NUMERICAL CONTINUATION, AND COMPUTATION OF NORMAL FORMS

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1 Introduction

In this chapter we give an overview of numerical methods for analyzing the solution behavior of the dynamical system

$$x'(t) = f(x(t), \alpha), \qquad x, f(x, \alpha) \in \mathbf{R}^n. \tag{1}$$

where, depending on the context, α denotes one or more parameters. Throughout we assume that f is as smooth as necessary. We do not consider Hamiltonian systems, reversible systems, and systems with symmetries in this chapter.

The emphasis in this chapter is on numerical *continuation* methods, as opposed to *simulation* methods. Time-integration of a dynamical system gives much insight into its solution behavior. However, once a solution type has been computed, for example a stationary solution (equilibrium) or a periodic solution (cycle), then continuation methods become very effective in determining the dependence of this solution on the parameter α. Moreover, continuation techniques can also be used when the solutions are not asymptotically stable. Knowledge of unstable solutions is often critical in the understanding of the global dynamics of a system.

We first review basic continuation techniques for following stationary and periodic solutions in one-parameter families of (1). We also describe algorithms for detecting codimension-1 bifurcations, namely folds and Hopf (or Andronov-Hopf) bifurcations, and methods for locating branch points. Branch switching techniques are also described.

Once a codimension-1 bifurcation has been located, it can be followed in two parameters, that is, with $\alpha \in \mathbf{R}^2$ in Equation (1). We give details on this for the case of folds, Hopf bifurcations, period-doubling bifurcations and torus bifurcations. We also describe techniques for the detection and continuation of codimension-2 bifurcations, including the cusp, and the Bogdanov-Takens (BT), generalized Hopf, fold-Hopf, and double-Hopf bifurcations. Efficiency is an important issue in the design of algorithms for following such singularities.

Particular attention is paid to connecting orbits, especially homoclinic orbits. Basic numerical techniques for the computation and continuation of such orbits are outlined. We also describe algorithms for the detection of higher codimension homoclinic bifurcations.

In many cases, detection of higher codimension bifurcations requires computation of certain *normal forms* for equations restricted to *center manifolds* at the critical parameter values. We provide explicit formulas for such coefficients for codimension-1 and 2 equilibrium bifurcations.

Finally, we discuss how to start continuation of local and global codimension-1 bifurcations from some codimension-2 equilibrium singularities.

For an excellent introduction to the subject of numerical continuation and bifurcation methods we refer to the fundamental paper of Keller (1977), which also appears, along with other important contributions, in Keller (1992). Comprehensive books on the subject include Kubiček & Marek (1983), Rheinboldt (1986), Keller (1987), Seydel (1994), Allgower & Georg (1996), and Govaerts (2000); see also Chapter 10 in Kuznetsov (1998). For tutorial introductions, see Beyn (1991), Doedel, Keller & Kernévez (1991*a*, 1991*b*) and Doedel (1999).

1.1 Notation

Throughout this chapter, t and α denote the time and parameter variables, respectively. The derivative with respect to the time variable t is denoted by $x' \equiv dx/dt$, where $x(t)$ is a function of t. We write $\dot{x} \equiv dx/d\alpha$ for the derivative of a function $x = x(\alpha)$ with respect to the parameter α.

If A is a matrix (or a vector represented by a one-column matrix), then A^* denotes the transposed matrix; if A has complex entries, then A^* is the transpose and complex-conjugated matrix. The scalar product of two vectors is then written as $\langle v, w\rangle = v^*w$, while the standard norm is defined by $\|v\| = \sqrt{\langle v, v\rangle}$. The identity matrix in $\mathbf{R}^n$ is denoted by I_n. We write $\mathcal{N}(A)$ and $\mathcal{R}(A)$ for the nullspace and the range, respectively, of a matrix A. We also use the notation $X = (x, \alpha)$ (or, more precisely, $X = (x^*, \alpha^*)^*$).

Let $f(x, \alpha)$ be a smooth function and (x_0, α_0) a given point. We then use the notation

$$f_x^0 v = f_x(x_0, \alpha_0)v, \qquad f_{xx}^0 vw = f_{xx}(x_0, \alpha_0)[v, w]$$

as a short hand for the derivatives of f with respect to x evaluated at (x_0, α_0) and, as a multilinear form, applied to the vectors v and w. Analogous expressions are used for derivatives with respect to α.

Let A and B be $n \times n$ matrices with elements $\{a_{ij}\}$ and $\{b_{ij}\}$, respectively, for $1 \le i, j \le n$, and set $m = \frac{1}{2}n(n-1)$. The *bialternate product* of A and B is an $m \times m$ matrix, denoted $A \odot B$, whose rows are labeled by the multi-index (p, q) $(p = 2, 3, \ldots, n;\ q = 1, 2, \ldots, p-1)$, whose columns are labeled by the multi-index (r, s) $(r = 2, 3, \ldots, n; s = 1, 2, \ldots, r-1)$, and whose elements are given by

$$(A \odot B)_{(p,q),(r,s)} = \frac{1}{2}\left\{ \begin{vmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{vmatrix} + \begin{vmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{vmatrix} \right\}.$$

2 Continuation of Stationary and Periodic Solutions

Here we consider the computation of one-parameter families of equilibria of (1), that is, solutions of

$$f(x, \alpha) = 0. \tag{2}$$

Such a continuum of solutions is often referred to as a *solution branch.* With $\alpha \in \mathbf{R}$ and $X = (x, \alpha)$ the above equation can be written as

$$f(X) = 0,$$

where $f : \mathbf{R}^{n+1} \to \mathbf{R}^n$. A solution X_0 of $f(X) = 0$ is called *regular* if f_X^0 has maximal rank, that is, if $\mathrm{Rank}(f_X^0) = n$. Near a regular solution one has a unique solution branch, as we now make precise.

Theorem 2.1 *Let X_0 be a regular solution of $f(X) = 0$. Then, near X_0, there exists a unique one-dimensional continuum of solutions $X(s)$ with $X(0) = X_0$.*

Proof. We have $X \equiv (x, \alpha)$ and $f_X^0 = (f_x^0 \mid f_\alpha^0)$. If $\mathrm{Rank}(f_x^0 \mid f_\alpha^0) = n$ then either f_x^0 is nonsingular and by the Implicit Function Theorem one has $x = x(\alpha)$ near x_0, or else one can interchange columns in f_x^0 to see that the solution can locally be parametrized by one of the components of x. Thus a unique *solution branch* passes through a regular solution. •

Remark. An example of the second case in the proof is the *simple fold* (or saddle-node bifurcation) described in Section 3.

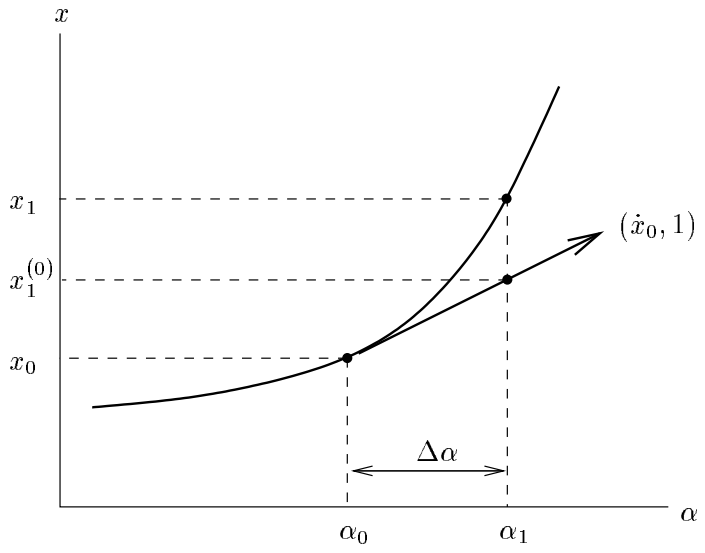

Figure 1: Graphical interpretation of parameter continuation.

2.1 Parameter continuation

Here we take α to be the continuation parameter. Suppose we have a solution x_0 of (2) at α_0, as well as its derivatives $\dot{x}_0$ with respect to the parameter α. We want to compute the solution x_1 at $\alpha_1 \equiv \alpha_0 + \Delta\alpha$. See Figure 1 for a graphical interpretation. To find x_1, we solve $f(x_1, \alpha_1) = 0$ for x_1 by Newton's method,

$$f_x(x_1^{(\nu)}, \alpha_1)\ \Delta x_1^{(\nu)} = -f(x_1^{(\nu)}, \alpha_1), \qquad x_1^{(\nu+1)} = x_1^{(\nu)} + \Delta x_1^{(\nu)}, \qquad \nu = 0, 1, 2, \cdots,$$

with $x_1^{(0)} = x_0 + \Delta\alpha\ \dot{x}_0$. If $f_x(x_1, \alpha_1)$ is nonsingular and $\Delta\alpha$ sufficiently small then the theory for Newton's method assures that this iteration will converge. After convergence, the new derivative vector $\dot{x}_1$ can be obtained by solving

$$f_x(x_1, \alpha_1)\dot{x}_1 = -f_\alpha(x_1, \alpha_1).$$

This equation follows from differentiating $f(x(\alpha), \alpha) = 0$ with respect to α at $\alpha = \alpha_1$. In practice, the calculation of $\dot{x}_1$ can be done at negligible computational cost, since the numerical decomposition of the final Jacobian matrix $f_x(x_1, \alpha_1)$ in Newton's method can be reused.

2.2 Pseudo-arclength continuation

This method, due to Keller (1977), allows the continuation of any regular solution, including folds. Geometrically, it is the most natural continuation method. Suppose we have a solution (x_0, α_0) of $f(x, \alpha) = 0$, as well as the normalized direction vector $(\dot{x}_0, \dot{\alpha}_0)$ of the solution branch at (x_0, α_0). Keller's method consists of solving the following equations for x_1 and α_1 :

$$f(x_1, \alpha_1) = 0, \qquad (x_1 - x_0)^* \dot{x}_0 + (\alpha_1 - \alpha_0)\dot{\alpha}_0 - \Delta s = 0.$$

See Figure 2 for a graphical interpretation. Newton's method for solving these equations can

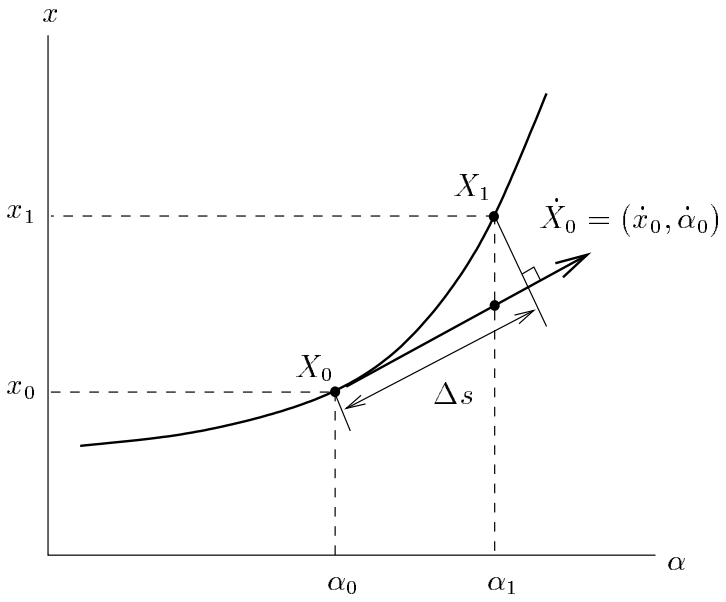

Figure 2: Graphical interpretation of Keller's method.

be written as

$$\begin{pmatrix} (f_x^1)^{(\nu)} & (f_\alpha^1)^{(\nu)} \\ \dot{x}_0^* & \dot{\alpha}_0 \end{pmatrix} \begin{pmatrix} \Delta x_1^{(\nu)} \\ \Delta \alpha_1^{(\nu)} \end{pmatrix} = - \begin{pmatrix} f(x_1^{(\nu)}, \alpha_1^{(\nu)}) \\ (x_1^{(\nu)} - x_0)^* \dot{x}_0 - (\alpha_1^{(\nu)} - \alpha_0)\dot{\alpha}_0 - \Delta s \end{pmatrix}. \tag{3}$$

Upon convergence, the next direction vector can be computed from

$$\begin{pmatrix} f_x^1 & f_\alpha^1 \\ \dot{x}_0^* & \dot{\alpha}_0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{\alpha}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that, in practice, the decomposed Jacobian of the final Newton iteration can be reused to compute the new direction vector. Furthermore, the normalization $\dot{x}_0^* \dot{x}_1 + \dot{\alpha}_0 \dot{\alpha}_1 = 1$ ensures that the orientation of the branch is preserved if Δs is sufficiently small. The new direction vector must be rescaled to have unit length. In practice, the step size Δs is updated regularly during the computation of a solution branch. In the simplest case the choice of the new Δs is based on the convergence of the Newton iteration (3). More sophisticated methods rely on local error estimates; see, for example, Deuflhard, Fiedler & Kunkel (1987), Rheinboldt (1986).

Theorem 2.2 *The Jacobian of Keller's equations is nonsingular at a regular solution.*

Proof. With $X = (x, \alpha) \in \mathbf{R}^{n+1}$, Keller's equations are

$$f(X_1) = 0, \qquad (X_1 - X_0)^* \dot{X}_0 - \Delta s = 0,$$

where $\| \dot{X}_0 \| = 1$. The matrix in Newton's method at $\Delta s = 0$ is $\begin{pmatrix} f_X^0 \\ \dot{X}_0^* \end{pmatrix}$. At a regular solution $\mathcal{N}(f_X^0) = \mathrm{Span}\{\dot{X}_0\}$. We must show that $\begin{pmatrix} f_X^0 \\ \dot{X}_0^* \end{pmatrix}$ is nonsingular. If, on the contrary, $\begin{pmatrix} f_X^0 \\ \dot{X}_0^* \end{pmatrix}$ is

singular, then $f_X^0 z = 0$ and $\dot{X}_0^* z = 0$, for some vector $z \neq 0$. Thus $z = c\dot{X}_0$, for some non-zero constant c. But then $0 = \dot{X}_0^* z = c\dot{X}_0^* \dot{X}_0 = c \parallel \dot{X}_0 \parallel^2 = c$, so $z = 0$, which is a contradiction. •

Remarks. The addition of Keller's continuation equation $(X_1 - X_0)^* \dot{X}_0 = \Delta s$ is one of the simplest examples of an *extended system* that regularizes an otherwise singular system. In the current context, the singularity would arise at a fold (see Section 3.2.1), if we did not use Keller's method. We shall discuss many other extended systems in this chapter.

The linear system (3) in Newton's method applied to Keller's equations has the form

$$\begin{pmatrix} A & c \\ b^* & d \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}.$$

If A is a sparse matrix, for example if A is tridiagonal, then the bordered system can be solved efficiently using *bordered LU-decomposition*; see, for example, Doedel, Keller & Kernévez (1991*a*).

2.3 Periodic solution continuation

Suppose we want to compute a periodic solution of Equation (1), and follow it as a parameter α changes. Note that the period T of the solution is not known *a priori*, and moreover T typically varies with α. To deal with this, we fix the interval of periodicity by the transformation $t \to t/T$. Then (1) becomes

$$x'(t) = T f(x(t), \alpha), \tag{4}$$

and we now seek solutions of period 1, that is,

$$x(0) = x(1)\ . \tag{5}$$

Note that the period T is one of the unknowns.

In our continuation context, assume that we have computed $(x_{k-1}(\cdot), T_{k-1}, \alpha_{k-1})$ and that we want to compute $(x_k(\cdot), T_k, \alpha_k)$. For simplicity of notation, let $(x(\cdot), T, \alpha) \equiv (x_k(\cdot), T_k, \alpha_k)$. Equations (4) and (5) do not uniquely specify x and T, since $x(t)$ can be translated freely in time, that is, if $x(t)$ is a periodic solution then so is $x(t + \sigma)$, for any σ. Thus, a "*phase condition*" is needed. An example is the Poincaré orthogonality condition

$$(x(0) - x_{k-1}(0))^*\ x'_{k-1}(0) = 0.$$

However, there is a numerically more suitable phase condition, that has the property of keeping sharp fronts or peaks in approximately the same position. To be more precise, let $\tilde{x}(t)$ be a solution. We want the phase-shifted solution that is closest to x_{k-1} in phase, that is, we want the solution that minimizes

$$D(\sigma) \equiv \int_0^1 \parallel \tilde{x}(t + \sigma) - x_{k-1}(t) \parallel^2\ dt.$$

The optimal solution $\tilde{x}(t + \hat{\sigma})$ satisfies the necessary condition $dD/d\sigma = 0$, that is,

$$\int_0^1 (\tilde{x}(t + \hat{\sigma}) - x_{k-1}(t))^*\ \tilde{x}'(t + \hat{\sigma})\ dt = 0.$$

Writing $x(t) \equiv \tilde{x}(t + \hat{\sigma})$ gives

$$\int_0^1 (x(t) - x_{k-1}(t))^*\ x'(t)\ dt = 0.$$

Integration by parts, using periodicity, gives

$$\int_0^1 x(t)^* \, x'_{k-1}(t) \, dt = 0. \tag{6}$$

The phase condition (6) is very suitable for numerical computations. It is not difficult to establish the following version of *Poincaré continuation.*

Theorem 2.3 *Let $(x_0(\cdot), T_0)$ define a solution of (4−6), when $\alpha = \alpha_0$. If the Floquet multiplier 1 is algebraically simple then $(x_0(\cdot), T_0)$ can be continued locally as a function of α.*

Remark. The *Floquet multipliers* are the eigenvalues of the *monodromy matrix* $V(1)$, where $V(t)$ is the fundamental solution matrix of the homogeneous linear equation, that is, $V(t)$ satisfies

$$V'(t) \;=\; T_0 \, f_x(x_0(t), \alpha_0) \, V(t), \qquad V(0) = I.$$

Due to periodicity, $V(1)$ always has an eigenvalue equal to 1, called the *trivial multiplier.* For the numerical computation of Floquet multipliers see Fairgrieve & Jepson (1991) and Doedel, Keller & Kernévez (1991*b*).

Above we have assumed α to be fixed. However, in practice we use Keller's method (see Section 2.2) to trace out a branch of periodic solutions. In particular, this allows calculation past folds along such a branch. In the current function space application, Keller's continuation equation takes the form

$$\int_0^1 (x(t) - x_{k-1}(t))^* \dot{x}_{k-1}(t) \, dt + (T - T_{k-1})\dot{T}_{k-1} + (\alpha - \alpha_{k-1})\dot{\alpha}_{k-1} = \Delta s. \tag{7}$$

The complete computational formulation then consists of Equations (4)-(7), which are to be solved for $x(\cdot), T$, and α. These equations correspond to a (generalized) boundary value problem (BVP), and are solved by a numerical boundary value technique. The most widely used discretization method for boundary value problems in ordinary differential equations is the method of orthogonal collocation with piecewise polynomials. In particular, orthogonal collocation is used in software such as COLSYS (Ascher, Christiansen & Russell (1979)), AUTO (Doedel (1981)), (Doedel, Champneys, Fairgrieve, Kuznetsov, Sandstede & Wang (1997)), COLDAE (Ascher & Spiteri (1995)), and CONTENT (Kuznetsov & Levitin (1997)). Its high accuracy (de Boor & Swartz (1973)) and its known mesh-adaption techniques (Russell & Christiansen (1978)) make this method particularly suitable for difficult problems.

3 Locating Codimension-1 Bifurcations

An equilibrium without eigenvalues having zero real part, or a periodic solution without nontrivial Floquet multipliers of unit modulus, is called *hyperbolic.* When following an stationary or periodic solution branch, one can encounter *bifurcation points*, where the solution loses hyperbolicity. In the case of equilibria, the Jacobian matrix f_x has at least one eigenvalue with zero real part at such a point, while for the case of a periodic solution there is at least one nontrivial Floquet multiplier of unit modulus. In this section we introduce test functions (sometimes called bifurcation functions) to detect codimension-1 bifurcations of stationary and periodic solutions to (1).

3.1 Test functions

Let $X = X(s)$ be a smooth, local parametrization of the curve

$$f(X) = 0, \quad f : \mathbf{R}^{n+1} \to \mathbf{R}^n,$$

such that $s = 0$ corresponds to a bifurcation point.

Definition 3.1 *A smooth scalar function $\psi : \mathbf{R}^{n+1} \to \mathbf{R}^1$ defined along the curve is called a* test function *for the corresponding bifurcation if $g(0) = 0$, where $g(s) = \psi(X(s))$.*

The test function ψ has a *regular zero* at the bifurcation point if $g'(0) \neq 0$. A bifurcation point is *detected* between two successive points X_0 and X_1 on the curve if a test function $\psi = \psi(X)$ has opposite signs at these points

$$\psi(X_0)\psi(X_1) < 0.$$

In such case, one can attempt to *locate* the bifurcation point by applying Newton's method to the system

$$\begin{cases} f(X) &= 0, \\ \psi(X) &= 0, \end{cases} \tag{8}$$

with initial point X_0, for example. If the solution branch $X(s)$ is regular near the bifurcation point, and if the test function is defined and differentiable in a neighborhood of the curve and has a regular zero at the bifurcation point, then Newton's method will converge, provided $\|X_0 - X_1\|$ is small. One-dimensional secant methods can be used to solve (8) under less restrictive conditions.

3.2 Locating codimension-1 equilibrium bifurcations

Consider the equilibrium curve (2) corresponding to a one-parameter system (1). There are two generic codimension-1 bifurcations that can be encountered along the equilibrium curve: the fold and the Hopf bifurcation. At a *fold point*, the Jacobian matrix f_x has an algebraically simple zero eigenvalue and no other eigenvalues on the imaginary axis. At a *Hopf point* f_x has a unique and algebraically simple pair of purely imaginary non-zero eigenvalues. In both cases, the topology of the phase portrait near the equilibrium changes when the parameter passes the bifurcation value (see Section 7).

3.2.1 Folds

The function

$$\psi_f(x, \alpha) = \lambda_1(x, \alpha)\lambda_2(x, \alpha) \cdots \lambda_n(x, \alpha),$$

where $\lambda_j(x, \alpha)$ are the eigenvalues of the Jacobian matrix $f_x(x, \alpha)$, is a test function for the fold bifurcation. Since

$$\psi_f(x, \alpha) = \det f_x(x, \alpha), \tag{9}$$

this test function can be efficiently computed without computing eigenvalues.

Definition 3.2 *A point (x_0, α_0) in a one-parameter system (1) is called* a simple fold *if*

$$\text{(S1)} \quad p^* f^0_{xx} qq \neq 0 \qquad \text{and} \qquad \text{(S2)} \quad p^* f^0_\alpha \neq 0,$$

where $f^0_x q = f^{0*}_x p = 0, \quad q^* q = p^* q = 1.$

Note that (S2) means $f^0_\alpha \notin \mathcal{R}(f^0_x)$. Simple folds are often called *simple quadratic folds.*

Theorem 3.1 *The test function* (9) *has a regular zero at a simple fold point.*

Proof. Near a simple fold the equilibrium branch (2) has the parametrization

$$x(\epsilon) = x_0 + \epsilon q + O(\epsilon^2), \qquad \alpha(\epsilon) = \alpha_0 - \frac{1}{2}\frac{p^* f^0_{xx} qq}{p^* f^0_\alpha}\epsilon^2 + O(\epsilon^3).$$

The Jacobian matrix $f_x(x(\epsilon), \alpha(\epsilon))$ has a simple eigenvalue $\lambda(\epsilon)$ with normalized eigenvector $v(\epsilon)$, such that $\lambda(0) = 0, v(0) = q$. Both $\lambda(\epsilon)$ and $v(\epsilon)$ are smooth. Differentiating the first equation of the eigenvalue problem

$$f_x(x(\epsilon), \alpha(\epsilon))v(\epsilon) - \lambda(\epsilon)v(\epsilon) = 0, \qquad p^* v(\epsilon) = 1,$$

with respect to ϵ at $\epsilon = 0$ and multiplying the result from the left by p^*, gives

$$\dot\lambda(0) = p^* f^0_{xx} qq \neq 0. \quad \bullet$$

The Jacobian matrix of (8) with $X = (x, \alpha)$ is nonsingular at a simple fold, so that Newton's method can be used to locate it.

Another way to detect a fold point is to monitor *extremum points* with respect to the parameter α on the equilibrium curve (2). At a simple fold point the α-component of the tangent vector to the curve (2) changes sign.

3.2.2 Hopf points

Consider the real-valued function

$$\psi_H(x, \alpha) = \prod_{i>j}(\lambda_i(x, \alpha) + \lambda_j(x, \alpha)), \tag{10}$$

where, as before, the $\lambda_j(x, \alpha)$ are the eigenvalues of the Jacobian matrix $f_x(x, \alpha)$. This function vanishes at a Hopf bifurcation point, where there is a pair of eigenvalues $\lambda_{1,2} = \pm i\omega_0$. Clearly, ψ_H is also zero if there is a pair of *real* eigenvalues

$$\lambda_1 = \kappa, \qquad \lambda_2 = -\kappa.$$

We have to exclude such points when looking for Hopf bifurcations.

Definition 3.3 *A Hopf point* (x_0, α_0) *in a one-parameter system* (1) *is called* simple *if*

$$\dot\mu(0) = \mathrm{Re}\,[p^* \dot A(\alpha_0) q] \neq 0,$$

where $q, p \in \mathbf{C}^n$ *satisfy*

$$f^0_x q = i\omega_0 q, \qquad f^{0*}_x p = -i\omega_0 p, \qquad p^* q = 1,$$

and

$$\dot A(\alpha) \equiv \frac{d}{d\alpha} f_x(x_e(\alpha), \alpha) = f_{xx}(x_e(\alpha), \alpha)\dot x_e(\alpha) + f_{x\alpha}(x_e(\alpha), \alpha),$$

and where $x_e(\alpha)$ *denotes the continuation of the equilibrium* x_0 *for* α *close to* α_0.

It is easy to show that $\dot{\mu}(0)$ is exactly the α-derivative of the real part of the critical pair of eigenvalues $\lambda_{1,2}(\alpha) = \mu(\alpha) \pm i\omega(\alpha)$, when it crosses the imaginary axis. The following theorem is then obvious, since

$$\mu(\alpha) = \frac{1}{2}(\lambda_1(\alpha) + \lambda_2(\alpha)).$$

Theorem 3.2 *The test function* (10) *has a regular zero at a simple Hopf point.*

The Jacobian of (8) is nonsingular at such a point, and Newton's method can be applied.

As in the fold case, one can compute (10) without explicit computation of the eigenvalues of f_x (Fuller 1968, Guckenheimer & Myers 1996), by using the bialternate product defined in Section 1 :

Theorem 3.3 (Stéphanos (1900)) *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

(i) $A \odot A$ has eigenvalues $\lambda_i \lambda_j$;
(ii) $2A \odot I_n$ has eigenvalues $\lambda_i + \lambda_j$;

where $i = 2, 3, \ldots, n;\ j = 1, 2, \ldots, i-1$.

Therefore, the test function (10) can be expressed as

$$\psi_H(x, \alpha) = \det(2f_x(x, \alpha) \odot I_n). \tag{11}$$

The definition of the bialternate product (see Section 1) leads to the following formula for the elements of $2A \odot I_n$:

$$(2A \odot I_n)_{(p,q),(r,s)} = \begin{cases} -a_{ps} & \text{if } r = q, \\ a_{pr} & \text{if } r \neq p \text{ and } s = q, \\ a_{pp} + a_{qq} & \text{if } r = p \text{ and } s = q, \\ a_{qs} & \text{if } r = p \text{ and } s \neq q, \\ -a_{qr} & \text{if } s = p, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, given the matrix A, the computation of the elements of $2A \odot I_n$ can be efficiently programmed. Also, in Section 3.4, we give an efficient method for computing a function whose value is proportional to (11), by solving an extended linear system.

3.3 Locating codimension-1 bifurcations of periodic solutions

We now describe test functions for the location of codimension-1 bifurcations of periodic solutions in terms of the Jacobian matrix $A = P_x$ of the Poincaré map associated with the periodic solution:

$$x \mapsto P(x, \alpha), \qquad x \in \mathbf{R}^{n-1},\ \alpha \in \mathbf{R}^1,$$

where x provides a smooth parametrization of a codimension-1 cross-section to the closed orbit. Let $\mu_1, \mu_2, \ldots, \mu_{n-1}$ be the multipliers of the periodic solution, i.e., the eigenvalues of A evaluated at the fixed point corresponding to the periodic solution. Adding $\mu_n = 1$ gives the set of Floquet multipliers introduced in Section 2.3.

There are three generic codimension-1 bifurcations of periodic solutions: fold, flip (period-doubling), and Neimark-Sacker (torus) bifurcations. At a *fold* point, the matrix A has a simple unit multiplier $\mu_1 = 1$; at a *flip* point there is a simple multiplier $\mu_1 = -1$; at a *torus* bifurcation A has a pair of non-real multipliers on the unit circle: $\mu_{1,2} = \exp(\pm i\theta_0), 0 < \theta_0 < \pi$. In each of these cases, we assume that the critical eigenvalues are the only eigenvalues of A on the unit circle.

The following test functions locate fold, flip, and Neimark-Sacker bifurcations, respectively:

$$\psi_f = \prod_{i=1}^{n-1}(\mu_i - 1), \tag{12}$$

$$\psi_{fl} = \prod_{i=1}^{n-1}(\mu_i + 1), \tag{13}$$

$$\psi_{NS} = \prod_{n>i>j}(\mu_i\mu_j - 1). \tag{14}$$

To detect a true Neimark-Sacker bifurcation, we must check that $\psi_{NS} = 0$ due to *nonreal* multipliers with unit product: $\mu_i\mu_j = 1$. As in the previous section, we can express the test functions (12)–(14) in terms of the Jacobian matrix, namely,

$$\begin{aligned} \psi_f &= \det(A - I_{n-1}), \\ \psi_{fl} &= \det(A + I_{n-1}), \\ \psi_{NS} &= \det(A \odot A - I_m), \end{aligned}$$

where $m = \frac{1}{2}(n-2)(n-1)$. The last formula follows from statement (i) of Stéphanos' theorem. Using the definition of the bialternate product we get

$$(A \odot A)_{(p,q),(r,s)} = a_{pr}a_{qs} - a_{qr}a_{ps}.$$

Actually, a fold point can be more easily detected as an extremum point of the α-component of the (discretized) periodic solution branch. For the flip and torus bifurcations, the following approach is often applicable. Recall the BVP (4-6) for computing a periodic solution. After discretization, for example using finite differences or collocation, these equations can be solved by Newton's method. Often the Newton matrix can be numerically decomposed such that the approximate monodromy matrix $V(1)$ (see Section 2.3) is implicitly obtained as a by-product, namely in the form $A_1V(1) = -A_0$, for certain nonsingular $n \times n$-matrices A_0 and A_1; see Doedel, Keller & Kernévez (1991*b*). The test functions for flip and torus bifurcation can then be expressed as

$$\begin{aligned} \psi_{fl} &= \frac{1}{\det A_1}\det(-A_0 + A_1), \\ \psi_{NS} &= \frac{1}{(\det A_1 \odot A_1)}\det(A_0 \odot A_0 - A_1 \odot A_1). \end{aligned}$$

To obtain the last test function, we used the identities $(AB) \odot (AB) = (A \odot A)(B \odot B)$ and $(A \odot A)^{-1} = A^{-1} \odot A^{-1}$. These test functions are used in CONTENT (Kuznetsov & Levitin 1997).

3.4 Test functions defined by bordering techniques

As we have seen, the detection of codimension-1 bifurcations of stationary and periodic solutions can be reduced to the detection of zeros of certain determinants along the corresponding branches. The numerical computation of a determinant of a square matrix is a simpler task than computing all its eigenvalues. However, scaling problems can arise when the system dimension is large. The following *bordering technique* avoids such scaling problems.

The idea is to construct a scalar function $g = g(s)$ that vanishes simultaneously with the determinant of a parameter-dependent $(n \times n)$-matrix $B(s)$. Suppose that the determinant vanishes at, say, $s = 0$, and also assume that the eigenvalue $\lambda = 0$ of $B_0 = B(0)$ is geometrically simple. Such a $g = g(s)$ can be computed as the last component of the solution vector to the $(n+1)$-dimensional *bordered system*:

$$\begin{pmatrix} B(s) & p_0 \\ q_0^* & 0 \end{pmatrix} \begin{pmatrix} w \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{15}$$

where $q_0, p_0 \in \mathbf{R}^n$ are certain fixed vectors.

Lemma 3.1 (Keller (1977)) *If $q_0 \notin \mathcal{R}(B_0^*)$ and $p_0 \notin \mathcal{R}(B_0)$, then the matrix*

$$M(s) = \begin{pmatrix} B(s) & p_0 \\ q_0^* & 0 \end{pmatrix}$$

is nonsingular for all sufficiently small $|s|$.

Therefore, generically, Equation (15) has a unique solution. In practical computations, the vectors q_0, p_0 should be *adapted* along the solution branch to make $M(s)$ as well-conditioned as possible. Note that $g(s)$ is proportional to $\det B(s)$. Indeed, by Cramer's rule

$$g(s) = \frac{\det B(s)}{\det M(s)},$$

so $g(0) = 0$. Moreover, if $s = 0$ is a regular zero of the determinant then it is also a regular zero of g. A more general result is given in Section 5.

The function $g(s)$ serves as a test function for the fold bifurcation if we let $B(s) = f_x(x(s), \alpha(s))$, where $(x(s), \alpha(s))$ is a regular parametrization of the equilibrium branch near the fold. For the detection of a Hopf bifurcation, one can use $g(s)$ for

$$B(s) = 2f_x(x(s), \alpha(s)) \odot I_n.$$

Taken literally, this method only applies when n is relatively small. To detect Hopf bifurcation in large systems, the method can be applied after reduction to an invariant subspace corresponding to the dominant eigenvalues; see, for example the discussion in Govaerts, Guckenheimer & Khibnik (1997). Alternatively, detection of Hopf bifurcation in such systems can be based on computing dominant eigenvalues using standard numerical linear algebra.

If A is the Jacobian matrix of the Poincaré map, then (15) with

$$\begin{aligned} B &= A - I_n, \\ B &= A + I_n, \\ B &= A \odot A - I_m, \end{aligned}$$

gives test functions for the fold, flip, and torus bifurcations, respectively.

Remark. The bordering approach can also be used to detect codimension-1 bifurcations of periodic solutions without explicit computation of the Jacobian matrix of the Poincaré map. We illustrate this below for the case of the flip (period-doubling) bifurcation.

Consider the BVP (4)-(6) for the continuation of a periodic solution. For given $(x(\cdot), T)$, introduce the following auxiliary BVP for $(v(\cdot), G)$ with fixed *bordering functions* φ_0, ψ_0:

$$\begin{cases} v'(t) = T f_x(x(t), \alpha)v(t) - G\varphi_0(t), \\ v(1) = -v(0), \\ \int_0^1 v^*(t)\psi_0(t)\ dt = 1 \end{cases} \tag{16}$$

(cf. Equation (15)). The functions φ_0 and ψ_0 are selected to make (16) uniquely solvable, which is always possible. The solution component G of (16) defines a functional

$$\Psi = G[x(\cdot), T, \alpha]$$

which can serve as a test function for the flip bifurcation. Indeed, if $G = 0$ then the first equation in (16) reduces to the variational equation of the periodic solution $x(t)$. The last equation in (16) normalizes the variational solution $v(t)$, and the boundary condition $v(1) = -v(0)$ corresponds to the multiplier $\mu = -1$ of the Poincaré map at the flip bifurcation.

4 Branch Switching

In this section we consider the computation of solution branches that emanate from certain bifurcation points. Specifically, we consider stationary solutions near simple branch points, and periodic solutions near Hopf and near period-doubling points.

4.1 The algebraic branching equation

Let $f : \mathbf{R}^{n+1} \to \mathbf{R}^n$ be as in Equation (2). A solution $X_0 \equiv X(s_0)$ of $f(X) = 0$ is called a *simple singular point* if $f_X^0 \equiv f_X(X_0)$ has rank $n-1$. In the parameter formulation, where $f_X^0 = (f_x^0 \mid f_\alpha^0)$, we have that $X_0 = (x_0, \alpha_0)$ is a simple singular point if, and only if,
(i) dim $\mathcal{N}(f_x^0) = 1, f_\alpha^0 \in \mathcal{R}(f_x^0)$, or (ii) dim $\mathcal{N}(f_x^0) = 2, f_\alpha^0 \notin \mathcal{R}(f_x^0)$.

Suppose we have a solution branch $X(s)$ of $f(X) = 0$, where s is some parametrization. Let $X_0 \equiv (x_0, \alpha_0)$ be a simple singular point. Then we must have

$$\mathcal{N}(f_X^0) = \text{Span}\{\phi_1, \phi_2\}, \qquad \mathcal{N}(f_X^{0*}) = \text{Span}\{\psi\}.$$

From differentiating $f(X(s)) = 0$, we also have $f^0 \equiv f(X_0) = 0$, $f_X^0 \dot{X}_0 = 0$, and $f_{XX}^0 \dot{X}_0 \dot{X}_0 + f_X^0 \ddot{X}_0 = 0$. Thus $\dot{X}_0 = \alpha\phi_1 + \beta\phi_2$, for some $\alpha, \beta \in \mathbf{R}^1$, and

$$\psi^* f_{XX}^0(\alpha\phi_1 + \beta\phi_2)(\alpha\phi_1 + \beta\phi_2) + \psi^* f_X^0 \ddot{X}_0 = 0.$$

Above, the second term is zero, and we can rewrite the equation as

$$c_{11}\alpha^2 + 2c_{12}\alpha\beta + c_{22}\beta^2 = 0, \tag{17}$$

where

$$c_{11} = \psi^* f_{XX}^0 \phi_1\phi_1, \qquad c_{12} = \psi^* f_{XX}^0 \phi_1\phi_2, \qquad c_{22} = \psi^* f_{XX}^0 \phi_2\phi_2.$$

Equation (17) is the *algebraic branching equation* (ABE); see Keller (1977). If the *discriminant* is positive, that is, if $c_{12}^2 - c_{11}c_{22} > 0$, then the ABE has two real nontrivial, linear independent solution pairs (α_1, β_1) and (α_2, β_2), which are unique up to scaling. In such case we have a *simple branch point*, that is, two distinct branches pass through X_0.

Remark. Branch points are not a generic phenomenon in one-parameter solution families: they disappear under generic perturbations of the equation $f(X) = 0$, giving rise to nonintersecting branches. However, in systems with certain symmetries or invariant planes, branch points appear in a persistent manner.

4.2 Branch switching at simple branch points

The direction vector of a bifurcating branch can be written $\dot{Y}_0 = \alpha\phi_1 + \beta\phi_2$, where (α, β) is a root of the ABE (17). (Here we write "$\dot{Y}_0$", in order to distinguish this direction vector from the direction vector $\dot{X}_0$ of the "given" branch at the branch point.) We can take $\phi_1 = \dot{X}_0$. Then $(\alpha_1, \beta_1) = (1, 0)$ must be a root of the ABE and we must have $c_{11} = 0$. If $\Delta > 0$ then also $c_{12} \neq 0$. The second root then satisfies $\alpha_2/\beta_2 = -c_{22}/2c_{12}$. To evaluate c_{12} and c_{22} we need the nullvectors. Note that ψ is a simple nullvector of f_X^{0*}. We already have chosen $\phi_1 = \dot{X}_0$. Choose $\phi_2 \perp \phi_1$. Then ϕ_2 is a nullvector of the matrix $\begin{pmatrix} f_X^0 \\ \dot{X}_0^* \end{pmatrix}$.

Remark. Note that the above matrix is also the Jacobian of Keller's continuation system at X_0. Its null space is indeed one-dimensional at a simple singular point. This implies that

$$\Psi_{BP} = \det \begin{pmatrix} f_X \\ \dot{X}^* \end{pmatrix} \tag{18}$$

is a test function for singular points. This test function has a regular zero at simple branch points (for a proof, see Keller (1987), or Kuznetsov (1995)).

Locating a branch point of the curve (2) using (18) might be difficult, since the domain of convergence for the Newton corrections (3) shrinks as one approaches such a point. This difficulty can be avoided by introducing $(p, \beta) \in \mathbf{R}^n \times \mathbf{R}^1$ and considering the *extended system* (see Moore (1980) and Mei (1989)):

$$\begin{cases} f(x,\alpha) + \beta p &= 0, \\ f_x^*(x,\alpha)p &= 0, \\ p^* f_\alpha(x,\alpha) &= 0, \\ p^* p - 1 &= 0. \end{cases} \tag{19}$$

A simple branch point (x_0, α_0) corresponds to a regular solution $(x, \alpha, \beta, p) = (x_0, \alpha_0, 0, p_0)$ to (19). Therefore, the standard Newton method can be applied directly to (19) to locate a simple branch point.

After determining the coefficients α_2 and β_2, one must scale the direction vector $\dot{Y}_0 \equiv \alpha_2\phi_1 + \beta_2\phi_2$ of the bifurcating branch so that $\| \dot{Y}_0 \| = 1$. The first solution X_1 on the bifurcating branch can now be computed from

$$f(X_1) = 0, \qquad (X_1 - X_0)^* \dot{Y}_0 - \Delta s = 0. \tag{20}$$

As initial approximation to X_1 we can use $X_1^{(0)} = X_0 + \Delta s\, \dot{Y}_0$. For a graphical interpretation see Figure 3. This method is implemented in several codes, including STAFF (Borisyuk 1981), PITCON (Rheinboldt & Burkardt 1983), and CONTENT (Kuznetsov & Levitin 1997).

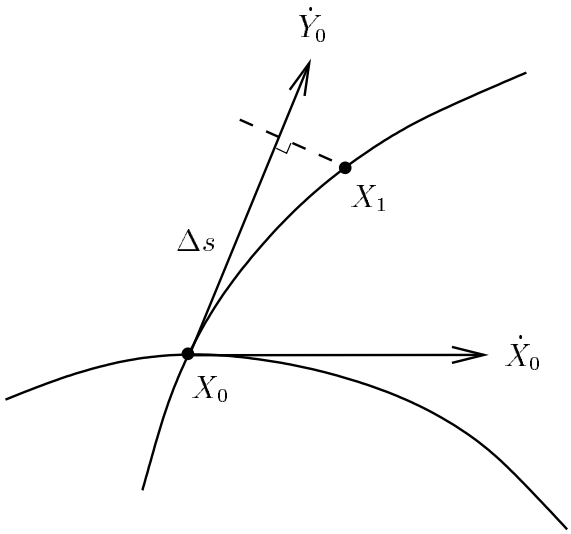

Figure 3: Switching branches using the correct branching direction.

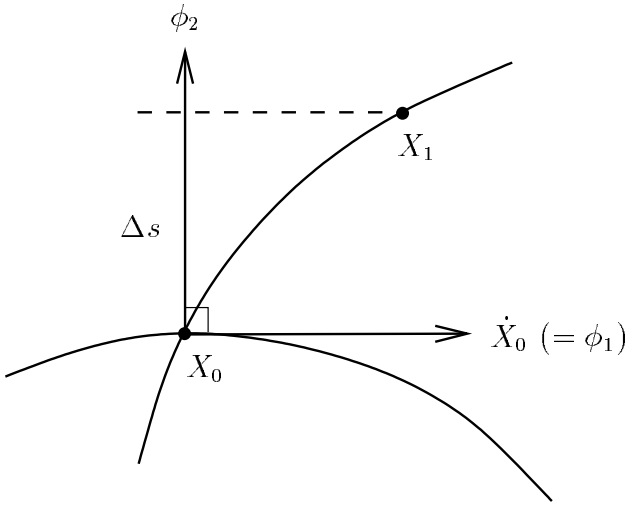

Figure 4: Switching branches using the orthogonal direction.

Instead of Equations (20), one can also use

$$f(X_1) = 0, \qquad (X_1 - X_0)^* \phi_2 - \Delta s = 0,$$

where ϕ_2 is the second null vector of f_X^0, with $\phi_2 \perp \phi_1$, $\phi_1 = \dot{X}_0$. For a graphical interpretation see Figure 4. This method is implemented in AUTO and works very well in many applications, although there may be situations where it fails. Its advantage is that it does not require the computation of second order derivatives.

4.3 Approximation of periodic solutions near a Hopf point

Let (x_0, α_0) be a simple Hopf bifurcation point of Equation (1). By the Hopf Bifurcation Theorem, this ensures the existence of a bifurcating branch of periodic solutions. Moreover, one has the following asymptotic estimates for periodic solutions near the Hopf bifurcation :

$$x(t;\epsilon) = x_0 + \epsilon\phi(t) + O(\epsilon^2), \qquad T(\epsilon) = T_0 + O(\epsilon^2), \qquad \alpha(\epsilon) = \alpha_0 + O(\epsilon^2).$$

Here ϵ locally parametrizes the periodic solution branch. $T(\epsilon)$ denotes the period, and $T_0 = 2\pi/\omega_0$. The function $\phi(t)$ is the normalized nonzero periodic solution of the linearized, constant coefficient problem

$$\phi'(t) = f_x^0 \ \phi(t).$$

To compute a first periodic solution $(x_1, T_1, \alpha_1) \equiv (x, T, \alpha)$, near a Hopf bifurcation (x_0, α_0), we solve Equations (4), (5), (7), together with a modified version of the phase condition (6). An initial approximation for Newton's method is $x^{(0)}(t) = x_0 + \Delta s \ \phi(t)$, $T^{(0)} = T_0$, $\alpha^{(0)} = \alpha_0$, where $\phi(t)$ is now a nonzero solution of the time-scaled, linearized equations

$$\phi'(t) = T_0 \ f_x^0 \ \phi(t), \qquad \phi(0) = \phi(1),$$

namely, $\phi(t) = \sin(2\pi t)w_s + \cos(2\pi t)w_c$, where (w_s, w_c) is a null vector in

$$\begin{pmatrix} -\omega_0 I_n & f_x^0 \\ f_x^0 & \omega_0 I_n \end{pmatrix} \begin{pmatrix} w_s \\ w_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \omega_0 = \frac{2\pi}{T_0}.$$

(This nullspace is actually two-dimensional, with $(-w_c, w_s)$ being a second independent null vector.)

For the phase equation we "align" $x(t)$ with $x_0 + \epsilon\phi(t)$. Following the derivation that led to Equation (6), we obtain

$$\int_0^1 x(t)^* \ \phi'(t) \ dt = 0 \ .$$

Finally, in the continuation equation (7) we have $\dot{\alpha}_0 = \dot{T}_0 = 0$.

4.4 Approximation of double-period solutions near a flip point

Let $x_0(t)$ be a T_0-periodic solution at $\alpha = \alpha_0$ with a simple Floquet multiplier $\mu_1 = -1$. Under certain genericity conditions, the Flip Bifurcation Theorem ensures the existence of a bifurcating branch of (approximately) double-period solutions to Equations (4-6) for nearby parameter values. Moreover, one has the following asymptotic estimates for these double-period solutions near the flip bifurcation:

$$x(t;\epsilon) = x_0(2t) + \epsilon\Phi(2t) + O(\epsilon^2), \qquad T(\epsilon) = 2T_0 + O(\epsilon^2), \qquad \alpha(\epsilon) = \alpha_0 + O(\epsilon^2).$$

Here ϵ locally parametrizes the branch of double-period orbits and

$$\Phi(t) = \begin{cases} \phi(t), & 0 \le t < 1, \\ -\phi(t-1), & 1 \le t \le 2, \end{cases}$$

where $\phi(t)$ is the solution to

$$\begin{cases} \phi'(t) = T_0 f_x(x_0(t), \alpha)\phi(t), \\ \phi(1) = -\phi(0), \\ \int_0^1 \psi_0^*(t)\phi(t)\, dt = 1, \end{cases}$$

that is, Equation (16) with $G = 0$.

Since the flip point is a simple branch point for the *second iterate* of the Poincaré map, one can also use the branch switching technique based on ABE described in Section 4.2.

5 Continuation of Codimension-1 Bifurcations

Suppose Equation (1) has a codimension-1 equilibrium bifurcation at $\alpha = \alpha_0$. Generically, there is a curve $\alpha = \alpha(s)$, with $\alpha \in \mathbf{R}^2$ and $s \in \mathbf{R}^1$, along which the equation has an equilibrium with the given bifurcation. The bifurcation curve, say, $\mathcal{B}$, can be computed as a projection of a certain curve Γ in a space of larger dimension onto the α-plane. Thus, we have to specify a continuation problem for Γ, that is, we shall define functions determining the curve in a certain higher-dimensional space.

5.1 Continuation of codimension-1 equilibrium bifurcations

5.1.1 Minimally augmented systems

In this approach we simply append the relevant test function to the equilibrium equation, thus obtaining a system of $n+1$ equations in the $(n+2)$-dimensional vector of unknowns (x, α). More precisely, we have the following continuation problem

$$\begin{cases} f(x, \alpha) &= 0, \\ \det f_x(x, \alpha) &= 0, \end{cases} \tag{21}$$

for the fold bifurcation, and the continuation problem

$$\begin{cases} f(x, \alpha) &= 0, \\ \det(2f_x(x, \alpha) \odot I_n) &= 0, \end{cases} \tag{22}$$

for the Hopf bifurcation, where $\odot$ stands for the bialternate product defined in Section 1. Each system consists of $n+1$ equations in $n+2$ variables. It is called a *minimally augmented system* since it only has one extra equation, compared to the equilibrium equations. If a bifurcation point is detected while continuing an equilibrium and located as a zero of the corresponding test function ψ_f or ψ_H, then we have all the necessary initial data to start the continuation of the bifurcation curve defined by (21) or (22).

The following theorem follows from the discussion in Section 3.

Theorem 5.1 *The Jacobian matrices of the minimally augmented systems* (21) *and* (22) *have rank* $n+1$ *at simple fold and simple Hopf points, respectively.*

Here "simple" means simplicity with respect to at least one parameter α_1 or α_2. The defining system (21) has been implemented in LOCBIF (Khibnik, Kuznetsov, Levitin & Nikolaev 1993).

5.1.2 Bordering techniques

As mentioned earlier, the defining systems (21) and (22) may have scaling problems. In addition it is generally not possible to express the Jacobian matrix explicitly in terms of the partial derivatives of $f(x,\alpha)$. Thus, we have to rely on numerical differentiation, even if the derivatives of f are known analytically. To overcome this difficulty, we replace the test function in (21) or (22) by a function $g(x,\alpha)$ that vanishes together with the given test function but whose derivatives can be expressed analytically. We have already used this *bordering technique* in Section 3.

The mathematical basis of the technique is provided by the following statement.

Theorem 5.2 (Govaerts & Pryce (1993)) *Let*

$$M = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$

be a nonsingular $(n+m)\times(n+m)$ *block matrix with* $A \in \mathbf{R}^{n\times n}$, $B, C \in \mathbf{R}^{n\times m}$, $D \in \mathbf{R}^{m\times m}$. *Let the inverse*

$$M^{-1} = \begin{pmatrix} P & Q \\ R^* & S \end{pmatrix}$$

be decomposed similarly. Let $p \le \min(n,m)$. *Then* A *has rank deficiency* p *if, and only if,* S *has rank deficiency* p.

In the case where $m \ll n$ and for small p this result is used to express that A has a desired rank deficiency.

In the fold case, instead of (21), we introduce a modified minimally augmented system

$$\begin{cases} f(x,\alpha) &= 0, \\ g(x,\alpha) &= 0, \end{cases} \tag{23}$$

where $g = g(x,\alpha)$ is computed as the last component of the solution vector to the $(n+1)$-dimensional *bordered system*:

$$\begin{pmatrix} f_x(x,\alpha) & p_0 \\ q_0^* & 0 \end{pmatrix} \begin{pmatrix} w \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{24}$$

for suitable vectors $q_0, p_0 \in \mathbf{R}^n$. If q_0 is close to the nullvector of $f_x(x,\alpha)$ and p_0 is close to the nullvector of $f_x^*(x,\alpha)$, then the matrix

$$M = \begin{pmatrix} f_x(x,\alpha) & p_0 \\ q_0^* & 0 \end{pmatrix}$$

is nonsingular at (x,α) and (24) has a unique solution. In practical computations, q_0 and p_0 are the nullvectors of f_x and f_x^*, respectively, at the previous point on the fold curve. For $g = 0$, system (24) implies

$$f_x w = 0, \qquad q_0^* w = 1.$$

Thus w is a scaled nullvector of $f_x(x,\alpha)$ and $\det f_x(x,\alpha) = 0$ as in (21). The derivatives of g with respect to (x,α) can be computed by differentiating (24). Let z denote a component of x or α. Then,

$$\begin{pmatrix} f_x(x,\alpha) & p_0 \\ q_0^* & 0 \end{pmatrix} \begin{pmatrix} w_z \\ g_z \end{pmatrix} + \begin{pmatrix} f_{xz}(x,\alpha) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $(w_z, g_z)^T$ can be found by solving the system

$$\begin{pmatrix} f_x(x,\alpha) & p_0 \\ q_0^* & 0 \end{pmatrix} \begin{pmatrix} w_z \\ g_z \end{pmatrix} = - \begin{pmatrix} f_{xz}(x,\alpha)w \\ 0 \end{pmatrix}. \tag{25}$$

This system has the same matrix M as (24), while the right-hand side involves the known vector w and the derivative f_{xz} of the Jacobian matrix f_x. The derivative g_z can be expressed explicitly if we introduce the solution $(v,h)^*$ to the *transposed system*

$$M^* \begin{pmatrix} v \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Multiplying (25) from the left by (v^*, h) and taking into account that $(v^*, h)M = (0, 1)$, gives

$$g_z = -v^* f_{xz}(x,\alpha)w$$

or

$$g_x = -v^* f_{xx}(x,\alpha)w, \qquad g_\alpha = -v^* f_{x\alpha}(x,\alpha)w. \tag{26}$$

These equations can be used to independently verify that the Jacobian matrix of (23),

$$\begin{pmatrix} f_x & f_\alpha \\ g_x & g_\alpha \end{pmatrix},$$

is nonsingular at a simple fold point by applying Lemma 3.1.

In the Hopf case, the modified minimally augmented system looks exactly like (23), with the function $g = g(x,\alpha)$ now computed by solving the bordered system

$$\begin{pmatrix} 2f_x(x,\alpha) \odot I_n & P_0 \\ Q_0^* & 0 \end{pmatrix} \begin{pmatrix} W \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{27}$$

This system is $(m+1)$-dimensional, where $2m = n(n-1)$, and is nonsingular if the vectors $Q_0, P_0 \in \mathbf{R}^m$ are the nullvectors of $2f_x \odot I_n$ and $(2f_x \odot I_n)^*$, respectively, at a nearby generic point on the Hopf curve. The partial derivatives g_z can be expressed in terms of f_{xz} as in the fold case.

Remark. Note that (27) is singular at a point where the equilibrium has two pairs of purely complex eigenvalues ("double-Hopf", see Section 7) regardless of the choice of vectors Q_0 and P_0. To overcome this difficulty, allowing the continuation to pass through such codimension-2 points, one can use *double bordering*. Instead of (27), double bordering uses the $(m+2)$-dimensional system

$$\begin{pmatrix} 2f_x(x,\alpha) \odot I_n & P_0 \\ Q_0^* & 0 \end{pmatrix} \begin{pmatrix} W \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ I_2 \end{pmatrix}. \tag{28}$$

where P_0, Q_0 are $m \times 2$ matrices, selected to ensure regularity of the system (28), while W is a $m \times 2$ matrix and G is a square 2×2 matrix. We then use

$$g = \det G$$

as the defining function in (23).

The system (23) with g computed via (24) and (28) is implemented in CONTENT.

5.2 Standard augmented systems

5.2.1 Folds

If one allows the dimension of the continuation space to be increased by more than one, then many more defining systems can be formulated for computing codimension-1 bifurcation curves.

For example, the following system of $2n+1$ scalar equations for the $2n+2$ components of (x, q, α),

$$\begin{cases} f(x,\alpha) &= 0, \\ f_x(x,\alpha)q &= 0, \\ q^*q_0 - 1 &= 0, \end{cases} \tag{29}$$

can be used to compute a fold bifurcation curve (Moore & Spence 1980). Here, $q_0 \in \mathbf{R}^n$ is a *reference vector* that is not orthogonal to the null-space of f_x. In practical computations, q_0 is usually the nullvector of f_x at the preceding point on the solution curve. The projection of the solution curve of (29) onto the parameter plane gives the fold curve $\mathcal{B}$. To start the continuation of a fold curve, we have to supply the nullvector q_0 in addition to the critical equilibrium and the parameter values.

Theorem 5.3 *Let (x_0, α_0) be a simple fold point of (1), and let q_0 denote a normalized nullvector of $f_x^0 = f_x(x_0, \alpha_0)$. Then the Jacobian matrix of the standard augmented system (29) has full rank, namely $2n+1$, at (x_0, α_0).*

Proof. It is sufficient to show that the $(2n+1) \times (2n+1)$-matrix

$$J = \begin{pmatrix} f_x^0 & 0 & f_\alpha^0 \\ f_{xx}^0 q_0 & f_x^0 & f_{x\alpha}^0 q_0 \\ 0^* & q_0^* & 0 \end{pmatrix}$$

is nonsingular, where α now refers to the one-dimensional parameter that arises in Definition 3.2 of a simple fold. Suppose J is singular, i.e.,

$$\begin{aligned} &(i) \quad f_x^0 u + w f_\alpha^0 = 0, \\ &(ii) \quad f_{xx}^0 q_0 u + f_x^0 v + w f_{x\alpha}^0 q_0 = 0, \\ &(iii) \quad q_0^* v = 0, \end{aligned}$$

for some nonzero $u, v \in \mathbf{R}^n$, $w \in \mathbf{R}^1$. Multiplying (i) by the left nullvector p_0 of f_x^0, we get

$$w p_0^* f_\alpha^0 = 0.$$

Using Property (S2) of a simple fold, it follows that $w = 0$ and hence

$$u = c_1 q_0, \qquad c_1 \in \mathbf{R}^1.$$

From (ii) we get

$$c_1 p_0^* f_{xx}^0 q_0 q_0 + p_0^* f_x^0 v = 0,$$

so $c_1 = 0$ by Property (S1) of the simple fold. Thus $u = 0$. From (ii) it now follows that

$$f_x^0 v = 0,$$

or $v = c_2 q_0$, $c_2 \in \mathbf{R}^1$. But then, by (iii), $c_2 = 0$. Thus, $u = v = 0$ and $w = 0$, which is a contradiction. •

Remark. The last equation in (29) may be replaced by the standard normalization condition $q^*q - 1 = 0$.

5.2.2 Hopf points

Next, consider the following system of $3n+2$ scalar equations for the $3n+2$ components consisting of (x, v, w, α) and ω:

$$\left\{ \begin{array}{rcl} f(x,\alpha) & = & 0, \\ f_x(x,\alpha)v + \omega w & = & 0, \\ f_x(x,\alpha)w - \omega v & = & 0, \\ v^*v_0 + w^*w_0 - 1 & = & 0, \\ v^*w_0 - v_0^*w & = & 0. \end{array} \right. \tag{30}$$

These equations are the real form of the complex system for (x, q, α, ω)

$$\left\{ \begin{array}{rcl} f(x,\alpha) & = & 0, \\ f_x(x,\alpha)q - i\omega q & = & 0, \\ q^*u_0 - 1 & = & 0, \end{array} \right. \tag{31}$$

that defines a necessary condition for Hopf bifurcation. Such systems were first introduced and analyzed by Griewank & Reddien(1983); see also Holodniok & Kubiček(1984), Beyn (1991), and Doedel, Keller & Kernévez (1991*a*). Here $q = v+iw \in \mathbf{C}^n$ is the critical complex eigenvector and $u_0 = v_0 + iw_0 \in \mathbf{C}^n$ is a vector that is *not orthogonal* to the critical eigenvector corresponding to $i\omega$. As in the case of a fold, u_0 is usually chosen as the critical eigenvector at the previously found solution point on the bifurcation curve. The projection of the solution curve Γ to (30) onto the (α_1, α_2)-plane defines the Hopf bifurcation boundary. To start the continuation of such a curve Γ from a Hopf point detected along an equilibrium curve, we also need to compute the Hopf frequency ω_0 and the two real vectors v_0 and w_0.

Theorem 5.4 *The Jacobian matrix of the augmented system* (30) *has full rank, namely* $3n+2$, *at a simple Hopf point* (x_0, α_0).

Proof. Let $f_x^0 = f_x(x_0, \alpha_0)$ and

$$f_x^0 q_0 = i\omega_0 q_0, \qquad [f_x^0]^* p_0 = -i\omega_0 p_0, \quad p_0^* q_0 = 1.$$

It is sufficient to prove that the Jacobian matrix of (31) with respect to (x, q, α, ω) evaluated at $(x_0, q_0, \alpha_0, \omega_0)$, namely,

$$J = \begin{pmatrix} f_x^0 & 0 & f_\alpha^0 & 0 \\ f_{xx}^0 q_0 & f_x^0 - i\omega_0 I_n & f_{x\alpha} q_0 & -iq_0 \\ 0^* & u_0^* & 0 & 0 \end{pmatrix},$$

has the trivial null-space. Here α denotes the one-dimensional parameter in Definition 3.3 of a simple Hopf point. Note that the real form of J gives the square Jacobian matrix of (30).

If J has a nontrivial null-space, then

$$\begin{array}{ll} (i) & f_x^0 u + \beta f_\alpha^0 = 0, \\ (ii) & f_{xx}^0 q_0 u + (f_x^0 - i\omega_0 I_n)s + \beta f_{x\alpha}^0 q_0 - i\delta q_0 = 0, \\ (iii) & u_0^* s = 0, \end{array}$$

for some $u \in \mathbf{R}^n$, $s \in \mathbf{C}^n$, $\beta, \delta \in \mathbf{R}^1$ with $\|u\|^2 + \|s\|^2 + \beta^2 + \delta^2 \neq 0$.

Denote by $x_e(\alpha)$ the continuation of x_0 for small $|\alpha|$, $x_e(\alpha_0) = x_0$. Then $f_x^0 \dot{x}_e + f_\alpha^0 = 0$, and Equation (i) implies

$$u = k\dot{x}_e(\alpha_0), \qquad \beta = k,$$

for some $k \in \mathbf{R}^1$. Substituting these into (ii) and multiplying by p_0^* from the left, gives

$$k[p_0^* \dot{A}(\alpha_0) q_0] + p_0^*(f_x^0 - i\omega_0 I_n)s - i\delta p_0^* q_0 = 0,$$

where, as in Definition 3.3,

$$\dot{A}(\alpha_0) = f_{xx}^0 \dot{x}_e(0) + f_{x\alpha}^0,$$

It follows that $k\mathrm{Re}[p_0^* \dot{A}(\alpha_0) q_0] = 0$. Taking the Definition 3.3 of the simple Hopf point into account, we get

$$k = 0,$$

so $u = 0,\ \beta = 0$. The equation (ii) now takes the form

$$(f_x^0 - i\omega_0 I_n)s - i\delta q_0 = 0,$$

implying $i\delta(p_0^* q_0) = 0$, which gives $\delta = 0$. From

$$(f_x^0 - i\omega_0 I_n)s = 0$$

we conclude that

$$s = zq_0, \quad z \in \mathbf{C}^1.$$

However, (iii) means $z(u_0^* q_0) = 0$, and therefore $z = 0$, since $u_0 \in \mathbf{C}^n$ is not orthogonal to q_0 by construction.

We have $u = 0, s = 0, \beta = \delta = 0$, which is a contradiction. •

Remark. The second and third equations in (30) imply

$$f_x^2 v + \omega^2 v = 0.$$

Thus w can be eliminated, thereby reducing the dimension of the augmented system (Roose & Hlavaček 1985). This gives the following $2n + 2$ equations in the $2n + 3$ variables (x, v, α, κ):

$$\left\{ \begin{array}{rcl} f(x,\alpha) & = & 0, \\ [f_x^2(x,\alpha) + \kappa I_n]\, v & = & 0, \\ v^* v - 1 & = & 0, \\ v^* l_0 & = & 0. \end{array} \right. \tag{32}$$

Here the reference vector $l_0 \in \mathbf{R}^n$ is chosen such that it is not orthogonal to the real two-dimensional eigenspace of f_x corresponding to the eigenvalues $\lambda_1 + \lambda_2 = 0,\ \lambda_1 \lambda_2 = \kappa$. A solution to (32) with $\kappa > 0$ corresponds to a Hopf bifurcation point with $\omega^2 = \kappa$, while that with $\kappa < 0$ correspond to a *neutral saddle* with two real eigenvalues $\lambda_{1,2} = \pm\sqrt{-\kappa}$. Unlike (30), the system (32) is also regular at a point where $\omega^2 = \kappa = 0$.

Since $[f_x^2(x,\alpha) + \kappa I_n]$ has rank defect 2 at points where $f_x(x,\alpha)$ has a pair of eigenvalues with $\lambda_1 + \lambda_2 = 0, \lambda_1 \lambda_2 = \kappa$, one can consider the following $(n+2)$-dimensional system similar to (28):

$$\left(\begin{array}{cc} [f_x^2(x,\alpha) + \kappa I_n] & P_0 \\ Q_0^* & 0 \end{array} \right) \left(\begin{array}{c} W \\ G \end{array} \right) = \left(\begin{array}{c} 0 \\ I_2 \end{array} \right).$$

At Hopf points, all elements of the (2×2)-matrix G vanish by Theorem 5.2. Therefore, the following defining system in the (x, α, κ)-space specifies a Hopf curve in the α-plane:

$$\left\{ \begin{array}{rcl} f(x,\alpha) & = & 0, \\ g_{11}(x,\alpha,\kappa) & = & 0, \\ g_{22}(x,\alpha,\kappa) & = & 0, \end{array} \right. \tag{33}$$

(cf. Werner (1996)). Systems similar to (29) and (30) are used in AUTO, while the systems (29), (32), and (33) are implemented in CONTENT.

5.3 Continuation of codimension-1 bifurcations of periodic solutions

Continuation of codimension-1 bifurcations of periodic solutions to Equation (1), requires two problem parameters ($\alpha \in \mathbf{R}^2$) in addition to the period. This is a more delicate problem than the corresponding problem for equilibria. If the system is not very stiff then we can eliminate the period by computing the Poincaré map and its Jacobian by numerical integration, and then apply continuation methods for fixed point bifurcations. This works satisfactorily in some cases; however, it fails if the periodic solution has very large or very small multipliers, which is often the case. In such situations, the BVP approach below is more reliable.

5.3.1 Folds

Recall Equations (4-6) for computing a periodic solution. We augment this BVP by the linearized, homogeneous equations

$$\left\{ \begin{array}{rcl} v'(t) - T f_x(x(t),\alpha)v(t) - \sigma f(x(t),\alpha) & = & 0, \\ v(1) - v(0) & = & 0, \\ \int_0^1 v^*(t)\dot{x}_0(t)\ dt & = & 0, \end{array} \right.$$

where, as in Section 2.3, $x_0(t)$ denotes a reference solution. Normalize $(v(\cdot), \sigma)$ by requiring

$$\int_0^1 v^*(t)v(t)\ dt + \sigma^2 = 1.$$

This resulting augmented system can be used for the continuation of fold bifurcations for periodic solutions. It is to be solved for the functions $x(t)$ and $v(t)$ defined on $[0,1]$, scalar variables T and σ, and two parameters α_1 and α_2. As in Section 2.3, the equations must be suitably discretized.

5.3.2 Period-doublings

For the period-doubling (flip) bifurcation, we augment Equations (4-6) by

$$\left\{ \begin{array}{rcl} v'(t) - T f_x(x(t),\alpha)v(t) & = & 0, \\ v(1) + v(0) & = & 0, \\ \int_0^1 v^*(t)v(t)\ dt - 1 & = & 0. \end{array} \right.$$

where the boundary condition $v(1) = -v(0)$ expresses that the Jacobian matrix of the Poincaré map has a multiplier $\mu = -1$. After suitable discretization this augmented system can be used to compute flip bifurcation curves of (1).

5.3.3 Tori (Neimark-Sacker)

For the continuation of the Neimark-Sacker bifurcation we introduce a complex eigenfunction $w(t)$ and a scalar variable θ parametrizing the critical multipliers $\mu_{1,2} = e^{\pm i\theta}$. The augmented system consists again of Equations (4-6), now augmented by

$$\left\{ \begin{array}{lcl} w'(t) - Tf_x(x(t),\alpha)w(t) & = & 0, \\ w(1) - e^{i\theta}w(0) & = & 0, \\ \int_0^1 w^*(t)w_0(t)\ dt - 1 & = & 0, \end{array} \right.$$

where $w_0(t)$ is a complex-valued reference function. Written in real variables, this augmented system can be discretized in order to continue generic Neimark-Sacker bifurcations.

5.3.4 Minimally augmented BVPs

The above standard augmented BVPs for continuing fold, flip, and Neimark-Sacker bifurcations of periodic solutions, involve double or triple the number of the differential equations in the underlying periodic solution continuation. It is also possible to derive *minimally augmented BVPs* to continue these bifurcations, using a *bordering technique* similar to that in the finite-dimensional case. We illustrate this approach for the flip bifurcation. The continuation of the flip bifurcation curve in two parameters can be reduced to the continuation of a solution of Equations (4-6) augmented by the single equation

$$G[u,T,\alpha] \ = \ 0, \tag{34}$$

where G is the test functional for the flip bifurcation defined in Section 3.4. The value of G is computed from the linear BVP for $(v(\cdot),G)$ with given *bordering functions* φ_0,ψ_0 and factor T

$$\left\{ \begin{array}{lcl} v'(t) - Tf_x(x(t),\alpha)v(t) + G\varphi_0(t) & = & 0, \\ v(1) + v(0) & = & 0, \\ \int_0^1 \psi_0^*(t)v(t)\ dt & = & 1. \end{array} \right. \tag{35}$$

The functions φ_0 and ψ_0 are selected to make (35) uniquely solvable. Equation (34) is the flip bifurcation condition. To apply the standard continuation technique, we have to use suitable finite-dimensional approximations. It is also possible to efficiently compute the derivatives of G with respect to u, T, and α. A similar approach is applicable to the continuation of the fold and Neimark-Sacker bifurcations.

6 Continuation of Codimension-1 Homoclinic Orbits

A *heteroclinic solution* $x(t)$ connecting two equilibria x_- and x_+ of an ODE system (1) satisfies

$$\lim_{t\to-\infty} x(t) = x_-, \qquad \lim_{t\to+\infty} x(t) = x_+. \tag{36}$$

We shall concentrate on the special case where $x_+ = x_- \equiv x_0$, called *homoclinic solutions*, as they have particular importance in global bifurcation theory. The approach is easily extended to the case where $x_+ \neq x_-$ provided a careful count is taken of the codimension of the connecting orbit, and the consequent number of free parameters α required.

For the case of a homoclinic orbit, their existence is of codimension-1, given the following nondegeneracy condition.

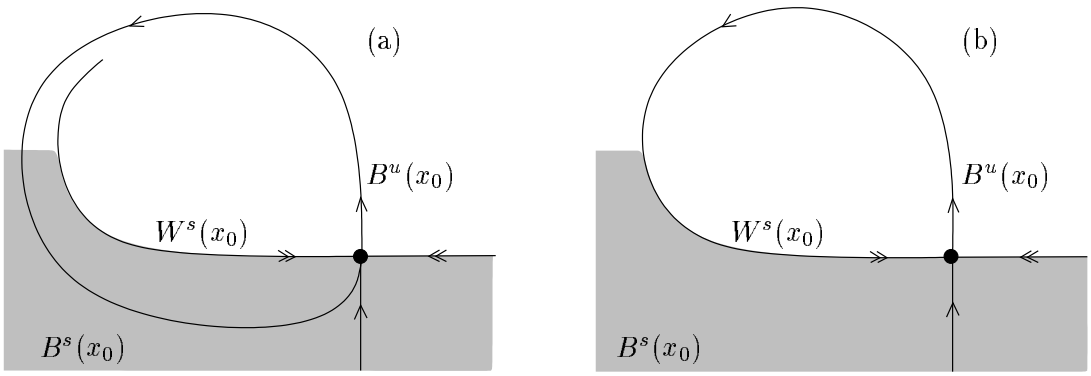

Figure 5: Illustrating (for $n = 2$, $n_c = 1$, $n_s = 1$, $n_u = 0$), the difference between (a) a codimension-1 central saddle-node homoclinic orbit, obeying Definition 6.2, and (b) a codimension-2 non-central saddle-node homoclinic orbit.

Definition 6.1 *A homoclinic orbit $x_h(t)$ to a hyperbolic equilibrium of* (1) *at parameter value $\alpha = \alpha_0$, with $\alpha \in \mathbf{R}^1$, is called* regular *or* nondegenerate *if the linear variational equation*

$$y'(t) = f_x(x_h(t), \alpha_0)y(t) + f_\alpha(x_h(t), \alpha_0)\beta, \quad \lim_{t\to\pm\infty}\{y(t), y'(t)\} \text{ exists}, \quad \beta \in \mathbf{R}^1$$

has the unique solution (up to scalar multiples) $\beta = 0$ and $y(t) = x_h'(t)$.

Remark. The final condition of this definition can be expressed geometrically in terms of the stable and unstable manifolds $W^s(x_0)$ and $W^u(x_0)$ of x_0 (see Section 8.2).

Another possibility of codimension-1 homoclinic connections is if $A = f_x(x_0, \alpha_0)$ has a simple zero eigenvalue and all other eigenvalues are off the imaginary axis. Then one only requires the transverse intersection between the center-stable and center-unstable manifolds of x_0 for a connection to occur. Since the sum of the dimensions of these manifolds is $n + 1$, such a connection will be of no extra codimension than that of a saddle-node equilibrium, i.e., codimension-1. In order to define a regularity condition for such a connection, we need the notions of the stable and unstable sets $B^s(x_0)$ and $B^u(x_0)$ being the subsets of the center-stable and center-unstable manifolds composed of trajectories that tend to x_0 as $t \to \pm\infty$ respectively. For precise definitions of these sets, see Deng (1990) and references therein; Figure 5 illustrates the case $n = 2$.

Definition 6.2 *A homoclinic orbit $x_h(t)$ to a saddle-node equilibrium x_0 at parameter value α_0 of* (1)*, with $\alpha \in \mathbf{R}^1$, is called* regular *or* nondegenerate *if $A = f_x(x_0, \alpha_0)$ has $n_c = 1$ simple eigenvalues at zero, $n_s \geq 0$ eigenvalues in the left-half plane and $n_u \geq 0$ eigenvalues in the right-half plane (counting multiplicity), with $n_u + n_s = n - 1$, and in addition the stable and unstable sets $B^s(x_0)$ and $B^u(x_0)$ intersect transversally along $x_h(t)$, i.e., for each $t \in \mathbf{R}^1$,*

$$\text{codim}\left(T_{x_h(t)}B^s(x_0) + T_{x_h(t)}B^u(x_0)\right) = 0, \tag{37}$$

Remarks.

1. In (37), for both tangent spaces to be defined, it is implicit that $x_h(t)$ does not lie in the boundary of either $B^u(x_0)$ or $B^s(x_0)$, that is $\{x_h(t)|t \in \mathbf{R}^1\} \not\subset W^s(x_0) \cup W^u(x_0)$. Orbits which violate this condition we refer to as *non-central saddle-node homoclinic orbits* (see Figure 5(b)).

2. Unlike Definition 6.1 above, we have not said anything about non-degeneracy with respect to the parameter, but this can be ensured by assuming that α is well-chosen so that the equilibrium undergoes a simple fold bifurcation upon varying α.

Indirect methods for computing homoclinic orbits include numerical shooting (Hassard 1980, Kuznetsov 1983, Rodríguez-Luis, Freire & Ponce 1990) and continuation of periodic solutions (as in Section 2) up to large period (Doedel & Kernévez 1986). In this section we shall focus on a direct formulation as a two-point boundary-value problem which may then be solved using continuation methods, as outlined in Section 2 above, to trace out codimension-1 paths of homoclinic solutions in a two-parameter plane.

6.1 A truncated boundary-value problem

Equation (1) subject to (36) defines a boundary-value problem on the real line, which cannot be solved directly for $t \in (-\infty, +\infty)$. There are two main approaches for defining problems on finite intervals. One is to use a different parametrization than time, say the arclength along the orbit, and the other is to truncate to $t \in (-T_-, T_+)$ for some suitably chosen $T_\pm$ and approximate boundary conditions. We shall concentrate on the truncation method, but the interested reader is referred to (Liu, Liu & Tang 1994, Moore 1995, Liu, Moore & Russell 1997, Bashir-Ali 1998) for recent developments in arclength methods.

Suppose now that $x_h(t)$ is a regular homoclinic orbit to the equilibrium x_0 at parameter value α_0. If x_0 is hyperbolic, there exists a unique equilibrium $x_e(\alpha)$ for all α close to α_0 such that $x_e(\alpha_0) = x_0$. If, on the other hand, x_0 is a saddle-node equilibrium, we set $x_e(\alpha) \equiv x_0$.

Consider the following boundary-value problem on an infinite interval

$$x'(t) = f(x(t), \alpha), \tag{38}$$

$$\lim_{t \to \pm\infty} x(t) = x_e(\alpha). \tag{39}$$

Note that any homoclinic solution to the equilibrium x_e is a solution of (38)–(39). Since any time shift of the solution $x(t)$ is still a solution, a condition is required to fix the phase. Suppose that some initial guess $\tilde{x}(t)$ for the solution is known. Then the following integral phase condition

$$\int_{-\infty}^{\infty} (x(t) - \tilde{x}(t))^* \, \tilde{x}'(t) \, dt = 0 \tag{40}$$

is a necessary condition for a minimum of the L_2-distance between x and $\tilde{x}$ over time shifts (cf. Equation (6)).

The boundary-value problem (38)–(40) defined on an infinite time-interval can be approximated by truncation to a finite interval $[-T_-, T_+]$, with suitable boundary conditions as follows (Beyn (1990*b*, 1990*a*)). Suppose that $A(\alpha) = f_x(x_e(\alpha), \alpha)$ has n_s eigenvalues (counting multiplicities) with negative real part, n_c eigenvalues with zero real part, and n_u eigenvalues with positive real part, so that

$$n_s + n_c + n_u = n.$$

In the hyperbolic case, $n_c = 0$, while if the equilibrium x_0 is a saddle-node, one has $n_c = 1$.

Note that (39) can be cast as

$$x(-T_-) \in B^u(x_e(\alpha)), \quad x(T_+) \in B^s(x_e(\alpha)).$$

Linearizing this condition about the equilibrium $x_e(\alpha)$, we obtain the *projection boundary conditions*

$$\begin{aligned} L_s(\alpha)(x(-T_-) - x_e(\alpha)) &= 0, \quad (41) \\ L_u(\alpha)(x(T_+) - x_e(\alpha)) &= 0, \quad (42) \end{aligned}$$

which replace (39). Here $L_s(\alpha)$ is a $n_s \times n$ matrix whose rows form a basis for the stable eigenspace of $A^*(\alpha)$. Accordingly, $L_u(\alpha)$ is a $n_u \times n$ matrix such that its rows form a basis for the unstable eigenspace of $A^*(\alpha)$. The boundary conditions (41) and (42) place the solution at the two end points in the center-unstable and center-stable eigenspaces of $A(\alpha)$, respectively. Finally, take the phase condition of the truncated problem to be

$$\int_{-T_-}^{T_+} (x(t) - \tilde{x}(t))^* \, \tilde{x}'(t) \, dt = 0. \quad (43)$$

It is not difficult to see that the truncated problem (38), (41)–(43) is a formally well-posed codimension-1 problem, when x_0 is *hyperbolic*, since one has n boundary conditions (41) and (42) plus one integral constraint (43). More precisely, we have the following result.

Theorem 6.1 (Beyn (1990*b*), Schecter (1995), Sandstede (1997)) *Let $x_h(t)$ be a regular homoclinic orbit of* (38) *to the equilibrium x_0 at $\alpha = \alpha_0$. Suppose $\tilde{x}(t)$ is such that* (40) *is satisfied with $x(t) = x_h(t)$, where*

$$\int_{-\infty}^{\infty} x_h'^*(t) \, \tilde{x}'(t) \, dt \neq 0.$$

Then there exist constants $\rho, C, T_* > 0$ such that, for any $T_-, T_+ > T_*$, there exists a solution $(\bar{x}, \bar{\alpha})$ to the truncated boundary-value problem (38), (41)–(43) *that is unique in the ball*

$$\left\{ (x, \alpha) \in C^1([-T_-, T_+], \mathbf{R}^n) \times \mathbf{R}^1 : \|x - x_h|_{[-T_-, T_+]}\|_1 + |\alpha| \leq \rho \right\},$$

and satisfies the error estimates

$$\begin{aligned} \|\bar{x} - x_h|_{[-T_-, T_+]}\|_1 &\leq C e^{-2\min\{\lambda^u T_-, |\lambda^s| T_+\}}, \quad (44) \\ |\bar{\alpha} - \alpha_0| &\leq C e^{-\min\{(2\lambda^u + |\lambda^s|)T_-, (2|\lambda^s| + \lambda^u)T_+\}}. \end{aligned}$$

Here $\|\cdot\|_1$ denotes the usual C^1-norm, and $\mathrm{Re}\{\lambda_i^s\} < \lambda^s < 0$ and $0 < \lambda^u < \mathrm{Re}\{\lambda_i^u\}$, where λ_i^s, $i = 1, \ldots n_s$, and λ_i^u, $i = 1, \ldots, n_u$ are the stable and unstable eigenvalues of $f_x(x_0, \alpha_0)$.

If x_0 is a *saddle-node*, then (41) and (42) give only $(n-1)$ boundary conditions. We then have the following theorem.

Theorem 6.2 (Schecter (1993), Sandstede (1997)) *Let $x_h(t)$ be a regular homoclinic orbit of* (38) *to the saddle-node equilibrium x_0 at $\alpha = \alpha_0$. Suppose $\tilde{x}(t)$ is such that* (40) *is satisfied with $x(t) = x_h(t)$, where*

$$\int_{-\infty}^{\infty} x_h'^*(t) \, \tilde{x}'(t) \, dt \neq 0.$$

Then there exist constants $\rho, C, T_* > 0$ such that, for any $T_-, T_+ > T_*$, there exists a solution $\bar{x}$ to the truncated boundary-value problem (38), (41)–(43) *with $\alpha = \alpha_0$ fixed that is unique in the ball*

$$\left\{ x \in C^1([-T_-, T_+], \mathbf{R}^n) : \|x - x_h|_{[-T_-, T_+]}\|_1 \leq \rho(T_-^{-1} + T_+^{-1}) \right\},$$

and satisfies the error estimate

$$\|\bar{x} - x_h|_{[-T_-, T_+]}\|_1 \leq C(T_-^{-2} + T_+^{-2}).$$

Proof. The statements follow readily from the proofs of Schecter (1995, Theorem 2.1) and Sandstede (1997, Theorem 3.1), where non-central homoclinic orbits to saddle-node equilibria have been addressed. •

In the case that the saddle-node homoclinic solution is continued in two parameters, we simultaneously need to compute the curve of saddle-node equilibria. This issue has been addressed in Section 5. For numerical experiments, we refer to Friedman (1993) and Bai & Champneys (1996).

6.2 Implementation details

The above codimension-1 truncated boundary-value problems for regular homoclinic orbits or central saddle-node homoclinic orbits can be solved using standard boundary-value codes. For example, a particular implementation is available in AUTO (Doedel, Champneys, Fairgrieve, Kuznetsov, Sandstede & Wang 1997), using a suite of routines for homoclinic continuation called HOMCONT (Champneys, Kuznetsov & Sandstede 1996). Here one can compute codimension-1 curves of homoclinic orbits with two free parameters, detect various codimension-2 points along such branches (see Section 8 below) and switch to continuation of higher codimension homoclinic bifurcations in three or more parameters. Some of the numerical issues involved in the continuation of solutions to such boundary-value problems include: efficient computation, ensuring smoothness with respect to α, of the boundary conditions (41), (42); the choice of T_- and T_+; and the accurate determination of starting solutions for two-parameter continuation.

The evaluation of (41), (42) requires a method for obtaining robust bases for the stable and unstable subspaces of $A(\alpha)^*$. A good choice is to use the Schur factorization of A^*, as in (Champneys et al. 1996) and (Doedel, Friedman & Kunin 1997). In the latter reference, smoothness with respect to α has been achieved by adding the coefficients of this factorization, subject to various defining equations, to the list of unknowns to be solved for by the continuation algorithm. This is likely to lead to little extra computational work provided n^2 is small compared to the size of the linear systems to be solved by the BVP-solver. A simpler but less robust method is to perform the Schur decomposition exactly at each continuation step using blackbox linear algebra methods; see (Champneys et al. 1996). An additional step is then made to normalize the stable and unstable subspaces in order to make them approximately smooth with respect to α (Beyn 1990*b*, App. C).

The choice of T_- and T_+ can be made adaptively during the continuation process using the error estimate (44) to achieve some desired tolerance, assuming the BVP to be solved exactly, (Beyn 1990*b*, Sect. 4).

Starting points for homoclinic orbit continuation may be periodic solutions computed to large period (Doedel & Kernévez 1986) or small amplitude solutions constructed near certain local codimension-2 bifurcations such as Bogdanov-Takens points (see Section 11.2.2). If neither of the above is available, a careful homotopy technique may be used to successively continue a small piece of the unstable manifold of x_0 at a parameter value away from the true homoclinic orbit, into a full solution of the truncated boundary-value problem. An account of this latter method is beyond the scope of this Handbook, but the interested reader is referred to Doedel, Friedman & Kunin (1997) for the theory and (Doedel, Friedman & Monteiro 1994, Doedel, Friedman & Guckenheimer 1994) for some applications.

7 Locating Codimension-2 Equilibrium Bifurcations

When investigating a two-parameter problem, one usually encounters higher-order degeneracies along codimension-1 bifurcation curves. Some of these degeneracies are determined by the Jacobian matrix, while others can only be detected taking into account nonlinear terms. For this reason we start this section with the nonlinear normal forms for codimension-1 equilibrium bifurcations, namely the fold and Hopf. Appropriate coefficients in these normal forms play the role of test functions for detecting codimension-2 bifurcations.

Codimension-2 equilibrium bifurcations are important, as they serve as organizing centers, from which several curves of codimension-1 bifurcations can emanate. For example, the Bogdanov-Takens (BT) point, discussed in this section, gives rise to curves of Hopf points, folds, and homoclinic orbits. The switching to such codimension-1 curves from a codimension-2 point is discussed in Section 11.

7.1 Normal forms for codimension-1 bifurcations

Suppose (1) has an equilibrium $x = x_0$ at $\alpha = \alpha_0$, where $\alpha \in \mathbf{R}$. Let $F(x) = f(x, \alpha_0)$ represent the multivariate Taylor series

$$\begin{aligned} F(x) &= A(x-x_0) + \frac{1}{2}B(x-x_0, x-x_0) + \frac{1}{6}C(x-x_0, x-x_0, x-x_0) \\ &\quad + \frac{1}{24}D(x-x_0, x-x_0, x-x_0, x-x_0) \\ &\quad + \frac{1}{120}E(x-x_0, x-x_0, x-x_0, x-x_0, x-x_0) + O(\|x-x_0\|^6), \end{aligned} \tag{45}$$

where

$$A = f_x^0, \qquad B(p,q) = f_{xx}^0 pq, \qquad C(p,q,z) = f_{xxx}^0 pqz.$$

To analyze codimension-1 bifurcations we need to take into account the linear, quadratic, and cubic terms. We also introduce the following multilinear terms of order 4 and 5 here, as they will be needed in Section 10.

$$D(p,q,z,v) = f_{x^{(4)}}^0 pqzv, \qquad E(p,q,z,v,w) = f_{x^{(5)}}^0 pqzvw,$$

where $p, q, z, v, w \in \mathbf{R}^n$. The dependence of A, B, C, D and E on (x_0, α_0) is not indicated to simplify notation. Assume further that $x_0 = 0, \alpha_0 = 0$.

If the solution $x = 0$, $\alpha = 0$ of (1) corresponds to a *fold bifurcation*, then the Jacobian matrix A has a simple zero eigenvalue $\lambda_1 = 0$ and no other critical eigenvalues. Let

$$Aq = 0, \qquad A^*p = 0,$$

be the nullvectors, normalized according to

$$\langle p, q\rangle = \langle q, q\rangle = 1.$$

Lemma 7.1 *The restriction of* (1) *at* $\alpha = 0$ *to the one-dimensional center manifold* W^c *has the form*

$$w' = aw^2 + O(|w|^3), \qquad w \in \mathbf{R}^1,$$

where the coefficient a *can be computed by the formula*

$$a = \frac{1}{2}\langle p, B(q,q)\rangle \equiv \frac{1}{2}p^* f_{xx}^0 qq. \tag{46}$$

If we have a simple fold with respect to the parameter α then the restriction of (1) to the (parameter-dependent) center manifold is locally topologically equivalent to the normal form

$$w' = \beta + aw^2,$$

where β is the unfolding parameter. This normal form predicts the collision of two equilibria when the parameter β passes through zero. Note that the unfolding parameter β can be expressed in terms of the original parameter α as

$$\beta = \langle p, f_\alpha(0,0)\rangle\alpha + O(\alpha^2)$$

(see Section 10).

If the equilibrium $x = 0$ of (1) has a *Hopf bifurcation* at $\alpha = 0$, then the Jacobian matrix $A = f_x(0,0)$ has a simple pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, and no other critical eigenvalues. Introduce two complex vectors

$$Aq = i\omega_0 q, \qquad A^*p = -i\omega_0 p,$$

and normalize them according to

$$\langle p, q\rangle = 1.$$

Lemma 7.2 *The restriction of* (1) *at* $\alpha = 0$ *to the two-dimensional center manifold is locally smoothly orbitally equivalent to the complex normal form*

$$w' = i\omega_0 w + l_1 w|w|^2 + O(|w|^4), \qquad w \in \mathbf{C}^1,$$

where the normal form coefficient l_1 *can be computed by the formula*

$$l_1 = \frac{1}{2}\, Re\, \langle p, C(q,q,\overline{q}) + B(\overline{q}, (2i\omega_0 I_n - A)^{-1}B(q,q)) - 2B(q, A^{-1}B(q,\overline{q})\rangle. \qquad (47)$$

If the Hopf point is simple and its *first Lyapunov coefficient* $l_1 \neq 0$, then the restriction of (1) to the (parameter-dependent) center manifold is locally topologically equivalent to the normal form

$$w' = (\beta + i\omega_0)w + l_1 w|w|^2.$$

This normal form describes a bifurcation of a unique periodic solution branch from the equilibrium $w = 0$, when the parameter β passes through the bifurcation value $\beta = 0$. The direction of the bifurcation is determined by the sign of l_1. The unfolding parameter β has the following asymptotic representation:

$$\beta = [\mathrm{Re}\langle p, \dot{A}(0)q\rangle]\alpha + O(\alpha^2),$$

where, as in Definition 3.3,

$$\dot{A}(\alpha) = \frac{d}{d\alpha} f_x(x_e(\alpha), \alpha),$$

and $x_e(\alpha)$ is the continuation of the equilibrium for small $|\alpha|$, $x_e(0) = 0$.

The formulas (46) and (47) were derived using the center-manifold reduction and subsequent normalization on the center manifold in (Kuznetsov 1995). Originally, an expression equivalent to (47) had been obtained by Lyapunov-Schmidt reduction and asymptotic expansions for the bifurcating periodic solution by Kopell and Howard in (Marsden & McCracken 1976) and by van Gils (1982). These formulas will be rederived in Section 10 below. The first algorithms to determine numerically the direction of Hopf bifurcation were developed by Hassard, Kazarinoff & Wan (1981) and implemented into the code BIFOR2.

7.2 Locating codimension-2 bifurcations

7.2.1 Codimension-2 points on the fold curve

While tracking a fold curve one can encounter the following singularities.

1. An additional real eigenvalue λ_2 meets the imaginary axis, with their geometric multiplicity remaining one, while the center manifold W^c becomes two-dimensional:

   $$\lambda_{1,2} = 0.$$

   These are the conditions for the *Bogdanov-Takens* (or *double-zero*) bifurcation.

   A test function to detect this bifurcation is given by

   $$\psi_{BT} = r^* q, \tag{48}$$

   where

   $$Aq = A^* r = 0, \qquad q^* q = r^* r = 1.$$

   Indeed, if $\psi_{BT} \neq 0$, then the zero eigenvalue is simple. If the standard augmented system (29) or the modified minimally augmented system (23),(24) is used to compute the fold branch, then the normalized nullvector q is known from the continuation. The function (48) has a regular zero at a generic Bogdanov-Takens point.

2. Two extra non-real eigenvalues $\lambda_{2,3}$ meet the imaginary axis, and W^c becomes three-dimensional:

   $$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm i\omega_0,$$

   for $\omega_0 > 0$. These conditions correspond to the *fold-Hopf* bifurcation, also known as the *Gavrilov-Guckenheimer* bifurcation. The Hopf-bifurcation test function ψ_H based on the bialternate product (see Equation (11)) can be used to detect this singularity. It is regular at a generic fold-Hopf point. However, ψ_H will also vanish at Bogdanov-Takens points.

3. The eigenvalue $\lambda_1 = 0$ remains simple and the only one on the imaginary axis ($\dim W^c = 1$), but the normal form coefficient a in Equation (46) vanishes:

   $$\lambda_1 = 0, \qquad a = 0.$$

   These are the conditions for a *cusp* bifurcation. One cannot detect this bifurcation by looking at eigenvalues of the equilibrium, because quadratic terms of $f(x, 0)$ are needed to compute a. The coefficient a can be used as a test function to detect this bifurcation:

   $$\psi_{CP} = a. \tag{49}$$

   Again, if the standard or the modified minimally augmented system is used to compute the fold branch, the nullvector q is known from the continuation. The function (49) has a regular zero at a generic cusp point.

7.2.2 Codimension-2 points on the Hopf curve

While following a Hopf bifurcation curve for Equation (1), one can encounter the following new singular points:

4. Two extra non-real eigenvalues $\lambda_{3,4}$ meet the imaginary axis and W^c becomes four-dimensional:
$$\lambda_{1,2} = \pm i\omega_1, \qquad \lambda_{3,4} = \pm i\omega_2,$$
with $\omega_{1,2} > 0$. These conditions define the *two-pair* or *double-Hopf* bifurcation.

   This bifurcation is most easily detectable if the double-bordered bialternate-product system (28) is used for the Hopf continuation. At this bifurcation, $2A \odot I_n$ has rank defect 2, so that all elements of the matrix G vanish. For example, the test function
$$\psi_{DH} = g_{22}$$
will have a regular zero at a generic double-Hopf point.

5. Finally, the first Lyapunov coefficient l_1 (Equation (47)) may vanish, while $\lambda_{1,2} = \pm i\omega_0$ remain simple and therefore $\dim W^c = 2$:
$$\lambda_{1,2} = \pm i\omega_0, \qquad l_1 = 0.$$
   At this point, a "soft" Hopf bifurcation turns into a "hard" one (or vice versa). It is often called a *generalized* Hopf bifurcation (or *Bautin bifurcation*). The test function is given by
$$\psi_{GH} = l_1.$$

The Bogdanov-Takens bifurcation can also be located along a Hopf bifurcation curve defined by the augmented system (32) (or (33)) as κ passes zero. At this point, two purely imaginary eigenvalues collide and we have a double zero eigenvalue. Following the curve further, we will continue a *neutral saddle* equilibrium with real eigenvalues $\lambda_1 = -\lambda_2$. The value of κ can be calculated if the defining system (27) (or (28)) is used to continue the Hopf curve. One has in that case
$$\kappa = \frac{(v^*Av)(w^*Aw) - (w^*Av)(v^*Aw)}{(v^*v)(w^*w) - (v^*w)^2},$$
where $v, w \in \mathbf{R}^n$ are two real vectors such that $Q = v \wedge w$, where $\wedge$ denotes the wedge product, and Q is a right nullvector of $2A \odot I_n$. We recall that the *wedge product* $v \wedge w$ of two vectors in $\mathbf{R}^n$ is a vector in $\mathbf{R}^{n(n-1)/2}$ indexed by pairs (i,j) where $1 \le j < i \le n$ such that $(v \wedge w)_{(i,j)} = v_j w_i - v_i w_j$. In the present case v, and w span the two-dimensional eigenspace that corresponds to the pair of eigenvalues with zero sum. This is an invariant subspace of A. An easy computation shows that $\psi_2 (= \kappa)$ is the product of the two zero-sum eigenvalues.

A fold-Hopf bifurcation can also occur while tracing a Hopf bifurcation curve. In this case it can be detected as a regular zero of
$$\psi_f = \det A.$$

Remark. In order to be able to use the above test functions to detect codimension-2 points, the underlying defining systems for the codimension-1 bifurcations must remain regular at the codimension-2 points. Otherwise, the continuation algorithms may not be able to pass such a point. The following two lemmas provide necessary information.

Lemma 7.3 *If (x, α) is a point corresponding to any generic codimension-2 equilibrium bifurcation of* (1)*, except the double-Hopf singularity, then rank $J = n + 1$, where J is the Jacobian matrix of the corresponding minimally augmented system* (21) *or* (22)*. A generic double-Hopf point is a simple branch point for* (22)*.*

Lemma 7.4 *The Jacobian matrix of the augmented system* (29) *has rank $2n + 1$ at generic Bogdanov-Takens and cusp bifurcation points of* (1)*, while the Jacobian matrix of the augmented system* (30) *has rank $3n + 2$ at generic Bautin, fold-Hopf and double-Hopf bifurcation points of* (1)*.*

The first interactive software to detect all codimension-2 points was LOCBIF (Khibnik 1990, Khibnik et al. 1993). Detection of all codimension-2 points as described above is implemented in CONTENT (Kuznetsov & Levitin 1997).

8 Locating Codimension-2 Homoclinic Bifurcations

In Section 6 we considered numerical methods for the two-parameter continuation of homoclinic orbits to equilibria. Suppose that we continue a branch of regular codimension-1 homoclinic orbits to (1) in two parameters, i.e., $\alpha \in \mathbf{R}^2$, so that a homoclinic loop to the equilibrium $x_e(s)$ exist whenever $\alpha = \alpha(s)$ with $s \in \mathbf{R}^1$; see Section 2.2 and Section 6. Here the one-dimensional parameter s is typically Keller's pseudo-arclength. We refer to these homoclinic solutions as the *primary* homoclinic orbits. Along this primary branch, codimension-2 homoclinic bifurcation points may arise. Such bifurcations may, for instance, lead to more complicated homoclinic connections such as so-called *n-homoclinic orbits* which follow the primary homoclinic loop n times. Another possibility is that the stability of the periodic solutions which accompany the primary homoclinic orbit changes. The issue addressed in this section is the detection of such codimension-2 homoclinic bifurcation points. We shall focus only on those known codimension-2 bifurcations that, at the critical parameter value, involve a unique finite-amplitude homoclinic orbit. Also, we confine ourselves to numerics. Details of the dynamics near each codimension-2 point are given elsewhere in the Handbook; see also the review papers (Fiedler 1996, Champneys & Kuznetsov 1994).

Codimension-2 homoclinic bifurcation points are detected along the primary branch $\alpha = \alpha(s)$ by locating zeroes of certain test functions; see Section 3 for the concept of test functions. The issue of defining these test functions is actually two-fold. First, consider the primary codimension-1 homoclinic orbits to the original problem (38)–(40) on the infinite time interval. A test function for a certain codimension-2 homoclinic bifurcation is a smooth function defined along the primary branch such that its regular zeroes correspond to the occurrence of the bifurcation. Afterwards, we need to define test functions for the truncated boundary-value problem (38), (41)–(43) on the finite time interval $(-T_-, T_+)$. We require that each such test function is a smooth function along the branch of primary homoclinic orbits to the truncated problem such that the limit of the test function exists as $T_-, T_+ \to \infty$, and is equal to the test function of the original problem on the infinite time interval. We call such test functions *well-defined.*

In the simplest cases, test functions are computable via eigenvalues of the equilibrium. In other cases, the homoclinic solution at the endpoints or solutions to the adjoint variational equation with appropriate boundary conditions are utilized. We address these two different

types of test functions in the following two sections. Test functions along branches of central saddle-node homoclinic orbits are considered in the last section.

The stable and unstable eigenvalues of $A = f_x(x_e(s), \alpha(s))$ are denoted by λ_i^s, $i = 1, \ldots n_s$ and λ_i^u, $i = 1, \ldots, n_u$. In addition, if a branch of central saddle-node homoclinic orbits is computed, we have $n_c = 1$, i.e., there is a simple eigenvalue $\lambda_1^c = 0$ at zero. We assume that the eigenvalues are ordered so that

$$\mathrm{Re}\{\lambda_{n_s}^s\} \leq \cdots \leq \mathrm{Re}\{\lambda_1^s\} < 0 < \mathrm{Re}\{\lambda_1^u\} \leq \cdots \leq \mathrm{Re}\{\lambda_{n_u}^u\}. \tag{50}$$

The stable (unstable) eigenvalues with real part closest to zero are termed the *leading* stable (unstable) eigenvalues.

8.1 Test functions utilizing eigenvalues

Let $n_c = 0$ so that $x_e(s)$ is hyperbolic. The following test functions have been shown in Champneys & Kuznetsov (1994) to be well-defined:

Resonant saddle:

$$\psi = \lambda_1^s + \lambda_1^u.$$

Neutral saddle, saddle-focus or bi-focus:

$$\psi = \mathrm{Re}\{\lambda_1^s\} + \mathrm{Re}\{\lambda_1^u\}.$$

Double real stable leading eigenvalue:

$$\psi = \begin{cases} (\mathrm{Re}\{\lambda_1^s\} - \mathrm{Re}\{\lambda_2^s\})^2, & \mathrm{Im}\{\lambda_1^s\} = 0, \\ (\mathrm{Im}\{\lambda_1^s\} - \mathrm{Im}\{\lambda_2^s\})^2, & \mathrm{Im}\{\lambda_1^s\} \neq 0. \end{cases} \tag{51}$$

A test function for *double real unstable leading eigenvalues* is obtained by replacing λ_j^s with λ_j^u in (51). We remark that zeroes of (51) are regular at generic double eigenvalues since the expressions in (51) represent the discriminant of the quadratic factor of the characteristic polynomial corresponding to this pair of eigenvalues.

Transitions to non-hyperbolic equilibria are detected in the following fashion. First, the truncated problem should be formulated in such a way that the branch of homoclinic orbits can be continued *through* parameter values at which eigenvalues cross the imaginary axis. Secondly, the labeling (50) has to be modified: the n_s leftmost and the n_u rightmost eigenvalues are

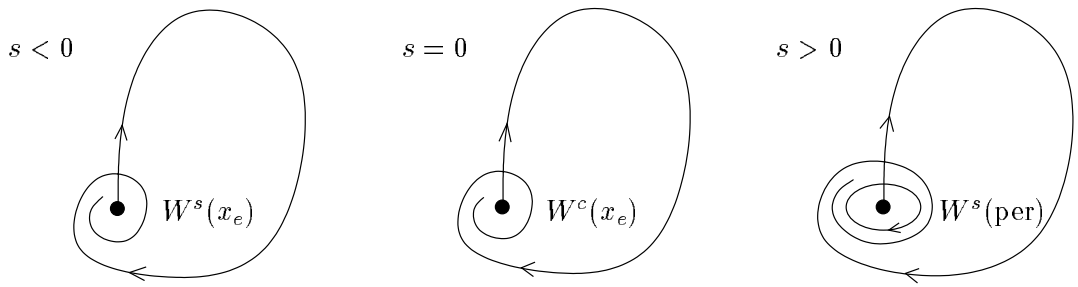

Figure 6: Continuation through a Shilnikov-Hopf bifurcation arising at $s = 0$. The homoclinic orbit exists for $s \leq 0$. For $s > 0$, the solution of the truncated problem is a connection between the equilibrium and a periodic solution with small amplitude.

labeled λ_i^s and λ_i^u, respectively, without regard to the sign of their real part. The test functions detecting fold and Hopf bifurcations are then defined as follows:

Non-hyperbolic equilibria (crossing of stable or unstable eigenvalues):

$$\psi = \mathrm{Re}\{\lambda_1^s\}, \qquad \psi = \mathrm{Re}\{\lambda_1^u\}.$$

We comment further on the first condition that the branch of homoclinic solutions to the truncated problem can be continued through the bifurcation point.

For the Shilnikov-Hopf bifurcation, where the equilibrium undergoes a Hopf bifurcation at, say, $s = 0$, there exists a solution to the truncated boundary-value problem (38), (41)–(43) beyond the codimension-2 bifurcation point. This solution corresponds to a heteroclinic connection between the equilibrium x_e and the bifurcating periodic solution of small-amplitude; see Figure 6 for an illustration. We refer to Champneys & Kuznetsov (1994) for more details.

A similar situation arises if we continue along a primary branch of homoclinic solutions towards a fold bifurcation. This scenario is explained in more detail in Section 8.3.

8.2 Test functions for homoclinic flip bifurcations

Homoclinic flip bifurcations are related to a discontinuous change of the exponential rate of convergence of either the homoclinic orbit itself or a certain solution to the adjoint variational equation along the homoclinic loop; see below.

Let $n_c = 0$ so that $x_e(s)$ is hyperbolic and assume that the leading eigenvalues are simple and real, i.e.,

$$\mathrm{Re}\{\lambda_{n_s}^s\} \leq \cdots \leq \mathrm{Re}\{\lambda_2^s\} < \lambda_1^s < 0 < \lambda_1^u < \mathrm{Re}\{\lambda_2^u\} \leq \cdots \leq \mathrm{Re}\{\lambda_{n_u}^u\}.$$

Hence, the matrix $A = f_x(x_e(s), \alpha(s))$ has normalized eigenvectors v_1^s and v_1^u associated with the eigenvalues λ_1^s and λ_1^u, respectively. Analogously, the adjoint matrix A^* has normalized eigenvectors w_1^s and w_1^u belonging to λ_1^s and λ_1^u. These quantities depend smoothly upon s.

In this situation, a codimension-1 homoclinic orbit is expected to converge to zero exponentially with rates λ_1^s and λ_1^u as $t \to \pm\infty$, respectively. An *orbit-flip bifurcation* occurs if the homoclinic orbit picks up a faster exponential rate, i.e., if it is contained in either the strong stable or the strong unstable manifold of the equilibrium; see Figure 7(a).

If the homoclinic solution is contained in the strong stable manifold, we have

$$\lim_{t\to\infty} e^{-\lambda_1^s t}\,(x(t) - x_e)^*\, w_1^s = 0,$$

see Sandstede (1993). The following test function detects this codimension-2 bifurcation for the truncated problem.

Orbit-flip (in the stable manifold):

$$\psi = e^{-\lambda_1^s T_+}\,(x(T_+) - x_e)^*\, w_1^s$$

The exponential factor guarantees that the test function converges to the test function of the original problem; without it, this limit would be identically equal to zero. Analogously, the test function

Orbit-flip (in the unstable manifold):

$$\psi = e^{\lambda_1^u T_-}\,(x(-T_-) - x_e)^*\, w_1^u$$

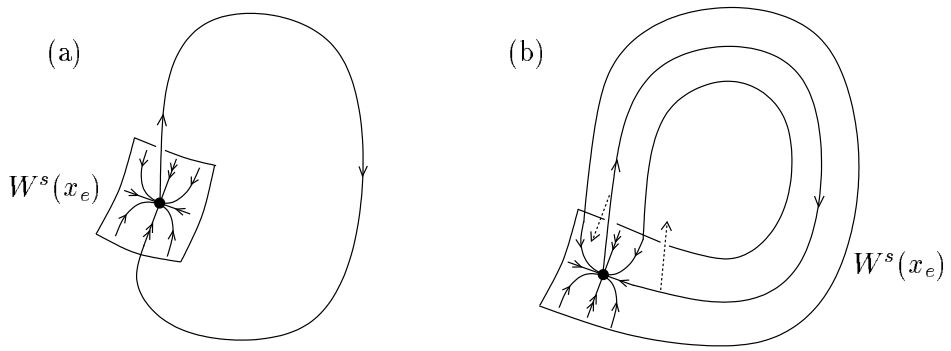

Figure 7: Illustrations of a homoclinic orbit at (a) an orbit-flip and (b) an inclination-flip bifurcation both occurring in the stable manifold. The dotted vectors shown in the right-hand picture are the solution $y_h(t) \perp W^s(x_e)$ of the adjoint variational equation.

detects homoclinic orbits in the strong unstable manifold.

Next, we focus on inclination-flip bifurcations. For any regular homoclinic orbit $x_h(t)$ to a hyperbolic equilibrium x_e, the *adjoint variational equation*

$$y'(t) = -f_x^*(x_h(t), \alpha_0)\, y(t) \tag{52}$$

has a unique bounded solution $y_h(t)$, which in fact converges to zero exponentially. This solution has the following geometric interpretation. Since the homoclinic orbit $x_h(t)$ is regular, we have that $T_{x_h(t)}W^s(x_e) \cap T_{x_h(t)}W^u(x_e)$ is one-dimensional by Definition 6.1. In particular, the codimension of $T_{x_h(t)}W^s(x_e) + T_{x_h(t)}W^u(x_e)$ is one. The aforementioned solution $y_h(t)$ to (52) satisfies

$$y_h(t) \perp (T_{x_h(t)}W^s(x_e) + T_{x_h(t)}W^u(x_e))$$

for all t. In analogy to the situation for homoclinic orbits, $y_h(t)$ is expected to decay exponentially with rates λ_1^u and λ_1^s as $t \to \pm\infty$. An *inclination-flip bifurcation* occurs if the solution $y_h(t)$ converges to zero with a higher exponential rate. We refer to Figure 7(b) for an illustration.

In order to detect inclination flips, we have to calculate the solution $y_h(t)$. Note that $y_h(t)$ can be regarded as a homoclinic orbit to (52). We can then follow the procedure outlined in Section 6 in order to derive an appropriate truncated boundary-value problem. Since (52) is non-autonomous, we do not need a phase condition. It is replaced by a condition which fixes the amplitude of $y_h(t)$; note that (52) is linear. For the boundary conditions, let $L_s^*(\alpha)$ be the $n_s \times n$ matrix whose rows form a basis for the stable eigenspace of $f_x(x_e(\alpha), \alpha)$. Similarly, $L_u^*(\alpha)$ is a $n_u \times n$ matrix such that its rows form a basis for the unstable eigenspace of $f_x(x_e(\alpha), \alpha)$. The truncated boundary-value problem for the computation of $y_h(t)$ is then given by

$$\begin{aligned} y'(t) + f_x^*(x(t), \alpha)\, y(t) + \varepsilon\, f(x(t), \alpha) &= 0, \\ L_s^*(\alpha) y(T_+) &= 0, \\ L_u^*(\alpha) y(-T_-) &= 0, \\ \int_{-T_-}^{T_+} (y(t) - \tilde{y}(t))^*\, \tilde{y}(t)\, dt &= 0, \end{aligned}$$

where $x = x(t)$ denotes the homoclinic solution, and $\tilde{y}(t)$ is such that

$$\int_{-T_-}^{T_+} y_h^*(t)\, \tilde{y}(t)\, dt \neq 0.$$

The truncated system is then solved with respect to (y, ε) for a given function $x(t)$ at the parameter value α. Here, $\varepsilon \in \mathbf{R}^1$ is an artificial parameter, which makes the boundary-value problem well-posed. We refer to Champneys et al. (1996) for more details.

Analogously to the case of an orbit-flip bifurcation for the homoclinic orbit, we then obtain the following test function for inclination-flip bifurcations.

Inclination-flip (of the stable manifold):

$$\psi = e^{-\lambda_1^s T_-}\, y^*(-T_-)\, v_1^s.$$

Inclination-flip (of the unstable manifold):

$$\psi = e^{\lambda_1^u T_+}\, y^*(T_+)\, v_1^u.$$

It can be shown that these test functions are well defined.

8.3 Test functions detecting non-central homoclinic orbits

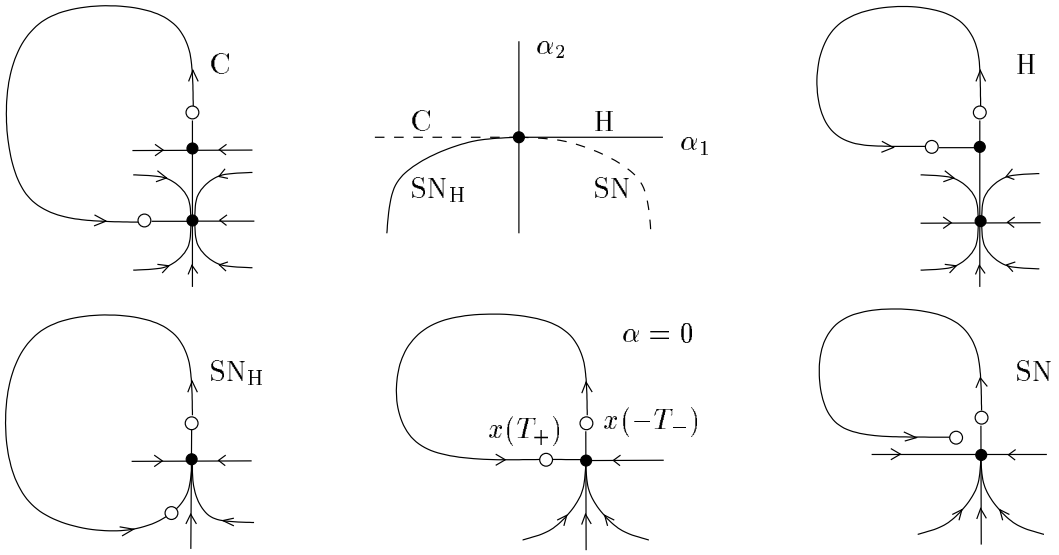

Figure 8: Continuation through a fold bifurcation arising at $\alpha = 0$. If a branch of central homoclinic orbits is continued, we have $\alpha \in \mathrm{SN_H}$. The algorithm will then continue through the fold bifurcation by computing solutions along the curve SN. On the other hand, if the branch H of homoclinic orbits to hyperbolic equilibria is computed, the algorithm continues along the branch C.

Suppose that a branch of central homoclinic orbits is continued in two parameters. Recall that in this case the truncated boundary-value problem is composed of (38), (41)–(43) plus conditions for the continuation of the saddle-node equilibrium; see Section 5 for the latter. One is then interested in detecting a transition to non-central homoclinic orbits.

Note that, by assumption, the equilibrium $x_e(s)$ has precisely one central eigenvalue $\lambda_1^c = 0$, i.e., $n_c = 1$. Let w_1^c be a normalized eigenvector corresponding to $\lambda_1^c = 0$ of the adjoint matrix $A^* = f_x^*(x_e(s), \alpha(s))$. The following test functions will then detect non-central saddle-node homoclinic orbits.

Non-central saddle-node homoclinic orbit (in center-stable and center-unstable manifold):

$$\psi = \frac{1}{T_+}(x(T_+) - x_e)^* \, w_1^c, \qquad \psi = \frac{1}{T_-}(x(-T_-) - x_e)^* \, w_1^c.$$

These functions measure the spectral projection of the end points of the approximate homoclinic orbit onto the one-dimensional center space. Note that, for the original problem on the infinite time interval, the branch $\mathrm{SN_H}$ of central homoclinic orbits cannot be continued beyond the point $\alpha = 0$ since no homoclinic loop exists along the branch SN; see Figure 8. For $(-T_-, T_+) = \mathbf{R}^1$, the above test functions are defined on account of Schecter (1993, Lemma 3.1), are smooth along $\mathrm{SN_H}$ including the point $\alpha = 0$, and have a smooth (artificial) extension along the curve SN. For finite time intervals, the test functions are also smooth and they converge to the test functions of the original problem as $T_-, T_+ \to \infty$ along the branch $\mathrm{SN_H}$ including $\alpha = 0$. Note, however, that the artifical solution of the truncated problem along the curve SN shown in Figure 8 has no limit as $T_- \to \infty$.

Finally, suppose that we continue along a primary branch H of homoclinic solutions to hyperbolic equilibria towards a fold bifurcation. At the fold bifurcation, another equilibrium is created. To be definite, we suppose that an unstable eigenvalue approaches zero; see Figure 8. Beyond the bifurcation point, the solution to the truncated boundary-value problem is then a heteroclinic orbit connecting the newly created equilibrium to the original equilibrium. In other words, the algorithms continues on the branch C shown in Figure 8; see Champneys et al. (1996) for more details.

9 Continuation of Codimension-2 Equilibrium Bifurcations

In this section we give regular defining systems based on bordering techniques for continuing codimension-2 equilibrium bifurcations of (1) in three parameters. Detailed proofs of regularity can be found in (Govaerts 2000). Test functions to detect codimension-3 bifurcations due to linear terms are also given. All defining and test functions are implemented in CONTENT (Kuznetsov & Levitin 1997). Earlier methods based on minimally augmented systems with determinants were proposed by Khibnik (1990) and implemented in LOCBIF (Khibnik et al. 1993).

9.1 Bogdanov-Takens

For computational purposes a Bogdanov-Takens point is characterized by the fact that the Jacobian matrix $A = f_x(x, \alpha)$ has a double eigenvalue zero with geometric multiplicity one and no other eigenvalues on the imaginary axis (the dependence on (x, α) will be suppressed in the following). In particular, the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$ satisfies

$$\left\{ \begin{array}{rcl} p(0) & = & 0, \\ p_\lambda(0) & = & 0, \end{array} \right. \tag{53}$$

and A has rank defect 1. Hence there exist vectors $v_1, w_1 \in \mathbf{R}^n$ such that

$$M(\lambda) = \begin{pmatrix} A - \lambda I_n & w_1 \\ v_1^* & 0 \end{pmatrix}$$

is nonsingular in a neighborhood of $\lambda = 0$. If we define $v(\lambda) \in \mathbf{R}^n$ and $g(\lambda) \in \mathbf{R}$ by solving

$$M(\lambda) \begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} 0_n \\ 1 \end{pmatrix}, \tag{54}$$

then

$$g(\lambda) = \frac{p(\lambda)}{\det M(\lambda)}.$$

Differentiating (54) with respect to λ we obtain

$$M(\lambda) \begin{pmatrix} v_\lambda \\ g_\lambda \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}, \tag{55}$$

$$M(\lambda) \begin{pmatrix} v_{\lambda\lambda} \\ g_{\lambda\lambda} \end{pmatrix} = \begin{pmatrix} 2v_\lambda \\ 0 \end{pmatrix}. \tag{56}$$

The defining equations for the Bogdanov-Takens (BT) curve are the equilibrium equation (2) supplemented by

$$\begin{cases} g(0) &= 0, \\ g_\lambda(0) &= 0. \end{cases}$$

These two equations are mathematically equivalent to (53), but they are better scaled and their derivatives can be computed more easily.

In practical computations v_1 and w_1 are chosen as approximations to the normalized right and left nullvectors of the local Jacobian matrix A. On a BT-curve, one can encounter the following singularities of certain linear terms:

1. Triple zero eigenvalue: $\psi_1 = 0$.
2. Hopf-BT : $\psi_2 = 0$.

Here

$$\begin{cases} \psi_1 &= g_{\lambda\lambda} \\ \psi_2 &= g_{22}, \end{cases}$$

where $g_{\lambda\lambda}$ is obtained from (56). The construction of g_{22} is more complicated and requires auxiliary data $v_{1b}, v_{2b}, w_{1b}, w_{2b} \in \mathbf{R}^m$, $2m = n(n-1)$, and scalars d_{12}, d_{21} such that the matrix

$$M_b = \begin{pmatrix} 2\,A \odot I_n & w_{1b} & w_{2b} \\ v_{1b}^* & 0 & d_{12} \\ v_{2b}^* & d_{21} & 0 \end{pmatrix}$$

is nonsingular. Then g_{22} is computed by solving

$$M_b \begin{pmatrix} V \\ G \end{pmatrix} = \begin{pmatrix} 0_m \\ I_2 \end{pmatrix},$$

where

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

The vectors v_{1b}, w_{1b} are chosen as normalized right and left nullvectors of $2A \odot I_n$, respectively, initialized and updated essentially like v_1 and w_1. Finally, the vectors

$$\begin{pmatrix} v_{2b} \\ d_{21} \end{pmatrix}, \begin{pmatrix} w_{2b} \\ d_{12} \end{pmatrix}$$

are updated jointly to make M_b as well-conditioned as possible.

9.2 Fold-Hopf

A fold-Hopf point is characterized by the fact that the Jacobian matrix $A = f_x(x, \alpha)$ has an algebraically simple eigenvalue zero, a pair of pure imaginary algebraically simple eigenvalues $\pm i\omega_0$, $\omega_0 > 0$, and no other eigenvalues on the imaginary axis. This implies that there exist vectors $v_1, w_1 \in \mathbf{R}^n$, $v_{1b}, v_{2b}, w_{1b}, w_{2b} \in \mathbf{R}^m$, $2m = n(n-1)$, and scalars d_{12}, d_{21} such that the matrices

$$M = \begin{pmatrix} A & w_1 \\ v_1^* & 0 \end{pmatrix}$$

and

$$M_b = \begin{pmatrix} 2\,A \odot I_n & w_{1b} & w_{2b} \\ v_{1b}^* & 0 & d_{12} \\ v_{2b}^* & d_{21} & 0 \end{pmatrix}$$

are nonsingular. The defining equations for the fold-Hopf curve are the equilibrium equation (2) and

$$\begin{cases} g & = & 0 \\ \det G & = & 0, \end{cases}$$

where the scalar g results from solving

$$M \begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} 0_n \\ 1 \end{pmatrix}$$

and the 2×2 matrix

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{57}$$

is obtained by solving the system

$$M_b \begin{pmatrix} V \\ G \end{pmatrix} = \begin{pmatrix} 0_m \\ I_2 \end{pmatrix}. \tag{58}$$

The bordering rows and columns in M and M_b are initialized and updated essentially as in the BT case.

Along the fold-Hopf curve, the following test functions can be computed :

$$\begin{cases} \psi_1 & = & g_{22}, \\ \psi_2 & = & g_\lambda(0), \end{cases}$$

where g_{22} is defined by (57), (58), and g_λ is obtained from (55).

The following linear singularities can be detected and located as regular zeros of the aforementioned test functions :

1. Fold + double-Hopf : $\psi_1 = 0$, $\psi_2 \neq 0$.
2. Hopf + BT : $\psi_1 = 0$, $\psi_2 = 0$.
3. Triple zero eigenvalue: $\psi_1 \neq 0$, $\psi_2 = 0$.

9.3 Double-Hopf

A double-Hopf point is characterized by the fact that the Jacobian matrix $A = f_x$ has two pairs of purely imaginary algebraically simple eigenvalues $\pm i\omega_1, \pm i\omega_2$, $\omega_1, \omega_2 > 0$ and no other eigenvalues on the imaginary axis. This implies that $2A \odot I_n$, where $A = f_x(x, \alpha)$, has rank defect 2 and there exist vectors $v_{1b}, v_{2b}, w_{1b}, w_{2b} \in \mathbf{R}^{n(n-1)/2}$ such that the matrix

$$M_b = \begin{pmatrix} 2\,A \odot I_n & w_{1b} & w_{2b} \\ v_{1b}^* & 0 & 0 \\ v_{2b}^* & 0 & 0 \end{pmatrix}$$

is nonsingular. The defining equations for the double-Hopf curve are the equilibrium equation (2) and

$$\begin{cases} g_{i_1 j_1} &= 0 \\ g_{i_2 j_2} &= 0 \end{cases}$$

where $g_{ij} = g_{ij}(u, \alpha)$ are the components of the matrix

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

obtained by solving the system

$$M_b \begin{pmatrix} V \\ G \end{pmatrix} = \begin{pmatrix} 0_m \\ I_2 \end{pmatrix}.$$

The vectors $v_{1b}, v_{2b}, w_{1b}, w_{2b}$ and the indices i_1, j_1, i_2, j_2 are updated along the curve in the following fashion. The vectors v_{1b}, v_{2b} are chosen to form an orthogonal basis of the right null space of $2A \odot I_n$, and w_{1b}, w_{2b} are chosen similarly for the left null space. The choice of (i_1, j_1), (i_2, j_2) is such that the space spanned by the gradient vector $g_{i_1 j_1 z}$ has the largest component orthogonal to the equilibrium surface and $g_{i_2 j_2 z}$ has the largest component orthogonal to the space spanned by both the tangent space to the equilibrium surface and $g_{i_1 j_1 z}$; here z ranges over the state variables and free parameters.

Along the double-Hopf curve, the following test functions can be computed :

$$\begin{cases} \psi_1 &= \det G^1, \\ \psi_2 &= \det A, \\ \psi_3 &= p^* q, \end{cases}$$

where the matrix G^1 is obtained by solving

$$M_b \begin{pmatrix} V^1 \\ G^1 \end{pmatrix} = \begin{pmatrix} V \\ 0 \end{pmatrix},$$

and $p \in \mathbf{R}^n$ and $q \in \mathbf{R}^n$ are obtained by solving

$$\begin{pmatrix} A & e_{p_2} \\ e_{p_1}^* & 0 \end{pmatrix} \begin{pmatrix} p \\ s \end{pmatrix} = \begin{pmatrix} 0_n \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} q^* & s \end{pmatrix} \begin{pmatrix} A & e_{p_2} \\ e_{p_1}^* & 0 \end{pmatrix} = \begin{pmatrix} 0_n^* & 1 \end{pmatrix}.$$

Above, e_{p_i} is the $(p_i)^{th}$ unit vector and s denotes a dummy real number. The values for p_1 and p_2 are obtained by looking for the smallest pivot elements in the decomposition with complete pivoting of the (Jacobian) matrix $A = f_x(x, \alpha)$.

The following linear singularities can be detected and located as regular zeros of the above defined test functions :

1. Resonant double-Hopf : $\psi_1 = 0$.
2. Fold + double-Hopf : $\psi_2 = 0, \psi_3 \neq 0$.
3. Hopf + BT : $\psi_2 = 0, \psi_3 = 0$.

9.4 Cusp

Since a cusp is a fold point where $A = f_x$ has rank defect 1, there exist vectors $v_1, w_1 \in \mathbf{R}^n$ such that

$$M = \begin{pmatrix} A & w_1 \\ v_1^* & 0 \end{pmatrix}$$

is nonsingular. In addition, there has to be a degeneracy in the second derivatives. The defining system of the cusp curve then consists of the equilibrium equation (2) together with

$$\begin{cases} g(x, \alpha) &= 0, \\ g^1(x, \alpha) &= 0, \end{cases} \tag{59}$$

where g is obtained by solving the bordered $(n+1)-$dimensional system

$$M \begin{pmatrix} v(x, \alpha) \\ g(x, \alpha) \end{pmatrix} = \begin{pmatrix} 0_n \\ 1 \end{pmatrix},$$

and g^1 is obtained by solving

$$M \begin{pmatrix} v^1(x, \alpha) \\ g^1(x, \alpha) \end{pmatrix} = \begin{pmatrix} -B(v, v) \\ 0 \end{pmatrix},$$

where $B(v, v) = f_{xx}(x, \alpha)vv$.

9.5 Generalized Hopf

9.5.1 Minimally augmented system

This method is suitable when the Jacobian matrix of the defining equations is computed numerically. The idea is, as in the simple Hopf case, that at a generalized Hopf point the bialternate-product matrix $2A \odot I_n$ is singular. Here $A = f_x(x, \alpha)$. The auxiliary data are vectors $v_{1b}, v_{2b}, w_{1b}, w_{2b} \in \mathbf{R}^n$ and scalars d_{12}, d_{21} such that the matrix

$$M_b = \begin{pmatrix} 2A \odot I_n & w_{1b} & w_{2b} \\ v_{1b}^* & 0 & d_{12} \\ v_{2b}^* & d_{21} & 0 \end{pmatrix}$$

is nonsingular.

The defining equations for the generalized Hopf (Bautin) curve are

$$\left\{ \begin{array}{rcl} f(x,\alpha) & = & 0 \\ \det G(x,\alpha) & = & 0 \\ l_1(x,\alpha) & = & 0, \end{array} \right. \tag{60}$$

where the matrix

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

is obtained by solving the system

$$M_b \begin{pmatrix} V \\ G \end{pmatrix} = \begin{pmatrix} 0_m \\ I_2 \end{pmatrix},$$

and where l_1 denotes the *first Lyapunov coefficient* defined as

$$l_1 \;=\; \frac{1}{2}\mathrm{Re}\,\langle p, C(q,q,\overline{q}) - 2B(q, A^{-1}B(q,\overline{q})) + B(\overline{q}, (2i\omega_0 I_n - A)^{-1}\, B(q,q))\rangle$$

(see formula (47)). The complex vectors $p, q \in \mathbf{C}^n$ satisfy

$$Aq = \lambda q, \quad A^* p = \bar{\lambda} p, \quad \langle \mathrm{Re}\; q, \mathrm{Re}\; q\rangle = \langle p, q\rangle = 1, \quad \langle \mathrm{Re}\, q, \mathrm{Im}\, q\rangle = 0,$$

where $\lambda = i\omega_0$ along the curve (60) and the last condition is added to ensure smoothness of the vector q. The multilinear functions $B(p,q)$ and $C(p,q,r)$ are defined by (45) and thought of as functions of (x,α).

9.5.2 Standard augmented system

We now describe another computational scheme for the generalized Hopf bifurcation. When available, symbolic derivatives up to order 4 can be used in this scheme. The continuation data consist of $8n + 5$ real numbers. As before, it is convenient to express some of the equations in complex form.

The idea is to express first that $A = f_x(x,\alpha)$ has an imaginary eigenvalue $i\omega_0$ with right eigenvector $q \in \mathbf{C}^n$ and left eigenvector $p \in \mathbf{C}^n$, and then to add the condition that the first Lyapunov value vanishes. To fix the right and left eigenvectors we add the normalization conditions $\langle q_0, q\rangle = \langle p, q\rangle = 1$, where $q_0 \in \mathbf{C}^n$ is the normalized right eigenvector q at a previously computed point on the curve. To simplify the expression for l_1, we introduce $v \in \mathbf{R}^n$ and $w \in \mathbf{C}^n$ as additional unknowns, where

$$v = A^{-1}B(q,\overline{q}), \qquad w = (2\,i\,\omega_0\, I_n - A)^{-1}\, B(q,q).$$

Thus, the continuation space consists of the state variables, the parameter variables, *and* the components of

$$(q, p, v, w, \omega_0, \lambda).$$

The defining equations for the generalized Hopf (Bautin) curve are then given by (2) supplemented by the complex system

$$\left\{\begin{array}{rcl} Aq - i\omega_0 q & = & 0 \\ A^T p + \lambda p & = & 0 \\ \langle q_0, q\rangle - 1 & = & 0 \\ \langle p, q\rangle - 1 & = & 0 \\ Av - B(q,\overline{q}) & = & 0 \\ (2i\omega_0 I_n - A)w - B(q,q) & = & 0 \\ \text{Re}\,\langle p, C(q,q,\overline{q}) - 2B(q,v) + B(\overline{q},w)\rangle & = & 0. \end{array}\right. \tag{61}$$

The complex variable λ is introduced artificially to regularize the system; formally, along the generalized Hopf curve, $\lambda = i\omega_0$. When written in real variables, (2) and (61) form a system of $8n+5$ equations with $8n+6$ variables including the three free parameters.

10 Normal Forms for Codimension-2 Equilibrium Bifurcations

Bifurcations of phase portraits of (1) are determined by the normal form coefficients at critical parameter values. For example, depending on the values of certain coefficients for the fold-Hopf and double-Hopf bifurcation, the system may exhibit quasi-periodic and "chaotic" behavior. In this section we show how such coefficients can be computed numerically, while the final section deals with the nondegeneracy of parametric unfolding. If the critical parameter values and the equilibrium position are known exactly then symbolic manipulation software such as MAPLE or Mathematica can be applied; see, for example, Sanders (1994).

10.1 List of codimension-2 normal forms

As we have seen in Section 7, there are five codimension-2 equilibrium bifurcations:

1. **The cusp** ($\lambda_1 = 0,\ a = 0$). Equation (1) restricted to the center manifold at the critical parameter values is given by

$$w' = bw^3 + O(w^4), \qquad w \in \mathbf{R}^1. \tag{62}$$

   If $b \neq 0$ and if the system (1) depends generically on the two parameters (α_1, α_2), then its restriction to the center manifold is locally topologically equivalent to the normal form

$$w' = \beta_1 + \beta_2 w + cw^3,$$

   where (β_1, β_2) are unfolding parameters that are related to (α_1, α_2) via an invertible smooth transformation. This normal form predicts a hysteresis phenomenon near the bifurcation.

2. **Bogdanov-Takens** ($\lambda_{1,2} = 0$). The restriction of (1) to the center manifold at the critical parameter values is locally smoothly equivalent to the normal form

$$\left\{\begin{array}{rcl} w_0' & = & w_1 \\ w_1' & = & aw_0^2 + bw_0 w_1 + O(\|w\|^3), \end{array}\right. \tag{63}$$

where $w = (w_0, w_1)^* \in \mathbf{R}^2$. If $ab \neq 0$, and if the parameters (α_1, α_2) enter (1) generically, then the restricted system is locally topologically equivalent to the normal form

$$\begin{cases} w_0' &= w_1 \\ w_1' &= \beta_1 + \beta_2 w_0 + a w_0^2 + b w_0 w_1. \end{cases}$$

An analysis of this normal form reveals a parameter plane curve of saddle homoclinic bifurcations emanating from the codimension-2 point. The unique periodic solution family born in the Hopf bifurcation approaches the homoclinic orbit and terminates there as its period $T \to \infty$.

3. **Generalized Hopf** ($\lambda_{1,2} = \pm i\omega_0$, $l_1 = 0$). The restriction of (1) to the center manifold at the critical parameter values is locally smoothly orbitally equivalent to the normal form

$$w' = i\omega_0 w + l_2 w|w|^4 + O(|w|^6), \quad w \in \mathbf{C}^1, \tag{64}$$

where the *second Lyapunov coefficient* l_2 is real. If $l_2 \neq 0$ then, generically, the restricted system (1) is locally topologically equivalent to the normal form

$$w' = (\beta_1 + i\omega_0)w + \beta_2 w|w|^2 + l_2 w|w|^4. \tag{65}$$

This normal form predicts the existence of a curve originating at the codimension-2 point in the parameter plane, where two periodic orbits collide and disappear through a non-hyperbolic periodic solution with a nontrivial multiplier $\mu_1 = 1$.

4. **Fold-Hopf** ($\lambda_1 = 0$, $\lambda_{2,3} = \pm i\omega_0$). The normalized restriction of (1) to the center manifold at the critical parameter values has the form

$$\begin{cases} w_0' &= \frac{1}{2}G_{200}w_0^2 + G_{011}|w_1|^2 + \frac{1}{6}G_{300}w_0^3 \\ & \quad + G_{111}w_0|w_1|^2 + O(\|(w_0, w_1, \overline{w}_1)\|^4) \\ w_1' &= i\omega_0 w_1 + G_{110}w_0 w_1 + \frac{1}{2}G_{210}w_0^2 w_1 + \frac{1}{2}G_{021}w_1|w_1|^2 \\ & \quad + O(\|(w_0, w_1, \overline{w}_1)\|^4). \end{cases} \tag{66}$$

Here $w_0 \in \mathbf{R}^1$ and $w_1 \in \mathbf{C}^1$, the coefficients G_{klm} in the first equation are real, while those in the second equation are complex. If $G_{200}G_{011} \neq 0$, then, generically, the restriction of (1) to the center manifold is locally smoothly orbitally equivalent to the system

$$\begin{cases} u' &= \beta_1 + bu^2 + c|z|^2 + O(\|(u, z, \overline{z})\|^4) \\ z' &= (\beta_2 + i\omega)z + duz + eu^2 z + O(\|(u, z, \overline{z})\|^4), \end{cases} \tag{67}$$

where ω, b, c, e are real functions of β, while d is a complex function of β with

$$\omega(0) = \omega_0, \qquad b(0) = \frac{1}{2}G_{200}, \qquad c(0) = G_{011}, \qquad d(0) = G_{110} - i\omega_0 \frac{G_{300}}{3G_{200}},$$

and

$$e(0) = \frac{1}{2}\,\mathrm{Re}\left[G_{210} + G_{110}\left(\frac{\mathrm{Re}\; G_{021}}{G_{011}} - \frac{G_{300}}{G_{200}} + \frac{G_{111}}{G_{011}}\right) - \frac{G_{021}G_{200}}{2G_{011}}\right].$$

In general, the O-terms in (67) cannot be truncated, since they affect the topology of the bifurcation diagram of the system near the bifurcation. Depending on the coefficients b, c, d, e the system can have two-dimensional invariant tori, chaotic dynamics, Neimark-Sacker bifurcations, and Shil'nikov homoclinic bifurcations.

5. **Double-Hopf** ($\lambda_{1,2} = \pm i\omega_1,\ \lambda_{3,4} = \pm i\omega_2$). Assume that

$$k\omega_1 \neq l\omega_2, \qquad k,l > 0, \qquad k+l \leq 5, \tag{68}$$

where k, l are integer numbers. The normalized restriction of system (1) to the center manifold has then the form

$$\left\{\begin{array}{rcl} w_1' &=& i\omega_1 w_1 + \frac{1}{2}G_{2100}w_1|w_1|^2 + G_{1011}w_1|w_2|^2 \\ && + \frac{1}{12}G_{3200}w_1|w_1|^4 + \frac{1}{2}G_{2111}w_1|w_1|^2|w_2|^2 + \frac{1}{4}G_{1022}w_1|w_2|^4 \\ && + O(\|(w_1,\overline{w}_1,w_2,\overline{w}_2)\|^6) \\ w_2' &=& i\omega_2 w_2 + G_{1110}w_2|w_1|^2 + \frac{1}{2}G_{0021}w_2|w_2|^2 \\ && + \frac{1}{4}G_{2210}w_2|w_1|^4 + \frac{1}{2}G_{1121}w_2|w_1|^2|w_2|^2 + \frac{1}{12}G_{0032}w_2|w_2|^4 \\ && + O(\|(w_1,\overline{w}_1,w_2,\overline{w}_2)\|^6), \end{array}\right. \tag{69}$$

where $(w_1, w_2)^* \in \mathbf{C}^2$, and $G_{jklm} \in \mathbf{C}^1$. Moreover, if

$$(\mathrm{Re}\ G_{2100})(\mathrm{Re}\ G_{1011})(\mathrm{Re}\ G_{1110})(\mathrm{Re}\ G_{0021}) \neq 0$$

and the critical eigenpairs cross the imaginary axis with nonzero velocities, then the system (1) is locally smoothly orbitally equivalent near the bifurcation to the system

$$\left\{\begin{array}{rcl} v_1' &=& (\beta_1 + i\omega_1)v_1 + \frac{1}{2}P_{11}v_1|v_1|^2 + P_{12}v_1|v_2|^2 \\ && + iR_1v_1|v_1|^4 + \frac{1}{4}S_1v_1|v_2|^4 + O(\|(v_1,\overline{v}_1,v_2,\overline{v}_2)\|^6), \\ v_2' &=& (\beta_2 + i\omega_2)v_2 + P_{21}v_2|v_1|^2 + \frac{1}{2}P_{22}v_2|v_2|^2 \\ && + \frac{1}{4}S_2v_2|v_1|^4 + iR_2v_2|v_2|^4 + O(\|(v_1,\overline{v}_1,v_2,\overline{v}_2)\|^6), \end{array}\right. \tag{70}$$

where $(v_1, v_2)^* \in \mathbf{C}^2$, and the coefficients P_{jk} and S_k are complex, while the numbers R_k are real. Moreover, the real parts of the critical values are given by the expressions

$$\mathrm{Re}\ P_{11} = \mathrm{Re}\ G_{2100}, \quad \mathrm{Re}\ P_{12} = \mathrm{Re}\ G_{1011}, \quad \mathrm{Re}\ P_{21} = \mathrm{Re}\ G_{1110}, \quad \mathrm{Re}\ P_{22} = \mathrm{Re}\ G_{0021},$$

and

$$\mathrm{Re}\ S_1 = \mathrm{Re}\ G_{1022} + \frac{1}{3}\mathrm{Re}\ G_{1011}\left[6\frac{\mathrm{Re}\ G_{1121}}{\mathrm{Re}\ G_{1110}} - 4\frac{\mathrm{Re}\ G_{0032}}{\mathrm{Re}\ G_{0021}} - \frac{(\mathrm{Re}\ G_{3200})(\mathrm{Re}\ G_{0021})}{(\mathrm{Re}\ G_{2100})(\mathrm{Re}\ G_{1110})}\right],$$

$$\mathrm{Re}\ S_2 = \mathrm{Re}\ G_{2210} + \frac{1}{3}\mathrm{Re}\ G_{1110}\left[6\frac{\mathrm{Re}\ G_{2111}}{\mathrm{Re}\ G_{1011}} - 4\frac{\mathrm{Re}\ G_{3200}}{\mathrm{Re}\ G_{2100}} - \frac{(\mathrm{Re}\ G_{2100})(\mathrm{Re}\ G_{0032})}{(\mathrm{Re}\ G_{1011})(\mathrm{Re}\ G_{0021})}\right].$$

As in the fold-Hopf case, the O-terms in (70) cannot be truncated, since they affect the topology of the bifurcation diagram of the system. Depending on the values of the normal form coefficients, the system can have invariant tori and chaotic dynamics, as well as Neimark-Sacker bifurcations and Shil'nikov homoclinic bifurcations.

Proofs of the results formulated above can be found in (Kuznetsov 1995) with relevant bifurcation diagrams and bibliographical references.

10.2 The normalization method

Our aim is to derive efficient formulas to numerically compute the coefficients of the normal forms (62), (63), (64), (66), and (69).

The following normalization technique is due to Coullet and Spiegel (Coullet & Spiegel 1983) (see also (Elphick, Tirapegui, Brachet, Coullet & Iooss 1987)). Suppose that the system (1) has the equilibrium $x = 0$ at $\alpha = 0$, and suppose that the Jacobian matrix $A = f_x(0,0)$ has n_c eigenvalues with zero real part (counting multiplicities). Let T^c be the corresponding generalized critical eigenspace of A. Write the system at $\alpha = 0$ as

$$x' = F(x), \qquad x \in \mathbf{R}^n, \tag{71}$$

with F given by (45), and restrict it to its n_c-dimensional invariant center manifold parametrized by $w \in \mathbf{R}^{n_c}$:

$$x = H(w), \qquad H : \mathbf{R}^{n_c} \to \mathbf{R}^n, \tag{72}$$

The restricted equation can be written as

$$w' = G(w), \qquad G : \mathbf{R}^{n_c} \to \mathbf{R}^{n_c}. \tag{73}$$

Substitution of (72) and (73) into (71) gives the following *homological equation*:

$$H_w(w)G(w) = F(H(w)). \tag{74}$$

We expand the functions G, H in (74) into multivariate Taylor series,

$$G(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} g_\nu w^\nu, \qquad H(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} h_\nu w^\nu,$$

and assume that the restricted equation (73) is put into the normal form up to a certain order. The coefficients g_ν of the normal form (73) and the coefficients h_ν of the Taylor expansion for $H(w)$ are unknown, but can be found from (74) by a recursive procedure, from lower to higher order terms. (Obviously, one has $\sum_{|\nu|=1} h_\nu w^\nu \in T^c$.) Collecting the coefficients of the w^ν-terms in (74) gives a linear system for the coefficient h_ν

$$Lh_\nu = R_\nu. \tag{75}$$

Here the matrix L is determined by the Jacobian matrix A and its critical eigenvalues. The right-hand side R_ν depends on the coefficients of G and H of order less than or equal to $|\nu|$ as well as on the terms of order less than or equal to $|\nu|$ of the Taylor expansion (45) of F. When R_ν involves only known quantities, the system (75) has a solution because either L is nonsingular or R_ν satisfies Fredholm's solvability condition

$$\langle p, R_\nu \rangle = 0,$$

where p is a nullvector of the adjoint matrix L^*, and $\langle p, q \rangle \equiv p^* q$. When R_ν depends on the unknown coefficient g_ν of the normal form, L is singular and the above solvability condition gives the expression for g_ν.

For all codimension-2 bifurcations, except Bogdanov-Takens, the invariant subspace of $L(L^*)$ corresponding to the zero eigenvalue is one-dimensional in $\mathbf{C}^n$, i.e., there are unique (up to scaling) nullvectors q and p,

$$Lq = 0, \qquad L^* p = 0, \qquad \langle p, q \rangle = 1,$$

and no generalized nullvectors. Then the unique solution h_ν to (75) satisfying $\langle p, h_\nu \rangle = 0$ can be obtained by solving the following nonsingular $(n+1)$-dimensional *bordered system*:

$$\begin{pmatrix} L & q \\ p^* & 0 \end{pmatrix} \begin{pmatrix} h_\nu \\ s \end{pmatrix} = \begin{pmatrix} R_\nu \\ 0 \end{pmatrix}.$$

We write $h_\nu = L^{INV} R_\nu$.

The Taylor expansion of $H(w)$ simultaneously defines the expansions of the center manifold, the normalizing transformation on it, and the normal form itself. Since we know *a priori* which terms are present in the normal form, the described procedure is a powerful tool to compute the normal form coefficients at the bifurcation parameter values. In the following sections, we summarize results obtained by this method with the help of the symbolic computation software MAPLE. Details can be found in (Kuznetsov 1997).

10.3 The cusp bifurcation

At such a bifurcation point, the system (1) has an equilibrium with a simple zero eigenvalue $\lambda_1 = 0$ and no other critical eigenvalues. Let $q, p \in \mathbf{R}^n$ satisfy

$$Aq = 0, \qquad A^*p = 0, \qquad \langle p, q \rangle = 1.$$

The restriction of (1) to the corresponding center manifold has the form

$$w' = aw^2 + bw^3 + O(w^4),$$

with unknown coefficients a and b. Applying the normalization technique described above, one gets

$$a = \frac{1}{2}\langle p, B(q,q) \rangle \ ,$$

which already appears in (46) for the fold bifurcation.

For the coefficient b we obtain

$$b = \frac{1}{6}\langle p, C(q,q,q) + 3B(q, h_2) \rangle,$$

where $h_2 = -A^{INV}[B(q,q) - \langle p, B(q,q) \rangle q]$, which can be computed by solving the nonsingular $(n+1)$-dimensional bordered system

$$\begin{pmatrix} A & q \\ p^* & 0 \end{pmatrix} \begin{pmatrix} h_2 \\ s \end{pmatrix} = \begin{pmatrix} -B(q,q) + \langle p, B(q,q) \rangle q \\ 0 \end{pmatrix}.$$

Recall that $a = 0$ at the cusp bifurcation. Thus the coefficient b in the normal form (62) can be expressed more compactly as

$$b = \frac{1}{6}\langle p, C(q,q,q) - 3B(q, A^{INV} B(q,q)) \rangle \ .$$

10.4 Bogdanov-Takens bifurcation

Here the equilibrium of system (1) has a double zero eigenvalue $\lambda_{1,2} = 0$, and there exist two real, linearly independent, (generalized) eigenvectors, $q_0, q_1 \in \mathbf{R}^n$, such that

$$Aq_0 = 0, \qquad Aq_1 = q_0.$$

Moreover, there exist corresponding vectors $p_0, p_1 \in \mathbf{R}^n$ of the transposed matrix A^*:

$$A^*p_1 = 0, \qquad A^*p_0 = p_1.$$

One can choose these vectors so that they satisfy

$$\langle q_0, p_0\rangle = \langle q_1, p_1\rangle = 1, \qquad \langle q_1, p_0\rangle = \langle q_0, p_1\rangle = 0.$$

Then one obtains

$$a = \frac{1}{2}\langle p_1, B(q_0, q_0)\rangle \qquad \text{and} \qquad b = \langle p_0, B(q_0, q_0)\rangle + \langle p_1, B(q_0, q_1)\rangle \ ,$$

for the coefficients a, b of the normal form (63).

10.5 Bautin (generalized Hopf) bifurcation

At such a bifurcation point the system (1) has an equilibrium with a simple pair of purely imaginary eigenvalues, $\lambda_{1,2} = \pm i\omega_0,\ \omega_0 > 0$, and no other critical eigenvalues. As in the simple Hopf case, we introduce two complex eigenvectors $q, p \in \mathbf{C}^n$:

$$Aq = i\omega_0 q, \qquad A^*p = -i\omega_0 p,$$

and normalize them according to

$$\langle p, q\rangle \equiv p^*q = 1.$$

The normalized restriction of (1) to the two-dimensional center manifold can be written as

$$w' = i\omega_0 w + \frac{1}{2}G_{21}w|w|^2 + \frac{1}{12}G_{32}w|w|^4 + O(|w|^6),$$

where $G_{jk} \in \mathbf{C}^1$.

An application of the normal form algorithm gives the cubic normal form coefficient

$$G_{21} = \langle p, C(q, q, \overline{q}) + B(\overline{q}, (2i\omega_0 I_n - A)^{-1}B(q, q)) - 2B(q, A^{-1}B(q, \overline{q}))\rangle, \tag{76}$$

while $l_1 = \frac{1}{2}\mathrm{Re}\ G_{21}$ is given by (47). The *second Lyapunov coefficient* is given by

$$l_2 = \frac{1}{12}\mathrm{Re}\ G_{32},$$

with

$$\begin{aligned} G_{32} \ = \ & \langle p, E(q, q, q, \overline{q}, \overline{q}) \\ & + D(q, q, q, \overline{h}_{20}) + 3D(q, \overline{q}, \overline{q}, h_{20}) + 6D(q, q, \overline{q}, h_{11}) \\ & + C(\overline{q}, \overline{q}, h_{30}) + 3C(q, q, \overline{h}_{21}) + 6C(q, \overline{q}, h_{21}) + 3C(q, \overline{h}_{20}, h_{20}) \\ & + 6C(q, h_{11}, h_{11}) + 6C(\overline{q}, h_{20}, h_{11}) \\ & + 2B(\overline{q}, h_{31}) + 3B(q, h_{22}) + B(\overline{h}_{20}, h_{30}) + 3B(\overline{h}_{21}, h_{20}) + 6B(h_{11}, h_{21})\rangle, \end{aligned}$$

where

$$\begin{aligned} h_{20} &= (2i\omega_0 I_n - A)^{-1} B(q,q), \\ h_{11} &= -A^{-1} B(q,\overline{q}). \end{aligned}$$

The complex vector h_{21} is found by solving the nonsingular $(n+1)$-dimensional complex system

$$\begin{pmatrix} i\omega_0 I_n - A & q \\ p^* & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} C(q,q,\overline{q}) + B(\overline{q}, h_{20}) + 2B(q, h_{11}) - G_{21} q \\ 0 \end{pmatrix},$$

while

$$\begin{aligned} h_{30} &= (3i\omega_0 I_n - A)^{-1} [C(q,q,q) + 3B(q, h_{20})], \\ h_{31} &= (2i\omega_0 I_n - A)^{-1} [D(q,q,q,\overline{q}) + 3C(q,q,h_{11}) + 3C(q,\overline{q},h_{20}) + 3B(h_{20},h_{11}) \\ &\qquad + B(\overline{q}, h_{30}) + 3B(q, h_{21}) - 3G_{21} h_{20}], \\ h_{22} &= -A^{-1} [D(q,q,\overline{q},\overline{q}) + 4C(q,\overline{q},h_{11}) + C(\overline{q},\overline{q},h_{20}) + C(q,q,\overline{h}_{20}) \\ &\qquad + 2B(h_{11},h_{11}) + 2B(q,\overline{h}_{21}) + 2B(\overline{q}, h_{21}) + B(\overline{h}_{20}, h_{20}) - 2h_{11}(G_{21} + \overline{G}_{21})]. \end{aligned}$$

10.6 Fold-Hopf bifurcation

At a fold-Hopf bifurcation point of (1) the Jacobian matrix $A = f_x(0,0)$ has a simple zero eigenvalue $\lambda_1 = 0$ and a pair of purely imaginary simple eigenvalues:

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm i\omega_0,$$

with $\omega_0 > 0$, and no other critical eigenvalues. Introduce two eigenvectors, $q_0 \in \mathbf{R}^n$ and $q_1 \in \mathbf{C}^n$,

$$Aq_0 = 0, \qquad Aq_1 = i\omega_0 q_1,$$

and two adjoint eigenvectors, $p_0 \in \mathbf{R}^n$ and $p_1 \in \mathbf{C}^n$, with

$$A^* p_0 = 0, \qquad A^* p_1 = -i\omega_0 p_1.$$

Normalize these vectors such that

$$\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 1.$$

The following orthogonality properties hold: $\langle p_1, q_0 \rangle = \langle p_0, q_1 \rangle = 0$. One obtains the following expressions for the quadratic coefficients in (66):

$$G_{200} = \langle p_0, B(q_0, q_0) \rangle, \qquad G_{110} = \langle p_1, B(q_0, q_1) \rangle, \qquad G_{011} = \langle p_0, B(q_1, \overline{q}_1) \rangle,$$

and the following formulas for the cubic coefficients in (66):

$$\begin{aligned} G_{300} &= \langle p_0, C(q_0,q_0,q_0) + 3B(q_0, h_{200}) \rangle, \\ G_{111} &= \langle p_0, C(q_0,q_1,\overline{q}_1) + B(q_0, h_{011}) + B(q_1, \overline{h}_{110}) + B(\overline{q}_1, h_{110}) \rangle, \\ G_{210} &= \langle p_1, C(q_0,q_0,q_1) + 2B(q_0, h_{110}) + B(q_1, h_{200}) \rangle, \\ G_{021} &= \langle p_1, C(q_1,q_1,\overline{q}_1) + 2B(q_1, h_{011}) + B(\overline{q}_1, h_{020}) \rangle, \end{aligned}$$

where

$$\begin{aligned}
h_{200} &= -A^{INV}[B(q_0,q_0) - \langle p_0, B(q_0,q_0)\rangle q_0], \\
h_{020} &= (2i\omega_0 I_n - A)^{-1} B(q_1,q_1), \\
h_{110} &= (i\omega_0 I_n - A)^{INV}[B(q_0,q_1) - \langle p_1, B(q_0,q_1)\rangle q_1], \\
h_{011} &= -A^{INV}[B(q_1,\overline{q}_1) - \langle p_0, B(q_1,\overline{q}_1)\rangle q_0].
\end{aligned}$$

Here the vectors h_{200} and h_{011} can be computed by solving the nonsingular $(n+1)$-dimensional real systems

$$\begin{pmatrix} A & q_0 \\ p_0^* & 0 \end{pmatrix} \begin{pmatrix} h_{200} \\ s \end{pmatrix} = \begin{pmatrix} -B(q_0,q_0) + \langle p_0, B(q_0,q_0)\rangle q_0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} A & q_0 \\ p_0^* & 0 \end{pmatrix} \begin{pmatrix} h_{011} \\ s \end{pmatrix} = \begin{pmatrix} -B(q_1,\overline{q}_1) + \langle p_0, B(q_1,\overline{q}_1)\rangle q_0 \\ 0 \end{pmatrix},$$

while the vector h_{110} can be found by solving the nonsingular $(n+1)$-dimensional complex system

$$\begin{pmatrix} i\omega_0 I_n - A & q_1 \\ p_1^* & 0 \end{pmatrix} \begin{pmatrix} h_{110} \\ s \end{pmatrix} = \begin{pmatrix} B(q_0,q_1) - \langle p_1, B(q_0,q_1)\rangle q_1 \\ 0 \end{pmatrix}.$$

10.7 Double-Hopf bifurcation

At the double-Hopf bifurcation the system (1) has an equilibrium for which the Jacobian matrix $A = f_x(0,0)$ has two pairs of purely imaginary simple eigenvalues:

$$\lambda_{1,4} = \pm i\omega_1, \qquad \lambda_{2,3} = \pm i\omega_2,$$

with $\omega_1 > \omega_2 > 0$, and no other critical eigenvalues. Assume that condition (68) holds. Since the eigenvalues are simple, there are two complex eigenvectors, $q_1, q_2 \in \mathbf{C}^n$, corresponding to these eigenvalues:

$$Aq_1 = i\omega_1 q_1, \qquad Aq_2 = i\omega_2 q_2.$$

Introduce the adjoint eigenvectors $p_1, p_2 \in \mathbf{C}^n$ by

$$A^* p_1 = -i\omega_1 p_1, \qquad A^* p_2 = -i\omega_2 p_2.$$

These eigenvectors can be normalized using the standard scalar product in $\mathbf{C}^n$,

$$\langle p_1, q_1\rangle = \langle p_2, q_2\rangle = 1.$$

The resonant cubic coefficients in the normal form (69) are given by

$$\begin{aligned}
G_{2100} &= \langle p_1, C(q_1,q_1,\overline{q}_1) + B(h_{2000},\overline{q}_1) + 2B(h_{1100},q_1)\rangle, \\
G_{1011} &= \langle p_1, C(q_1,q_2,\overline{q}_2) + B(h_{1010},\overline{q}_2) + B(h_{1001},q_2) + B(h_{0011},q_1)\rangle, \\
G_{1110} &= \langle p_2, C(q_1,\overline{q}_1,q_2) + B(h_{1100},q_2) + B(h_{1010},\overline{q}_1) + B(\overline{h}_{1001},q_1)\rangle, \\
G_{0021} &= \langle p_2, C(q_2,q_2,\overline{q}_2) + B(h_{0020},\overline{q}_2) + 2B(h_{0011},q_2)\rangle,
\end{aligned}$$

where

$$\begin{aligned}
h_{1100} &= -A^{-1}B(q_1,\overline{q}_1),\\
h_{2000} &= (2i\omega_1 I_n - A)^{-1}B(q_1,q_1),\\
h_{1010} &= [i(\omega_1+\omega_2)I_n - A]^{-1}B(q_1,q_2),\\
h_{1001} &= [i(\omega_1-\omega_2)I_n - A]^{-1}B(q_1,\overline{q}_2),\\
h_{0020} &= (2i\omega_2 I_n - A)^{-1}B(q_2,q_2),\\
h_{0011} &= -A^{-1}B(q_2,\overline{q}_2).
\end{aligned}$$

All matrices involved in the above formulas are nonsingular, due to the conditions (68) on the critical eigenvalues.

Expressions for the fifth-order coefficients in (69) are given in (Kuznetsov 1997).

11 Branch Switching at Codimension-2 Bifurcations

Suppose that the system (1) has a codimension-2 equilibrium $x = 0$ at $\alpha = 0$. Generically, one expects curves of codimension-1 bifurcations to emerge from the origin in the α-plane. In this section we describe methods to start the continuation of such curves based on the information available at the singularity. As we mentioned in Section 10, there are not only codimension-1 equilibria that emanate from codimension-2 points but, depending on the type, there may also be codimension-1 families of periodic and homoclinic orbits.

The analogous problem of switching from codimension-1 bifurcations to codimension-0 equilibria or periodic solutions was treated in Section 4. Here we will not consider switching between bifurcations of the same codimension, which is typical for nongeneric parameter situations (e.g. in systems with symmetries).

One way to set up a computational procedure is to consider the normal form of the codimension-2 singularity including the parameters $\beta \in \mathbf{R}^2$

$$w' = G(w,\beta), \qquad G : \mathbf{R}^{n_c+2} \to \mathbf{R}^{n_c}. \tag{77}$$

For all five codimension-2 bifurcations these normal forms are listed in Section 10. Suppose that an exact or approximate formula is available that gives the emanating codimension-1 bifurcations for the normal form (77). In order to transfer this to the original equation (1) we need a relation

$$\alpha = K(\beta), \qquad K : \mathbf{R}^2 \to \mathbf{R}^2 \tag{78}$$

between the unfolding parameters β and the given parameters α and, moreover, we need to extend the center manifold parametrization (72) with respect to β

$$x = H(w,\beta), \qquad H : \mathbf{R}^{n_c+2} \to \mathbf{R}^n. \tag{79}$$

Taking (78) and (79) together as $(x,\alpha) = (H(w,\beta),K(\beta))$ yields the center manifold for the suspended system $x' = f(x,\alpha), \alpha' = 0$. The homological equation (74) now turns into

$$H_w(w,\beta)G(w,\beta) = f(H(w,\beta),K(\beta)) \tag{80}$$

and the method of Section 10.2 extends readily to this case. We assume the Taylor series of G to be known as

$$G(w,\beta) = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu!\mu!} g_{\nu\mu} w^\nu \beta^\mu,$$

and the Taylor series of H and K to be unknown

$$H(w,\beta) = \sum_{|\nu|+|\mu|\geq 1} \frac{1}{\nu!\mu!} h_{\nu\mu} w^\nu \beta^\mu, \quad K(\beta) = \sum_{|\mu|\geq 1} \frac{1}{\mu!} k_\mu \beta^\mu.$$

We insert these expansions into equation (80) and apply a recursive procedure as in Section 10.2. For $\mu = 0$ this reproduces the coefficients from Section 10 while the coefficients with $|\mu| \geq 1$ yield the necessary data on the parameter dependence. Again, as in Section 10, we emphasize that this approach requires the knowledge of the normal form (77) and derives necessary conditions for the transformations.

We also notice that the Taylor terms are not always determined uniquely in this way and that the number of terms needed depends on the type of codimension-1 singularity we want to follow. For example, folds and Hopf points usually need fewer Taylor terms than periodic and homoclinic orbits. In the following sections we give some results for the cusp, Bogdanov-Takens, and Bautin bifurcations and comment on the remaining cases.

11.1 Switching at a cusp point

The cusp is the simplest case for branch switching because there is a smooth and regular curve of folds passing through it, and no other codimension-1 points are nearby. To be more specific, suppose that a cusp of (1) has been located at some (x_0, α_0) by solving the augmented system which consists of (2) and (59). The tangent vector to the curve of folds passing through it is then given by $(q, 0, 0)$ where $f_x^0 q = 0$. Note that according to (26) we have

$$g_x(x_0,\alpha_0)q = -p^* f_{xx}(x_0,\alpha_0)qq = 0$$

and hence $(q, 0, 0)$ spans the null space of the Jacobian of (f, g). We may then continue the fold branch with the predictor

$$(x_0,\alpha_0) + \varepsilon(q,0,0) \quad \text{for some small} \quad \varepsilon \neq 0.$$

To obtain more information on the location of the fold curve in the parameter plane we use the normal form

$$G(w,\beta) = \beta_1 + \beta_2 w + cw^3 + \ldots.$$

For simplicity, let $x_0 = 0, \alpha_0 = 0$ and consider the expansions

$$\begin{aligned} K(\beta) &= K_1\beta + O(\|\beta\|^2), \\ H(w,\beta) &= H_{01}\beta + qw + \tfrac{1}{2}H_{20}w^2 + H_{11}w\beta + \tfrac{1}{6}H_{30}w^3 + O(\|\beta\|^2 + w^4) \\ f(x,\alpha) &= Ax + A_1\alpha + \tfrac{1}{2}B(x,x) + B_1(x,\alpha) + \tfrac{1}{6}C(x,x,x) + O(\|\alpha\|^2 + \|x\|^4). \end{aligned}$$

Using the method described above one finds as in Section 10.3 that

$$H_{20} = -A^{INV}B(q,q), \quad c = \frac{1}{6}p^*(3B(H_{20},q) + C(q,q,q))$$

and $H_{01}, H_{11}, H_{20}, H_{30}, K_1$ are in addition determined through the following bordered systems

$$\begin{pmatrix} A & A_1 \\ p^*Bq & p^*B_1q \\ p^* & 0 \end{pmatrix} \begin{pmatrix} H_{01} \\ K_1 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$
\begin{aligned}
H_{11} &= A^{INV}([H_2\, q] - B(q, H_{01}) - B_1(q, K_1)),\\
H_{30} &= A^{INV}(6cq - 3B(H_{20}, q) - C(q, q, q)).
\end{aligned}
$$

It turns out that the first bordered system is nonsingular for generic cusps and can be reduced to one left and two right solves with A^{INV} and to a small 2×2 system.

Since the curve of folds for the truncated normal form is given by

$$
w = \varepsilon, \quad \beta_1 = 2c\varepsilon^3, \quad \beta_2 = -3c\varepsilon^2
$$

we obtain an $O(\varepsilon^4)$ approximation of the fold curve in the original system as follows

$$
\begin{aligned}
\alpha &= c\varepsilon^2 K_1 \begin{pmatrix} 0 \\ -3 \end{pmatrix} + c\varepsilon^3 K_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + O(\varepsilon^4)\\
x &= \varepsilon q + \varepsilon^2 \left(cH_{01} \begin{pmatrix} 0 \\ -3 \end{pmatrix} + \tfrac{1}{2} H_{20} \right) + \varepsilon^3 \left(cH_{01} \begin{pmatrix} 2 \\ 0 \end{pmatrix} + cH_{11} \begin{pmatrix} 0 \\ -3 \end{pmatrix} + \tfrac{1}{6} H_{30} \right) + O(\varepsilon^4).
\end{aligned}
$$

11.2 Switching at a Bogdanov-Takens point

11.2.1 Switching to folds and Hopf points

According to Section 7.2, any generic BT-point (x_0, α_0) in a two-parameter system is a regular solution of the minimally augmented fold equations $f = 0, g_F = 0$ (see (23), (24)) as well as of the corresponding Hopf equations $f = 0, g_H = 0$ (see (23), (27)). In this sense there is no problem of starting either of the two branches by solving the appropriate system.

However, if one replaces the large system (27) by the Hopf system (30), then the BT-point becomes a simple (actually symmetry-breaking) branch point for (30) and the branch switching methods of Section 4 apply, cf. (Spence, Cliffe & Jepson 1989). Starting Hopf curves at BT-points was suggested by (Roose 1987) and the use of bordered systems in (Griewank & Reddien 1989).

We provide here some higher-order approximation for the functions in (78), (79) which will be used in the next subsection for the fold and Hopf branches as well as for the homoclinic branch.

Let $x = 0,\ \alpha = 0$ be a BT-point; then choose vectors q_0, q_1, p_0, p_1 and compute the coefficients a, b in the normal form

$$
G(w, \beta) = \begin{pmatrix} w_1 \\ \beta_1 + \beta_2 w_0 + a w_0^2 + b w_0 w_1 + \dots \end{pmatrix},
$$

as in Section 10.4. We use the homological equation (80) with the expansions

$$
\begin{aligned}
K(\beta) &= K_1\beta + \tfrac{1}{2}K_2\beta_2^2 + O(\beta_1^2 + |\beta_1\beta_2| + |\beta_2|^3)\\
H(w, \beta) &= H_{01}\beta + [q_0, q_1]w + \tfrac{1}{2}H_{20,0}w_0^2 + H_{20,1}w_0w_1 + \tfrac{1}{2}H_{02,1}\beta_2^2\\
&\quad + O(w_1^2 + |w_0w_1| + |w_0|^3 + \beta_1^2 + |\beta_0\beta_1| + |\beta_2|^3)\\
f(x, \alpha) &= Ax + A_1\alpha + \tfrac{1}{2}B(x, x) + B_1(x, \alpha) + \tfrac{1}{2}B_2\alpha^2 + O(\|x\|^3 + \|\alpha\|^3)
\end{aligned}
\tag{81}
$$

The linear terms K_1, H_{01} are easily determined through

$$
\begin{aligned}
K_1 &= (\gamma_1^2 + \gamma_2^2)^{-1} \begin{pmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{pmatrix} \quad \text{where} \quad (\gamma_1, \gamma_2) = p_1^* A_1,\\
H_{01} &= A^{INV}([q_1, 0] - A_1K_1).
\end{aligned}
\tag{82}
$$

Here we have defined A^{INV} by solving the bordered nonsingular matrix $\begin{pmatrix} A & p_1 \\ q_0^* & 0 \end{pmatrix}$ as in Section 10.2. Note that

$$p_1^*([q_1,0] - A_1K_1) = (1,0) - (p_1^*A_1)K_1 = 0.$$

This equation does not define K_1 uniquely and the above choice of K_1 was made for convenience.

Let us introduce columns in $K_1 = [K_{1,0}, K_{1,1}]$ and in $H_{01} = [H_{01,0}, H_{01,1}]$ respectively. Then the fold curve for the normal form

$$w_0 = \varepsilon, w_1 = 0, \beta_1 = a\varepsilon^2, \beta_2 = -2a\varepsilon$$

transforms into

$$\alpha = -2a\varepsilon K_{1,1} + O(\varepsilon^2), \quad x = \varepsilon(q_0 - 2aH_{01,1}) + O(\varepsilon^2).$$

and the Hopf curve $w_0 = w_1 = 0, \beta_1 = 0, \beta_2 = -\varepsilon < 0$ into

$$\alpha = -\varepsilon K_{1,1} + O(\varepsilon^2), \quad x = -\varepsilon H_{01,1} \quad \text{with} \quad \varepsilon > 0.$$

With A^{INV} as defined above, one finds for the quadratic terms

$$\begin{array}{rclrcl} H_{20,0} & = & A^{INV}(2aq_1 - B(q_0,q_0)) & H_{20,1} & = & A^{INV}(bq_1 + H_{20,0} - B(q_0,q_1)) \\ K_2 & = & -(p_1^*z)K_{1,0} & H_{02,1} & = & -A^{INV}(z + A_1K_2) \end{array} \tag{83}$$

where

$$z = B(H_{01,1}, H_{01,1}) + 2B_1(H_{01,1}, K_{1,1}) + B_2(K_{1,1}, K_{1,1}).$$

Using the expressions for a and b it is easily verified that the right-hand sides in the first two equations are in the range of A. With the formulae above and (81) one can write down an $O(\varepsilon^3)$ approximation of the fold and the Hopf curve.

11.2.2 Switching to homoclinic orbits

The key to the construction of the homoclinic orbits in the normal-form system is a blowup transformation which anticipates the cuspoidal shape of the phase curves in the (w_0, w_1)-plane. Introduce new parameters τ, ε and phase variables ξ, η via

$$\beta_2 = \tau\varepsilon^2, \quad \beta_1 = \frac{1}{4a}(\tau^2 - 1)\varepsilon^4$$

$$w_0(t) = \frac{\varepsilon^2}{4a}(\xi(\frac{\varepsilon}{2}t) - 2\tau), \quad w_1(t) = \frac{\varepsilon^3}{8a}\eta(\frac{\varepsilon}{2}t).$$

This transforms $w' = G(w, \beta)$ into

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} \eta \\ \xi^2 - 4 \end{pmatrix} + \frac{\varepsilon}{2a}\begin{pmatrix} 0 \\ b\xi\eta - 2b\tau\eta \end{pmatrix}. \tag{84}$$

At $\varepsilon = 0$ this system is Hamiltonian and has a homoclinic orbit given explicitly by

$$(\xi_0, \eta_0)(t) = 2(1 - 3\,\mathrm{sech}^2(t), 6\,\mathrm{sech}^2(t)\tanh(t)).$$

One can now apply well-known techniques due to (Pontryagin 1934) and (Melnikov 1963) to obtain periodic and homoclinic orbits for the perturbed Hamiltonian system. Alternatively, according to (Hale 1983), one may treat this problem as a bifurcation problem in suitable function spaces. For this purpose take τ as a parameter and write (84) as an operator equation

$$F(z,\tau)=0, \quad \text{where} \quad z=(\xi,\eta,\varepsilon)\in C^1_{\text{bounded}}\times\mathbf{R}^1.$$

Setting $z_0=(\xi_0,\eta_0,0)$ we have $F(z_0,\tau)=0$ for all $\tau\in\mathbf{R}^1$ and a computation shows that a simple (in fact, a pitchfork) branch point occurs at $\tau_0=-\frac{5}{7}$. This proves the existence of a branch of nontrivial homoclinic orbits for the system (84) and leads to a first order approximation of the bifurcating branch as

$$z=(\xi_0,\eta_0,\varepsilon), \quad \tau=\tau_0.$$

For the normal form system we obtain an approximate homoclinic at

$$\beta_1=-\frac{6}{49a}\varepsilon^4+O(\varepsilon^6), \quad \beta_2=-\frac{5}{7}\varepsilon^2+O(\varepsilon^4)$$

$$w_0(t)=\frac{\varepsilon^2}{4a}\left(\xi_0\left(\frac{\varepsilon}{2}t\right)+\frac{10}{7}\right)+O(\varepsilon^3), \quad w_1(t)=\frac{\varepsilon^3}{8a}\eta_0\left(\frac{\varepsilon}{2}t\right)+O(\varepsilon^4).$$

Finally, with the data collected in (82), (83) and using (78), (79), (81) we arrive at a homoclinic predictor for the original system

$$\begin{aligned}\alpha &= -\tfrac{5}{7}\varepsilon^2K_{1,1}+\tfrac{\varepsilon^4}{49}\left(-6K_{1,0}+\tfrac{25}{2}K_2\right)+O(\epsilon^6)\\ x(t) &= \varepsilon^2\left(-\tfrac{5}{7}H_{01,1}+\tfrac{1}{4a}\left(\xi_0\left(\tfrac{\varepsilon}{2}t\right)+\tfrac{10}{7}+O(\varepsilon)\right)q_0\right)+\tfrac{\varepsilon^3}{8a}\eta_0\left(\tfrac{\varepsilon}{2}t\right)q_1+O(\varepsilon^4).\end{aligned}$$

This predictor can also be used to set up a phase fixing condition and thus start the continuation of a branch of codimension-1 homoclinics with the methods from Section 6. Starting homoclinic orbits at Bogdanov-Takens bifurcations was first considered in (Rodríguez-Luis et al. 1990) and (Beyn 1994).

11.3 Switching at Bautin (generalized Hopf) bifurcation

Since for this bifurcation there exists a continuation of the critical equilibrium $x_0(\alpha)$ for all sufficiently small $\|\alpha\|$, with $x_0(0)=0$, we can write (1) in a coordinate system with the origin at $x_0(\alpha)$ as

$$x'=A(\alpha)x+F(x,\alpha), \quad F:\mathbf{R}^{n+2}\to\mathbf{R}^n,$$

where $F=O(\|x\|^2)$. This allows one to avoid expanding the center manifold and the normal form into Taylor series in the parameters.

We have

$$F(x,\alpha)=\frac{1}{2}B(x,x,\alpha)+\frac{1}{6}C(x,x,x,\alpha)+O(\|x\|^4),$$

$$H(w,\bar w,\alpha)=wq(\alpha)+\bar w\bar q(\alpha)+\frac{1}{2}h_{20}(\alpha)w^2+h_{11}(\alpha)w\bar w+\frac{1}{2}h_{02}(\alpha)\bar w^2+O(|w|^3), \tag{85}$$

and

$$w'=\lambda(\alpha)w+c_1(\alpha)w|w|^2+c_2(\alpha)w|w|^4+O(|w|^6),$$

where $\lambda(\alpha)=\mu(\alpha)+i\omega(\alpha),\ \mu(0)=0,\omega(0)=\omega_0>0,$

$$A(\alpha)q(\alpha)=\lambda(\alpha)q(\alpha),\ A^*(\alpha)p(\alpha)=\bar\lambda(\alpha)p(\alpha),$$

and p, q are normalized in the standard way for all $\|\alpha\|$ small.

From the homological equation (80) we now get

$$c_1(\alpha) = \frac{1}{2}\langle p(\alpha)C(q(\alpha), q(\alpha), \bar{q}(\alpha), \alpha) + B(\bar{q}(\alpha), h_{20}(\alpha), \alpha) + 2B(q(\alpha), h_{11}(\alpha), \alpha)\rangle,$$

where

$$\begin{aligned} h_{20}(\alpha) &= [2\lambda(\alpha)I_n - A(\alpha)]^{-1}B(q(\alpha), q(\alpha), \alpha), \\ h_{11}(\alpha) &= [(\lambda(\alpha) + \bar{\lambda}(\alpha))I_n - A(\alpha)]^{-1}B(q(\alpha), \bar{q}(\alpha), \alpha). \end{aligned}$$

Note that $c_1(\alpha)$ is not uniquely defined and the above choice gives a $c_1(0)$ that coincides with (76) from Section 10.5.

To arrive at the normal form with real coefficients of the nonlinear terms

$$w' = (\mu(\alpha) + i\omega(\alpha))w + l_1(\alpha)w|w|^2 + l_2(\alpha)w|w|^4 + O(|w|^6), \tag{86}$$

one has to reparametrize time (Kuznetsov 1995) according to

$$dt = \left(1 - \frac{\text{Im } c_1(\alpha)}{\omega(\alpha)}|w|^2 + O(|w|^3)\right) d\tau,$$

which gives

$$l_1(\alpha) = \text{Re } c_1(\alpha) - \frac{\mu(\alpha)}{\omega(\alpha)}\text{Im } c_1(\alpha). \tag{87}$$

The unfolding parameters of the normal form (65) can then be expressed as

$$\begin{cases} \beta_1 &= \mu(\alpha), \\ \beta_2 &= l_1(\alpha), \end{cases} \tag{88}$$

where $l_1(\alpha)$ is defined by (87). The equations (88) can be written as

$$\beta = K_1\alpha + O(\|\alpha\|^2), \quad \alpha, \beta \in \mathbf{R}^2,$$

where K_1 is the 2×2 Jacobian matrix of (88) that is nonsingular at a generic Bautin point. Thus,

$$\alpha = K_1^{-1}\beta + O(\|\beta\|^2).$$

This allows relating the equation for the curve of fold periodic solutions in the truncated normal form (86), namely

$$|w| = \varepsilon, \quad \beta_1 = l_2\varepsilon^4, \quad \beta_2 = -2l_2\varepsilon^2,$$

to the original parameters. We also obtain an asymptotic estimate for the periodic solution using (85). Recall that an expression for $l_2(0)$ given in Section 10.5. Relating the Hopf curve in the truncated normal form with that in the original system is easy, since it is defined by $\beta_1 = 0, \beta_2 = \varepsilon, w = 0$.

11.4 Other Codimension-2 cases

A more delicate situation appears at a fold-Hopf bifurcation point (Section 10 and (Kuznetsov 1995)). Assuming an approximate normal form as in Section 10.1, asymptotic formulae for a curve of Shilnikov homoclinic orbits and for a curve of torus bifurcations have been derived in (Gaspard 1993) and applied to a specific example, the Rössler model. Note that the Shilnikov orbits appear in an exponentially narrow wedge in the parameter plane and their precise form depends on the higher-order terms of the approximate normal form. In addition there may be curves of heteroclinic tangencies of periodic solutions and torus bifurcations. Similar phenomena arise at double-Hopf bifurcations (cf. (Kuznetsov 1995)). A complete set of formulae suitable for switching to homoclinic and heteroclinic orbits in the fold-Hopf and the double-Hopf case seems not to be available.

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