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UvA Lecture 2:

One-Parameter Bifurcations of Fixed Points of Maps

Contents:

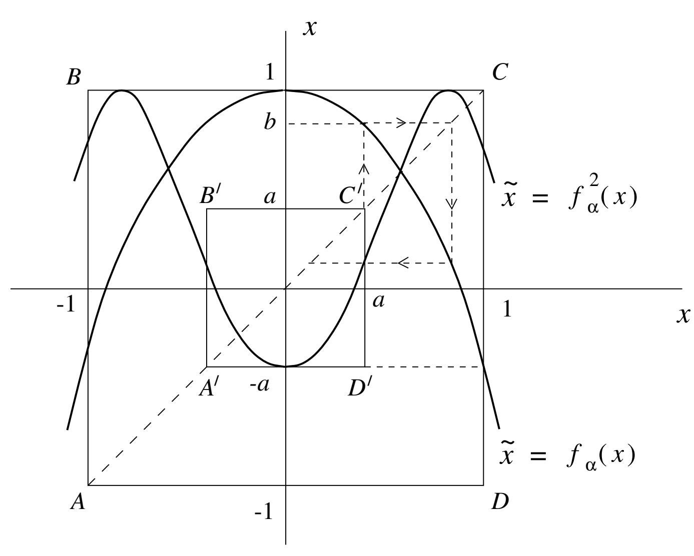
1. Figenbaum's universality

Consider a space $\mathcal{Y}$ of scalar maps

$$x \mapsto f(x), \quad f : \mathbf{R}^1 \to \mathbf{R}^1,$$

satisfying:

(1) $f(x)$ is an even smooth function, $f : [-1, 1] \to [-1, 1]$;
(2) $f'(0) = 0$, $x = 0$ is the only maximum, $f(0) = 1$;
(3) $f(1) = -a < 0$;
(4) $b = f(a) > a$;
(5) $f(b) = f^2(a) < a$.



The function $f_\alpha(x) = 1 - \alpha x^2$ belongs to the class $\mathcal{Y}$ for $\alpha > 1$.

For any $f \in \mathcal{Y}$ define the **doubling operator**:

$$(Tf)(x) = -\frac{1}{a} f(f(-ax)),$$

where $a = -f(1)$. One can check that $Tf \in \mathcal{Y}$.

Lemma 1 *The maps Tf and f^2 are topologically equivalent.*

Hence, if Tf has a periodic orbit of period N, f^2 has a periodic orbit of the same period and f therefore has a periodic orbit of period $2N$.

Th. 1 (Fixed-point existence) *The map $T : \mathcal{Y} \to \mathcal{Y}$ has a fixed point $\varphi \in \mathcal{Y}$: $T\varphi = \varphi$.*

It has been found numerically that

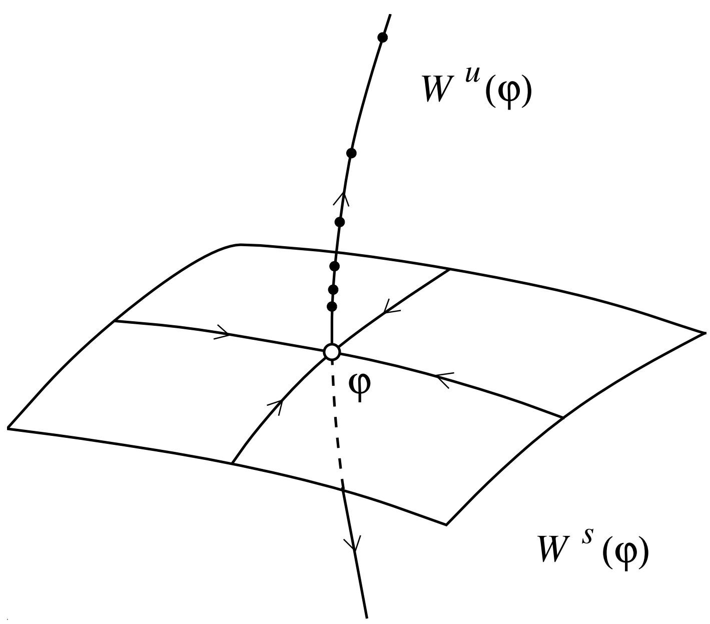
$$\begin{aligned}\varphi(x) &= 1 - 1.52763\ldots x^2 + 0.104815\ldots x^4 \\ &\quad + 0.0267057\ldots x^6 + \ldots .\end{aligned}$$

Th. 2 (Saddle properties of the fixed point) *The linear part L of the doubling operator T at its fixed point φ has only one eigenvalue with $|\mu| > 1$, namely $\mu_F = 4.6692\ldots$:*

$$L\psi = \mu_F \psi, \quad \psi \in \mathcal{Y},$$

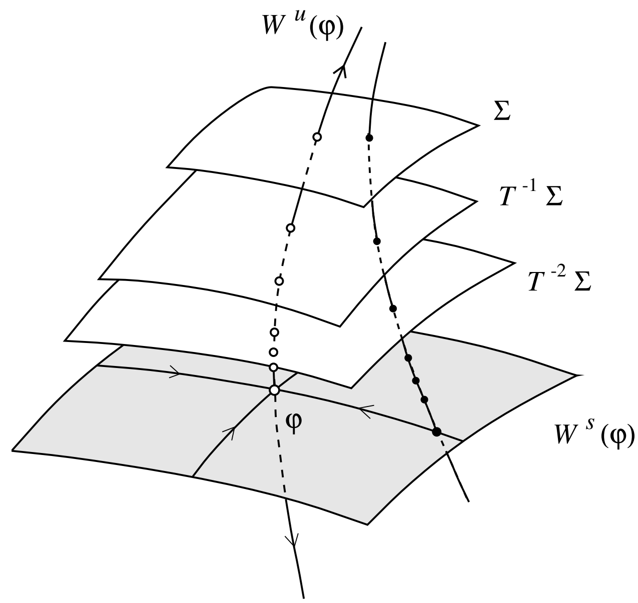
The rest of the spectrum of L is located strictly inside the unit circle.

Therefore, the fixed point φ has a codim 1 stable invariant manifold $W^s(\varphi)$ and a one-dimensional unstable invariant manifold $W^u(\varphi)$.



Consider all maps from $\mathcal{Y}$ having a fixed point with multiplier $\mu = -1$. Such maps form a codim 1 manifold $\Sigma \subset \mathcal{Y}$.

Th. 3 (Manifold intersection) *The manifold* Σ *intersects the unstable manifold* $W^u(\varphi)$ *transversally.*



The preimages $T^{-k}\Sigma$ will accumulate on $W^s(\varphi)$ as $k \to \infty$. $T^{-1}\Sigma$ is composed of maps having a cycle of period two with a multiplier -1, $T^{-2}\Sigma$ is formed by maps having a cycle of period four with a multiplier -1, etc.

Any generic one-parameter dynamical system

$$x \mapsto f_\alpha(x)$$

with $f_\alpha \in \mathcal{Y}$ defines a curve Λ in $\mathcal{Y}$. If this curve is sufficiently close to $W^u(\varphi)$, it will intersect **all** the preimages $T^{-k}\Sigma$.

Let ξ be a coordinate along $W^u(\varphi)$, and let ξ_k denote the coordinate of the intersection of $W^u(\varphi)$ with $T^{-k}\Sigma$. The doubling operator **restricted** to the unstable manifold has the form

$$\xi \mapsto \mu_F \xi + O(\xi^2)$$

with the inverse given by

$$\xi \mapsto \frac{1}{\mu_F}\xi + O(\xi^2).$$

Since

$$\xi_{k+1} = \frac{1}{\mu_F}\xi_k + O(\xi_k^2),$$

we have

$$\frac{\xi_k - \xi_{k-1}}{\xi_{k+1} - \xi_k} \to \mu_F$$

as $k \to \infty$, and so does the sequence of the bifurcation parameter values of f_α.

4. Neimark-Sacker bifurcation

Example 4 ("Normal form") Consider

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1^2+x_2^2)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $\theta = \theta(\alpha), a = a(\alpha)$, and $b = b(\alpha)$ are smooth functions, $a(0) < 0$; and $0 < \theta(0) < \pi$.

Introduce the complex variable $z = x_1 + ix_2$ and set $d = a + ib$. Then

$$z \mapsto e^{i\theta} z(1 + \alpha + d|z|^2) = \mu z + cz|z|^2,$$

where $\mu = \mu(\alpha) = (1+\alpha)e^{i\theta(\alpha)}$ and $c = c(\alpha) = e^{i\theta(\alpha)} d(\alpha)$. Using $z = \rho e^{i\varphi}$, we obtain the following polar form of the map:

$$\begin{cases} \rho \mapsto \rho(1+\alpha+a(\alpha)\rho^2) + \rho^4 R_\alpha(\rho) \\ \varphi \mapsto \varphi + \theta(\alpha) + \rho^2 Q_\alpha(\rho), \end{cases}$$

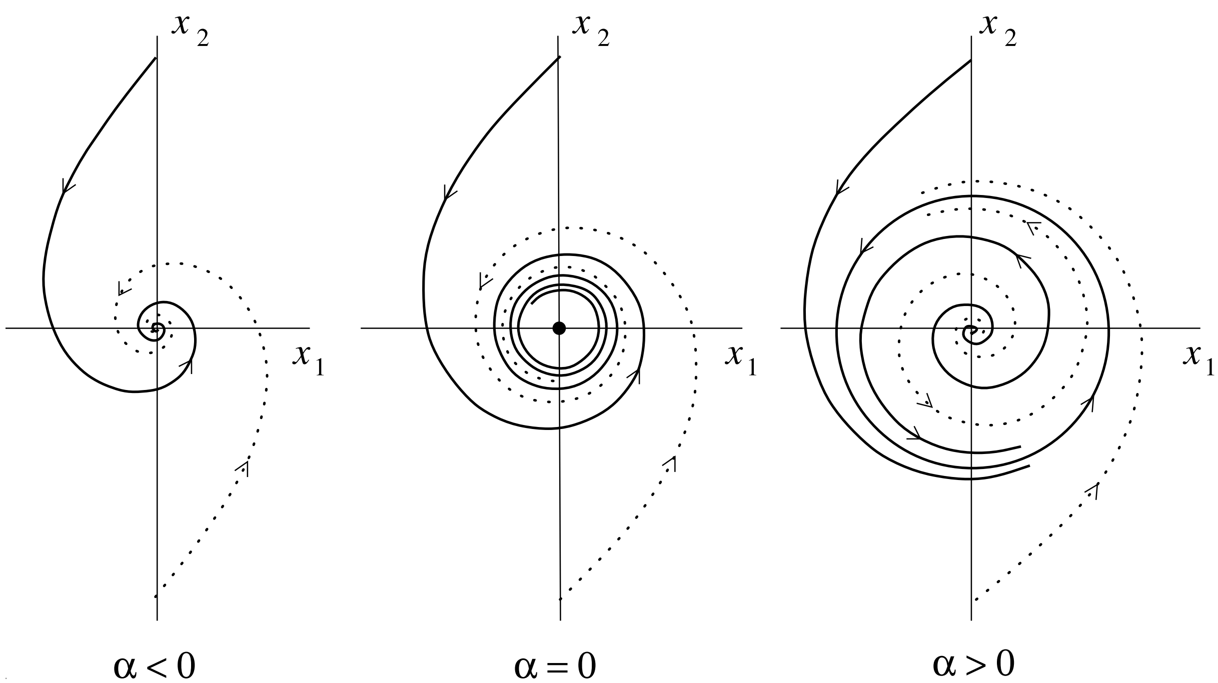
for smooth functions R and Q. This system has a fixed point at the origin for all α with multipliers

$$\mu_{1,2}(\alpha) = (1+\alpha)e^{\pm i\theta(\alpha)}.$$

For $\alpha \leq 0$ the origin is stable. For $\alpha > 0$ the ρ-map has an additional stable fixed point

$$\rho_0(\alpha) = \sqrt{-\frac{\alpha}{a(\alpha)}} + O(\alpha),$$

corresponding to a **closed invariant curve** of the planar map.



Let $\Delta\varphi = \theta(\alpha) + \rho_0^2 Q_\alpha(\rho_0)$. If

$$\frac{\Delta\varphi}{2\pi} = \frac{p}{q}$$

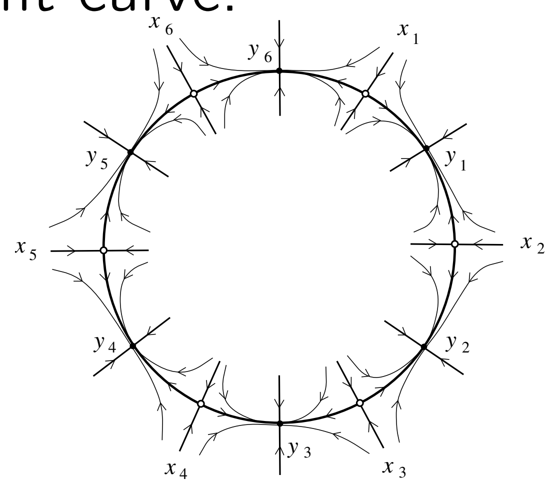
with integer p and q, all points on the invariant circle are **cycles** of period q of the pth iterate of the map. If the ratio is irrational, all orbits are **dense** in the circle.

Lemma 2 *The map*

$$\tilde{z} = e^{i\theta(\alpha)} z(1 + \alpha + d(\alpha)|z|^2) + O(|z|^4),$$

where $d(\alpha) = a(\alpha) + ib(\alpha)$; $a(\alpha), b(\alpha)$, *and* $\theta(\alpha)$ *are smooth real-valued functions,* $a(0) < 0, 0 < \theta(0) < \pi$, *has a stable closed invariant curve for all sufficiently small* $\alpha > 0$. $\square$

Orbit structure on the closed invariant curve is different from that of Example 4. Generically, there is only a **finite** number of cycles on the closed invariant curve.



The cycles exist for $\alpha \in (\alpha_1^{(j)}, \alpha_2^{(j)})$, $j = 1, 2, \ldots$, and disappear at $\alpha_{1,2}^{(j)}$ through the fold bifurcation. The bifurcating invariant closed curve has **finite smoothness** that increases as $\alpha \to 0$.

Consider a planar map

$$x \mapsto A(\alpha)x + F(x,\alpha),\ x \in \mathbf{R}^2, \alpha \in \mathbf{R}^1,$$

where $F = O(\|x\|^2)$ is smooth, and $A(\alpha)$ has two multipliers

$$\mu_{1,2}(\alpha) = r(\alpha)e^{\pm i\psi(\alpha)}$$

with $r(0) = 1, \psi(0) = \theta_0, 0 < \theta_0 < \pi$. One has $r(\alpha) = 1 + \beta(\alpha)$, for some $\beta = \beta(\alpha), \beta(0) = 0$. Suppose $\beta'(0) \neq 0$, then β can be used as a new parameter, and we have $\mu_1 = \mu(\beta), \mu_2 = \bar{\mu}(\beta)$,

$$\mu(\beta) = (1+\beta)e^{i\theta(\beta)},\ \theta(0) = \theta_0.$$

Write

$$x \mapsto A(\beta)x + F(x,\beta).$$

Lemma 3 *By introducing a complex variable z, the map can be written for sufficiently small $|\beta|$ as*

$$z \mapsto \mu(\beta)z + g(z,\bar{z},\beta),$$

where $g = O(|z|^2)$ is a smooth function of $(z,\bar{z},\beta)$.

Proof:

Let $q(\beta), p(\beta) \in \mathbf{C}^2$ be complex vectors such that

$$A(\beta)q(\beta) = \mu(\beta)q(\beta), \quad A^T(\beta)p(\beta) = \overline{\mu(\beta)}p(\beta)$$

Normalize them according to

$$\langle p(\beta), q(\beta) \rangle = 1,$$

where $\langle p, q \rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2$.
Any vector $x \in \mathbf{R}^2$ can be uniquely represented for small α as

$$x = zq(\beta) + \bar{z}\bar{q}(\beta).$$

We have an explicit formula for z, namely

$$z = \langle p(\beta), x \rangle,$$

since $\langle p(\beta), \bar{q}(\beta) \rangle = 0$. Indeed,

$$\langle p, \bar{q} \rangle = \langle p, \frac{1}{\bar{\mu}} A\bar{q} \rangle = \frac{1}{\bar{\mu}} \langle A^T p, \bar{q} \rangle = \frac{\mu}{\bar{\mu}} \langle p, \bar{q} \rangle.$$

Therefore

$$\left(1 - \frac{\mu}{\bar{\mu}}\right) \langle p, \bar{q} \rangle = 0$$

with $\mu \neq \bar{\mu}$ because for all sufficiently small $|\beta|$ we have $\theta(\beta) > 0$. The complex variable z satisfies the equation

$$z \mapsto \mu(\beta) z + \langle p(\beta), F(zq(\beta) + \bar{z}\bar{q}(\beta), \beta) \rangle,$$

having the required form with

$$g(z, \bar{z}, \beta) = \langle p(\beta), F(zq(\beta) + \bar{z}\bar{q}(\beta), \beta) \rangle.$$

Write g as a formal Taylor series in two complex variables (z and $\bar{z}$):

$$g(z, \bar{z}, \alpha) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(\alpha) z^k \bar{z}^l,$$

where

$$g_{kl}(\beta) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} \langle p(\beta), F(zq(\beta) + \bar{z}\bar{q}(\beta), \beta) \rangle \bigg|_{z=0},$$

for $k + l \geq 2,\ k, l = 0, 1, \ldots$.

Lemma 4 *The map*

$$z \mapsto \mu z + \frac{g_{20}}{2} z^2 + g_{11} z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + O(|z|^3),$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}, g_{ij} = g_{ij}(\beta)$*, can be transformed by an invertible parameter-dependent change of complex coordinate*

$$z = w + \frac{h_{20}}{2} w^2 + h_{11} w\bar{w} + \frac{h_{02}}{2}\bar{w}^2,$$

for all sufficiently small $|\beta|$*, into a map without quadratic terms:*

$$w \mapsto \mu w + O(|w|^3),$$

provided that

$$e^{i\theta_0} \neq 1 \quad and \quad e^{3i\theta_0} \neq 1.$$

Proof:

The inverse transformation is given by

$$w = z - \frac{h_{20}}{2} z^2 - h_{11} z\bar{z} - \frac{h_{02}}{2}\bar{z}^2 + O(|z|^3).$$

In the new coordinate w, the map takes the form

$$\begin{aligned}\tilde{w} = \mu w &+ \frac{1}{2}(g_{20} + (\mu - \mu^2)h_{20})w^2 \\ &+ (g_{11} + (\mu - |\mu|^2)h_{11})w\bar{w} \\ &+ \frac{1}{2}(g_{02} + (\mu - \bar{\mu}^2)h_{02})\bar{w}^2 \\ &+ O(|w|^3).\end{aligned}$$

Thus, by setting

$$h_{20} = \frac{g_{20}}{\mu^2 - \mu},\ h_{11} = \frac{g_{11}}{|\mu|^2 - \mu},\ h_{02} = \frac{g_{02}}{\bar{\mu}^2 - \mu},$$

we "kill" all the quadratic terms, if the denominators are nonzero for all sufficiently small $|\beta|$ including $\beta = 0$. Indeed,

$$\begin{aligned}\mu^2(0) - \mu(0) &= e^{i\theta_0}(e^{i\theta_0} - 1) \neq 0, \\ |\mu(0)|^2 - \mu(0) &= 1 - e^{i\theta_0} \neq 0, \\ \bar{\mu}(0)^2 - \mu(0) &= e^{i\theta_0}(e^{-3i\theta_0} - 1) \neq 0,\end{aligned}$$

due to our restrictions on θ_0. $\square$

Def. 1 *The conditions*

$$e^{ik\theta_0} = 1, \quad k = 1, 2, 3, 4$$

are called **strong resonances**.

Lemma 5 *The map*

$$z \mapsto \mu z + \frac{g_{30}}{6} z^3 + \frac{g_{21}}{2} z^2 \bar{z} + \frac{g_{12}}{2} z \bar{z}^2 + \frac{g_{03}}{6} \bar{z}^3 + O(|z|^4),$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}, g_{ij} = g_{ij}(\beta)$, *can be transformed by an invertible parameter-dependent change of coordinates*

$$z = w + \frac{h_{30}}{6} w^3 + \frac{h_{21}}{2} w^2 \bar{w} + \frac{h_{12}}{2} w \bar{w}^2 + \frac{h_{03}}{6} \bar{w}^3,$$

for all sufficiently small $|\beta|$, *into a map with only one cubic term:*

$$w \mapsto \mu w + c_1 w^2 \bar{w} + O(|w|^4),$$

$c_1 = \frac{1}{2} g_{21}$, *provided that*

$$e^{2i\theta_0} \neq 1 \quad and \quad e^{4i\theta_0} \neq 1.$$

Proof:

The inverse transformation is given by

$$w = z - \frac{h_{30}}{6} z^3 - \frac{h_{21}}{2} z^2 \bar{z} - \frac{h_{12}}{2} z \bar{z}^2 - \frac{h_{03}}{6} \bar{z}^3 + O(|z|^4).$$

$$
\begin{aligned}
\tilde{w} &= \lambda w + \frac{1}{6}(g_{30} + (\mu - \mu^3)h_{30})w^3 \\
&+ \frac{1}{2}(g_{21} + (\mu - \mu|\mu|^2)h_{21})w^2\bar{w} \\
&+ \frac{1}{2}(g_{12} + (\mu - \bar{\mu}|\mu|^2)h_{12})w\bar{w}^2 \\
&+ \frac{1}{6}(g_{03} + (\mu - \bar{\mu}^3)h_{03})\bar{w}^3 + O(|w|^4).
\end{aligned}
$$

Thus, by setting

$$
h_{30} = \frac{g_{30}}{\mu^3 - \mu},\ h_{12} = \frac{g_{12}}{\bar{\mu}|\mu|^2 - \mu},\ h_{03} = \frac{g_{03}}{\bar{\mu}^3 - \mu},
$$

we can annihilate all cubic terms in the resulting map except the $w^2\bar{w}$ -term, since all the involved denominators are nonzero for all sufficiently small $|\beta|$. One can also try to eliminate the $w^2\bar{w}$-term by setting

$$
h_{21} = \frac{g_{21}}{\mu(1 - |\mu|^2)}.
$$

However, the denominator vanishes at $\beta = 0$ **for all** θ_0. To obtain a transformation that is smoothly dependent on β, set $h_{21} = 0$, which results in

$$
c_1 = \frac{g_{21}}{2}. \ \square
$$

Combining Lemmas 4 and 5, we obtain

$$\begin{aligned} w &\mapsto (1+\beta)e^{i\theta(\beta)}w + c_1 w|w|^2 + O(|w|^4) \\ &= e^{i\theta(\beta)}(1+\beta+dw|w|^2) + O(|w|^2), \end{aligned}$$

where $a(0) = \text{Re } d(0) = \text{Re}(e^{-i\theta_0}c_1(0))$. One has

$$\begin{aligned} a(0) &= \text{Re}\left(\frac{e^{-i\theta_0}g_{21}^0}{2}\right) - \frac{1}{2}|g_{11}^0|^2 - \frac{1}{4}|g_{02}^0|^2 \\ &- \text{Re}\left(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})}g_{20}^0 g_{11}^0\right). \end{aligned}$$

Th. 4 *For any generic two-dimensional one-parameter system $x \mapsto f(x,\alpha)$, having a fixed point $x_0(\alpha)$ with multipliers*

$$\mu_{1,2}(\alpha) = r(\alpha)e^{\pm i\theta(\alpha)},$$

$r(0) = 1, 0 < \theta(0) < \pi$, *there is a neighborhood of $x_0(0)$ in which a unique closed invariant curve bifurcates from the fixed point as α passes through zero.* $\square$

Genericty conditions:

$(C.1)$ $r'(0) \neq 0$;

$(C.2)$ $e^{ik\theta(0)} \neq 1$ for $k = 1,2,3,4$;

$(C.3)$ $a(0) \neq 0$.