclen: 25.5306
      optM: [11 struct]
    man_ds: [11 struct]
      name: 'stable'
```

At the end of this session the stable manifold is contained in the $[2 \times 4000]$ array
`manstable.points` and the unstable manifold is contained in the $[2 \times 4000]$ array `manunstable.points`.

3   Finding intersections of stable and unstable manifolds

Once a stable and an unstable manifold are selected, the **Compute Intersections** button becomes available. We produce a list of intersections between the two manifolds by pressing this button. The end of the computations is announced by the message `Intersections computed between unstable and stable manifolds`. Afterwards, press

the **select** button above the **Plot** button to select such an intersection as a candidate connecting orbit for continuation.
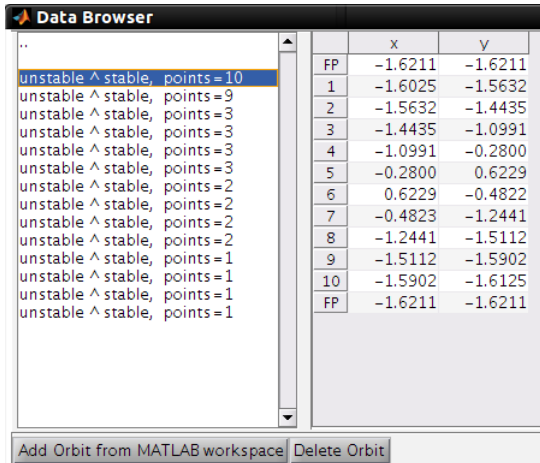

Figure 7: The intersection between the unstable and stable manifolds has produced multiple candidates for a connecting orbit. Select the connecting orbit by double-clicking.
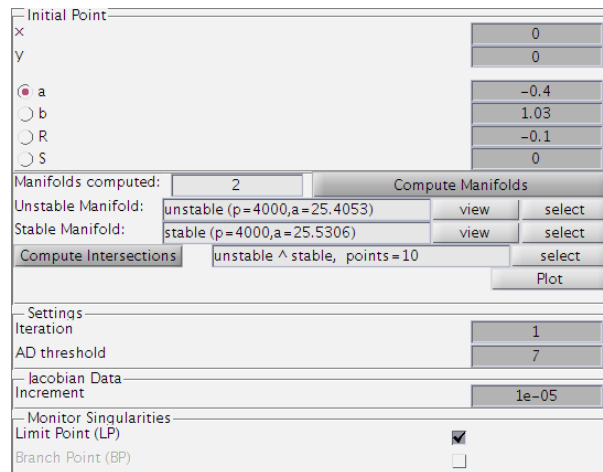

Figure 8: The Starter window after configuring a homoclinic continuation.

A browser window pops up that lists the available connecting orbits produced by the intersection (Figure 7). Select the largest intersection (with 10 points) by double-clicking.

The unstable manifold, stable manifold and connecting orbit can be plotted by pressing the **Plot** button (Figure 9). The MatContM menu in the right upper corner can be used to fine tune the produced image, while the MATLAB Print or Edit menus allow to print or edit it (Figure 10).
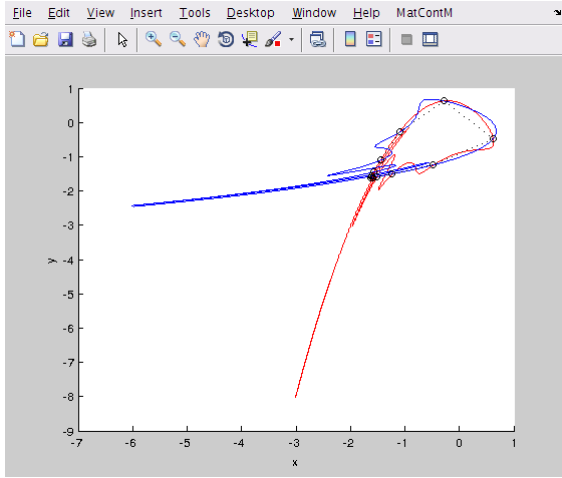
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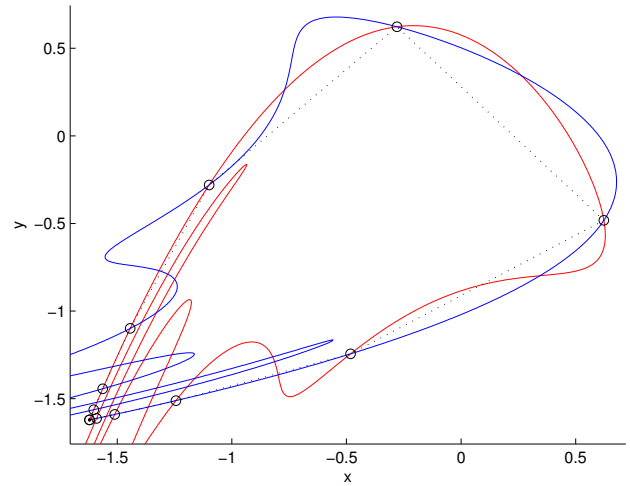Figure 9: The plot button produces an image of the stable and unstable manifold and the connecting orbit.

Figure 10: The plot can be manipulated by the tools in the `MatContM` menu and afterwards exported to an image file.

4 Continuation of homoclinic orbits in one parameter

Once the connecting orbit has been selected, we are ready for continuation. Select a as the free parameter. The `Starter` window should now look like the window depicted in Figure 8. The values of x and y do not play a role in the continuation of a connecting orbit. The "initial point" is now the selected homoclinic orbit.

We start the continuation by going to the main window menu and selecting: **Compute|Forward**. We find one *limit point* on the curve (Figure 11), where the computations should be resumed. When finished, press **View Curve** to open the data browser and then press **View CurveData** to view the numerical output of the continuation algorithm.
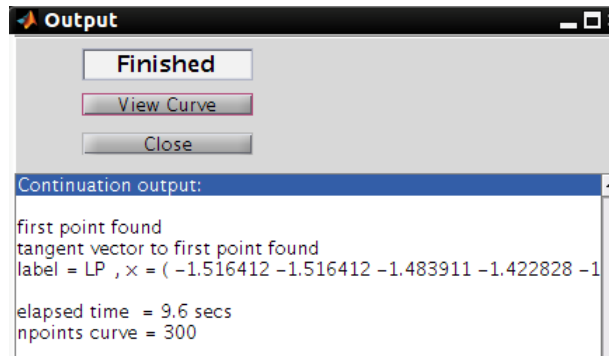
Figure 11: Output of the homoclinic connection continuation.

Notice how each point of the continuation curve represents an orbit. The last rows of the x matrix are reserved for parameters. Now the last row contains values of the free parameter a.

5 Continuation of homoclinic tangencies in two parameters

Go back to Data Browser and select the limit point that was detected in Figure 11. We are now able to perform a homoclinic tangency continuation starting from the detected limit point.

We need to select two free parameters in the Starter window, parameters a and b. The homoclinic connecting orbit associated with the limit point is automatically selected and is named LP_HO-points (Figure 13). When in the Starter window after selecting the LP_HO point (Figure 13), we can compute the stable and unstable manifold by pressing the **Compute Manifolds** button. We can use the same settings as in Figure 6, but use other names for the manifolds. The default value for the fixed point for computing the manifolds is the fixed point associated with LP_HO-points. Compute the stable and unstable manifold and select them in the Starter window, after deleting the previously computed manifolds stable and unstable manifolds. Press the **Plot** button to generate the figure. We do not need to compute an intersection because the orbit is already provided by LP_HO-points The corresponding connecting orbit is represented in Figure 12 by black circles and black dashed lines. The stable and unstable manifold of the saddle fixed point are also visualized.
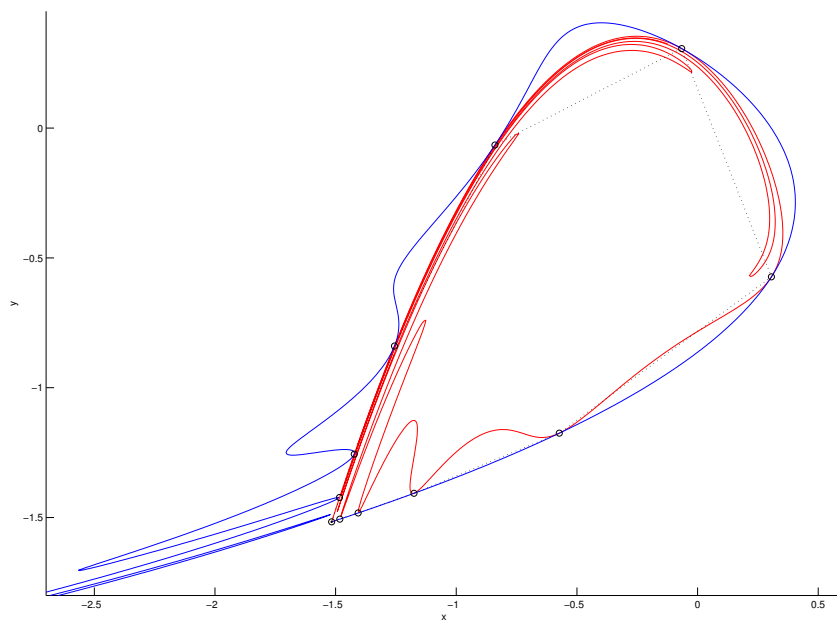
Figure 12: Homoclinic tangency.

Figure 13: **Starter** window of a
homoclinic tangency continuation.

We advise to set `MaxNumPoints` to 80 in the `Continuer` window to reduce the computation time. We are ready to start the homoclinic tangency continuation by selecting **Compute|Forward**. Next, select **Compute|Backward**.

Open a `Plot2D` window to visualize the computations in the parameter plane. Use the configuration seen in Figure 14. Select **MatContM|Redraw diagram** in the plot window. The result is shown in Figure 15. Notice that the tangency curve (red) forms a wedge. This wedge actually consists of two tangency curves (upper and lower) meeting at a sharp point at the left side. The continuation process accidentally hopped onto the other tangency curve when it encountered the sharp point on the left. This does not always happen, especially when stricter continuation thresholds are selected.

Figure 14

Figure 15

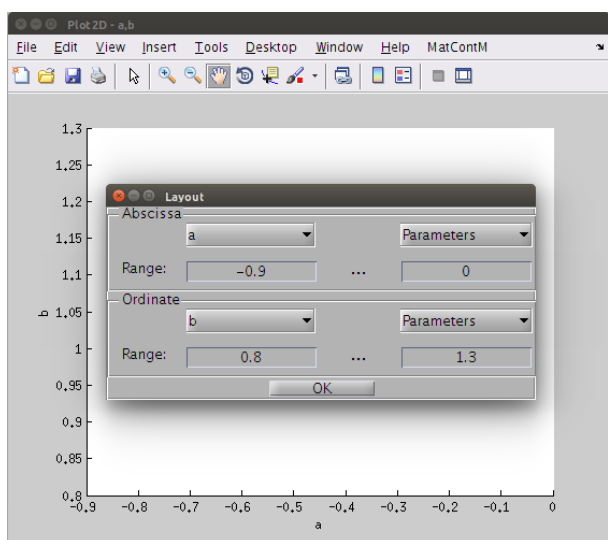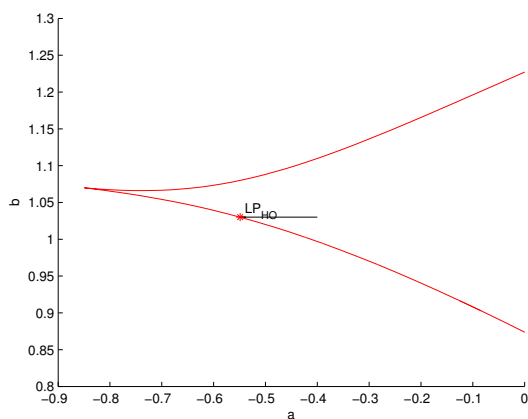# 6 Continuation of a heteroclinic orbit obtained through the MATLAB workspace

We will now compute a heteroclinic bifurcation curve using a connecting orbit that was obtained outside MatContM[4]. For example, enter in the MATLAB command window[5]:

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[4]You can skip this section, if not interested in such low-level functionality.

[5]You can copy/paste this into Matlab if you are using a pdf reader.

```
C=[ 0.46661702380495,0.51950188769804 ,0.48317182408537, ...
    0.47311690109635 ,0.6123,0.841195,1.22990435,0.6419816, ...
0.1347307,-0.14345697, -0.33379938281268,-0.38062126993587, ...
    -0.40424194997464,-0.41621339121235,-0.42230324914752, ...
    -0.42861702380495;0.46661702380495,0.37639405084666, ...
     0.43915841488454, 0.45696220552142, 0.2067,-0.276064, ...
    -1.3327006,-1.0930203,-0.7998431,-0.6233856,-0.49516248970159, ...
    -0.46255049612568,-0.44591635361118,-0.43743735402657, ...
    -0.43311132511221,-0.42861702380495];
```

The orbit is stored as a matrix under the variable name 'C' and is now available in the MATLAB workspace. We will now enter this orbit into MatContM. First, make sure you have selected a `Connecting Orbit` as initial point type (Figure 2) and select the `HE-Curve` (Figure 3). Now open the `Starter` window and press the **select** button above the **plot** button. You will now see a list of the available connecting orbits for selection. You can add `C` by clicking on the button **Add Orbit from MATLAB workspace** below .

Enter the variable name `C`, the orbit now appears in the list and is available for selection. After selecting the orbit, the name will appear in the `Starter` window. For this example, make sure that `Iteration` is set to 2 and b is selected as the free parameter. Set the values of parameter a to 0.3, b to -1.057, R to -0.5 and S to 0.
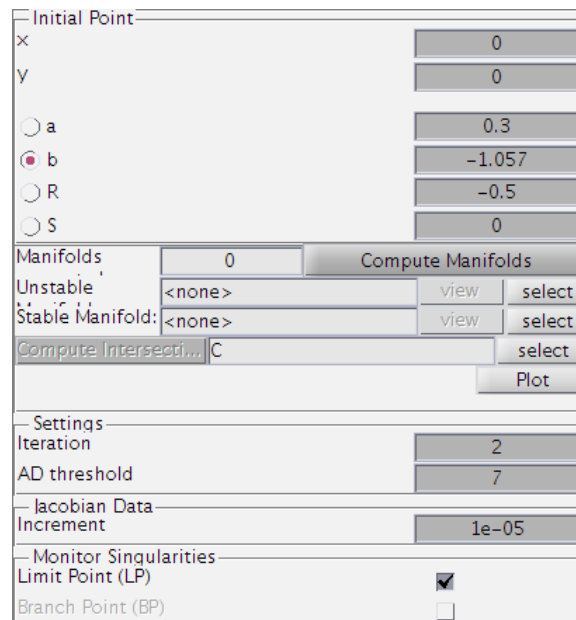

Figure 16: `Starter` window of a heteroclinic connection continuation.

The `Starter` window should now look like Figure 16. Proceed by selecting: **Compute| Backward**. You should detect at least one limit point. You can then select that point

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for a heteroclinic tangency continuation.

7   Additional Problems

A. Shear map

Consider the following planar map

$$\Psi = \psi \circ \varphi,$$

where

$$\varphi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \lambda^u x \\ \lambda^s y \end{pmatrix}$$

and

$$\psi : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - cf(x+y) \\ y + cf(x+y) \end{pmatrix}$$

with

$$f(z) = \begin{cases} 0 & \text{for } z \le 1, \\ (z-1)^2 & \text{for } z > 1. \end{cases}$$

Fix $\lambda^s = 0.4$, $\lambda^u = 2.0$ in Ψ and compute the stable and unstable invariant manifolds of the origin for $c = 0.5$, $0.75, 0.9$, and 1.0. Find the critical parameter value c^* corresponding to a homoclinic tangency.

B. McMillan map

Consider the following planar map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x + \dfrac{2\mu y}{1 + y^2} \end{pmatrix} \tag{2}$$

where $\mu > 1$, and its perturbation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x + \dfrac{2\mu y}{1 + y^2} + \varepsilon(\beta x + \gamma y) \end{pmatrix} \tag{3}$$

where $\varepsilon > 0$ and (β, γ) are parameters. Study stable and unstable invariant manifolds of the saddle fixed point $(x, y) = (0, 0)$ of the maps, as well as intersections of these manifolds.

(i) Study the map (2) :

   1. Prove that the set of points $(x, y) \in \mathbb{R}^2$ satisfying

$$x^2 y^2 + x^2 + y^2 - 2\mu xy = C$$

   is invariant w.r.t. map (2) for any real constant C. Illustrate this by numerical simulations in MatContM.

2. Prove that the origin $(x, y) = (0, 0)$ is a saddle fixed point of (2) and compute its stable and unstable invariant manifolds to verify that they form a 'figure-of-eight'. This is a very degenerate case.

(ii) Study the map (3) for $\epsilon = 0.05$ and $\gamma = 1.9$ when $(\beta, \mu) \in [-0.5, 2.0] \times [1, 4.5]$ using MatContM:

1. For $(\beta, \mu) = (0.1, 2.0)$, compute and plot branches of the stable and unstable invariant manifolds of the saddle $(0, 0)$ that emanate from it into the positive quadrant. *Warning*: These manifolds eventually pass through other quadrants of the (x, y)-plane.

2. Locate with MatContM intersection points of the invariant manifolds. *Hint*: You have to accurately compute rather long branches of the manifolds to ensure that they intersent sufficiently many times near the saddle. For that, you may need to tune parameters of the algorithm.

3. Continue the obtained approximation of the primary homoclinic orbit to the saddle w.r.t. parameter β and detect two limit points. Verify that at the corresponding parameter values the stable and the unstable invariant manifolds of $(0, 0)$ are (almost) tangent.

4. Continue the found LPs w.r.t. parameters (β, μ) and describe the domain of existence of Poincaré homoclinic structure. What happens as $(\beta, \mu) \to (0, 1)$?

C. Euler scheme for Lorenz system

Compute in MatContM the 1D-unstable invariant manifold of the saddle fixed point $(0, 0, 0)$ in the explicit *Euler scheme* for the Lorenz system:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + h\sigma(y - x) \\ y + h(rx - y - xz) \\ z + h(xy - bz) \end{pmatrix}$$

with $\sigma = 10, b = 8/3, r = 8.37$, and $h = 0.1$.