Yuri A. Kuznetsov

UvA Lecture 3:

Center Manifolds and Normal Forms

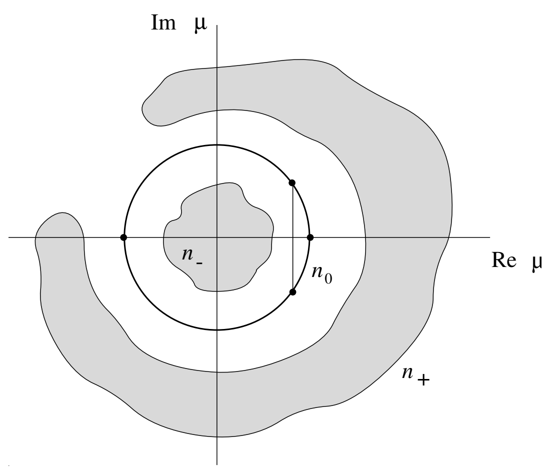
Contents:

1. Critical center manifolds

Consider a map

$$x \mapsto \tilde{x} = f(x), \quad x, \tilde{x} \in \mathbf{R}^n, \tag{1}$$

where f is smooth, $f(0) = 0$. Let the fixed point $x_0 = 0$ be **nonhyperbolic** having $n_0 > 0$ multipliers with $|\mu| = 1$.



Let T^c denote the linear (generalized) eigenspace corresponding to the union of the n_0 eigenvalues of $A = f_x(0)$ on the unit circle.

Th. 1 (Center Manifold Theorem) *There is a locally defined smooth n_0-dimensional invariant manifold $W^c_{loc}(0)$ of (1) that is tangent to T^c at $x = 0$.*

Def. 1 *The manifold W^c_{loc} is called the* **center** (*or* **centre**) **manifold**.

Lemma 1 *There is a nonsingular $n \times n$ matrix S such that*

$$S^{-1}AS = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},$$

where $n_0 \times n_0$ matrix B has all its eigenvalues on the unit circle, while $(n_+ + n_-) \times (n_+ + n_-)$ matrix C has no eigenvalue on the unit circle.

Write (1) as

$$x \mapsto Ax + F(x),$$

with $\|F(x)\| = O(\|x\|^2)$. Let

$$x = S \begin{pmatrix} u \\ v \end{pmatrix}, \quad u \in \mathbf{R}^{n_0}, v \in \mathbf{R}^{n_+ + n_-}.$$

Then map (1) takes the form

$$\begin{cases} u & \mapsto & Bu + g(u,v), \\ v & \mapsto & Cv + h(u,v), \end{cases} \tag{2}$$

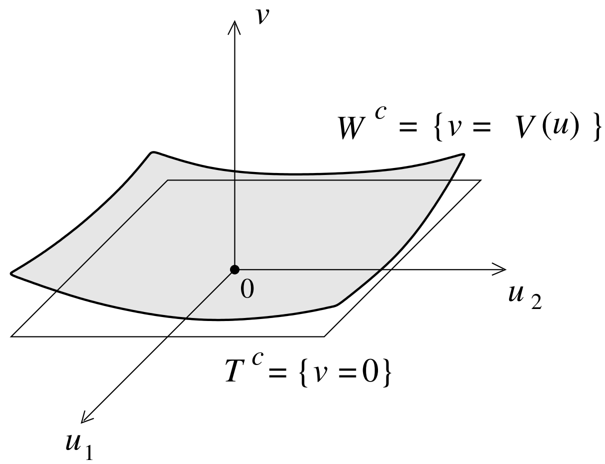
where $g, h = O(\|u\|^2 + \|v\|^2)$. One has

$$S^{-1}F\left(S\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}.$$

For map (2), the center manifold has a local representation

$$W^c = \{(u,v) \ : \ v = V(u)\},$$

where $V : \mathbf{R}^{n_0} \to \mathbf{R}^{(n_+ + n_-)}$ is smooth and $V(u) = O(\|u\|^2)$.



Th. 2 (Reduction Principle) *The map* (2) *is locally topologically equivalent near the origin to the map*

$$\begin{cases} u & \mapsto & Bu + g(u, V(u)) \\ v & \mapsto & Cv. \end{cases}$$

Def. 2 *The map*

$$u \mapsto Bu + g(u, V(u))$$

is called the **restriction** *of map* (2) *to its center manifold.*

2. Parameter-dependent center manifolds

Consider now a map

$$x \mapsto f(x, \alpha), \quad x \in \mathbf{R}^n, \alpha \in \mathbf{R}^1. \tag{3}$$

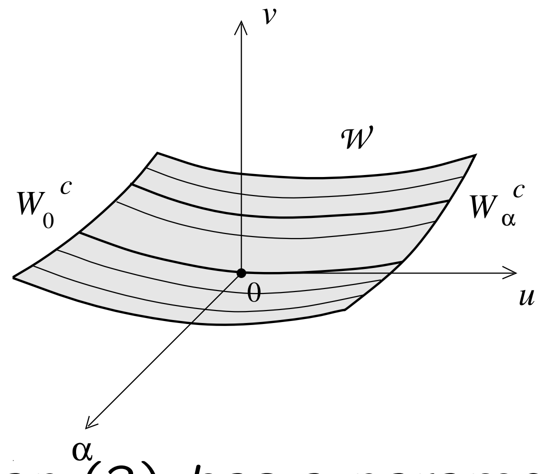
Suppose that at $\alpha = 0$ the fixed point $x = 0$ has n_0 eigenvalues on the unit circle. Introduce the **extended system**:

$$\begin{cases} \alpha & \mapsto & \alpha \\ x & \mapsto & f(x, \alpha). \end{cases}$$

There exists a center manifold of the extended system:

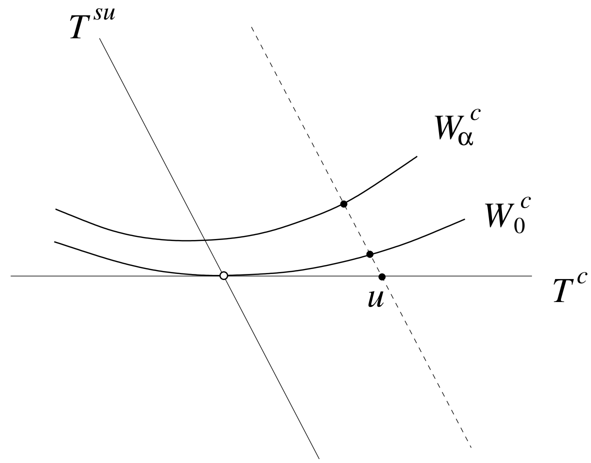
$$\mathcal{W}^c \subset \mathbf{R} \times \mathbf{R}^n, \ \dim \mathcal{W}^c = n_0 + 1.$$

Since $\alpha \mapsto \alpha$, the manifold $\mathcal{W}^c$ is foliated by n_0-dimensional invariant manifolds $W^c_\alpha = \mathcal{W}^c \cap \Pi_\alpha$.



Lemma 2 *Map (3) has a parameter-dependent local invariant manifold W^c_α coinciding at $\alpha = 0$ with its critical center manifold.*

Introduce local coordinates $u \in \mathbf{R}^{n_0}$ in W^c_α, for example, by its projection to T^c.



The restriction of (3) to W^c_α then can be written as

$$u \mapsto \Phi(u, \alpha). \tag{4}$$

Let matrices B and C be associated to $A = f_x(0,0)$ as in Lemma 1.

Th. 3 (Shoshitaishvili, 1972) *The map* (3) *is locally topologically equivalent near* $(x, \alpha) = (0,0)$ *to the map*

$$\begin{cases} u & \mapsto & \Phi(u, \alpha) \\ v & \mapsto & Cv. \end{cases}$$

Moreover, (4) *can be replaced by any locally topologically equivalent map.*

3. Computation of critical normal forms

3.1. Normalization technique

Write the map at $\alpha = 0$ as

$$\tilde{x} = f(x), \quad x \in \mathbf{R}^n, \tag{5}$$

and restrict it to its n_0-dimensional CM:

$$x = H(w), \quad H : \mathbf{R}^{n_0} \to \mathbf{R}^n, \tag{6}$$

The restricted map becomes

$$\tilde{w} = G(w), \quad G : \mathbf{R}^{n_0} \to \mathbf{R}^{n_0}. \tag{7}$$

The invariancy of CM, $\tilde{x} = H(\tilde{w})$, gives the **homological equation**:

$$f(H(w)) = H(G(w)). \tag{8}$$

Now write

$$f(x) = Ax + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\|x\|^4),$$

expand the functions G, H into Taylor series with unknown coefficients,

$$G(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} g_\nu w^\nu, \quad H(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} h_\nu w^\nu,$$

and assume that the restricted map (7) is put into the **normal form** up to a certain order.

Collecting the coefficients of the w^ν-terms in the homological equation (8) gives a linear system for h_ν

$$L_\nu h_\nu = R_\nu. \tag{9}$$

When R_ν involves only known quantities, the linear system has a solution because either L_ν is nonsingular, or R_ν satisfies the **Fredholm solvability condition**

$$\langle p, R_\nu \rangle = 0,$$

where p is a null-vector of the adjoint matrix $\overline{L}_\nu^T$:

$$L_\nu q = 0, \quad \overline{L}_\nu^T p = 0, \quad \langle p, q \rangle = 1,$$

and for $p, q \in \mathbf{C}^n$ the **scalar product** is defined by

$$\langle p, q \rangle = \sum_{k=1}^{n} \bar{p}_k q_k.$$

When R_ν depends on the unknown coefficient g_ν of the normal form, L_ν is singular and the solvability condition gives the expression for g_ν.

3.2. Fold bifurcation

Let $q, p \in \mathbf{R}^n$ satisfy

$$Aq = q, \quad A^p = p, \quad \langle p, q \rangle = 1.$$

Expand

$$f(H) = AH + \frac{1}{2}B(H,H) + O(\|H\|^3),$$

and parametrize the center manifold:

$$H(w) = wq + \frac{1}{2}h_2 w^2 + O(w^3), \quad w \in \mathbf{R}^1,\ h_2 \in \mathbf{R}^n.$$

The critical normal form is

$$\tilde{w} = G(w) = w + bw^2 + O(w^3).$$

The equation $f(H(w)) = H(G(w))$ reads as

$$\begin{aligned} A(wq + \frac{1}{2}h_2 w^2 + \cdots) + \frac{1}{2}B(wq + \cdots, wq + \cdots) + \cdots \\ = \ (w + bw^2 + \cdots)q + \frac{1}{2}h_2(w + \cdots)^2 + \cdots \end{aligned}$$

The w^2-terms give the equation for h_2:

$$(A - I_n)h_2 = -B(q,q) + 2bq.$$

It is singular and its solvability implies

$$b = \frac{1}{2}\langle p, B(q,q) \rangle$$

3.3. Flip bifurcation

Let $q, p \in \mathbf{R}^n$ satisfy

$$Aq = -q, \quad A^T p = -p, \quad \langle p, q \rangle = 1.$$

Expand

$$f(H) = AH + \frac{1}{2}B(H,H) + \frac{1}{6}C(H,H,H) + O(\|H\|^4),$$

and parametrize the center manifold as

$$H(w) = wq + \frac{1}{2}h_2 w^2 + \frac{1}{6}h_3 w^3 + O(w^4),$$

where $w \in \mathbf{R}^1$, $h_{2,3} \in \mathbf{R}^n$. The critical normal form is

$$\tilde{w} = G(w) = -w + cw^3 + O(w^4).$$

The w^2-terms in the homological equation

$$f(H(w)) = H(G(w))$$

give for h_2:

$$(A - I_n)h_2 = -B(q,q).$$

Since $\mu = 1$ is not an eigenvalue of A, the matrix $(A - I_n)$ is nonsingular. Thus,

$$h_2 = -(A - I_n)^{-1} B(q,q).$$

The w^3-terms in the homological equation

$$f(H(w)) = H(G(w))$$

give the linear system for h_3:

$$(A + I_n)h_3 = 6cq - C(q, q, q) - 3B(q, h_2).$$

This system is singular, since $(A + I_n)q = 0$, so it has a solution only if

$$\langle p, 6cq - C(q, q, q) - 3B(q, h_2)\rangle = 0,$$

which implies

$$c = \frac{1}{6}\langle p, C(q, q, q)\rangle + \frac{1}{2}\langle p, B(q, h_2)\rangle.$$

Taking into account $h_2 = -(A - I_n)^{-1}B(q, q)$, we get the invariant formula for the flip normal form coefficient:

$$c = \frac{1}{6}\langle p, C(q, q, q)\rangle - \frac{1}{2}\langle p, B(q, (A - I_n)^{-1}B(q, q))\rangle.$$

Notice that all expressions can be evaluated in the original basis.

3.3. Neimark-Sacker bifurcation

Introduce two complex eigenvectors:

$$Aq = e^{i\theta_0}q, \quad A^T p = e^{-i\theta_0}p, \quad \langle p, q\rangle = 1,$$

where

$$\langle p, q\rangle = \sum_{k=1}^{n} \overline{p}_k q_k.$$

The homological equation takes the form

$$f(H(w, \overline{w})) = H(G(w, \overline{w})),$$

where

$$\begin{aligned} H(w, \overline{w}) &= wq + \overline{w}\,\overline{q} \\ &+ \sum_{1\leq j+k\leq 3} \frac{1}{j!k!} h_{jk} w^j \overline{w}^k + O(|w|^4), \end{aligned}$$

$$F(H) = AH + \frac{1}{2}B(H, H) + \frac{1}{6}C(H, H, H) + O(\|H\|^4).$$

and

$$G(w, \overline{w}) = e^{i\theta_0}w + \frac{1}{2}G_{21}w|w|^2 + O(|w|^4).$$

Quadratic terms give

$$\begin{aligned} h_{20} &= (e^{2i\theta_0}I_n - A)^{-1}B(q,q), \\ h_{11} &= (I_n - A)^{-1}B(q,\overline{q}). \end{aligned}$$

While the $w^2\overline{w}$-terms give the singular system

$$\begin{aligned} (e^{i\theta_0}I_n - A)h_{21} &= C(q,q,\overline{q}) + B(\overline{q},h_{20}) \\ &+ 2B(q,h_{11}) - G_{21}q. \end{aligned}$$

The solvability of this system is equivalent to

$$\langle p, C(q,q,\overline{q}) + B(\overline{q},h_{20}) + 2B(q,h_{11}) - G_{21}q\rangle = 0,$$

so the cubic normal form coefficient can be expressed as

$$\begin{aligned} G_{21} &= \langle p, C(q,q,\overline{q}) + B(\overline{q},(e^{2i\theta_0}I_n - A)^{-1}B(q,q)) \\ &+ 2B(q,(I_n - A)^{-1}B(q,\overline{q}))\rangle, \end{aligned}$$

Then the direction of the Neimark-Sacker bifurcation is determined by

$$a = \frac{1}{2}\text{Re }(e^{-i\theta_0}G_{21}).$$