The Dynamics of Reaction-Diffusion Patterns

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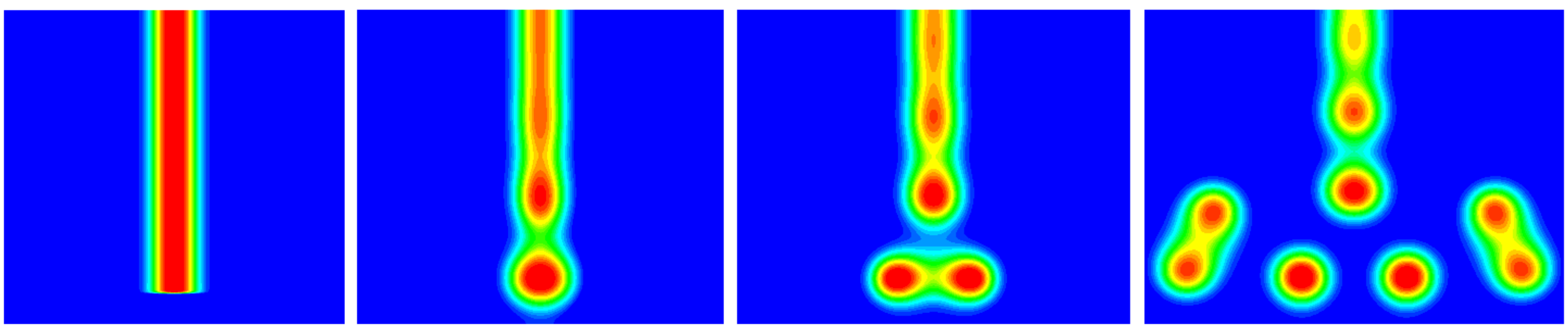
(Rob Gardner, Tasso Kaper, Yasumasa Nishiura, Keith Promislow, Bjorn Sandstede)



STRUCTURE OF THE TALK

- Motivation
- Topics that won’t be discussed
- Analytical approaches
- Patterns close to equilibrium
- Localized structures
- Periodic patterns & Busse balloons
- Interactions
- Discussion and more ...

MOTIVATION

Reaction-diffusion equations are perhaps the most 'simple' PDEs that generate complex patterns

⇕

Reaction-diffusion equations serve as (often over-) simplified models in many applications

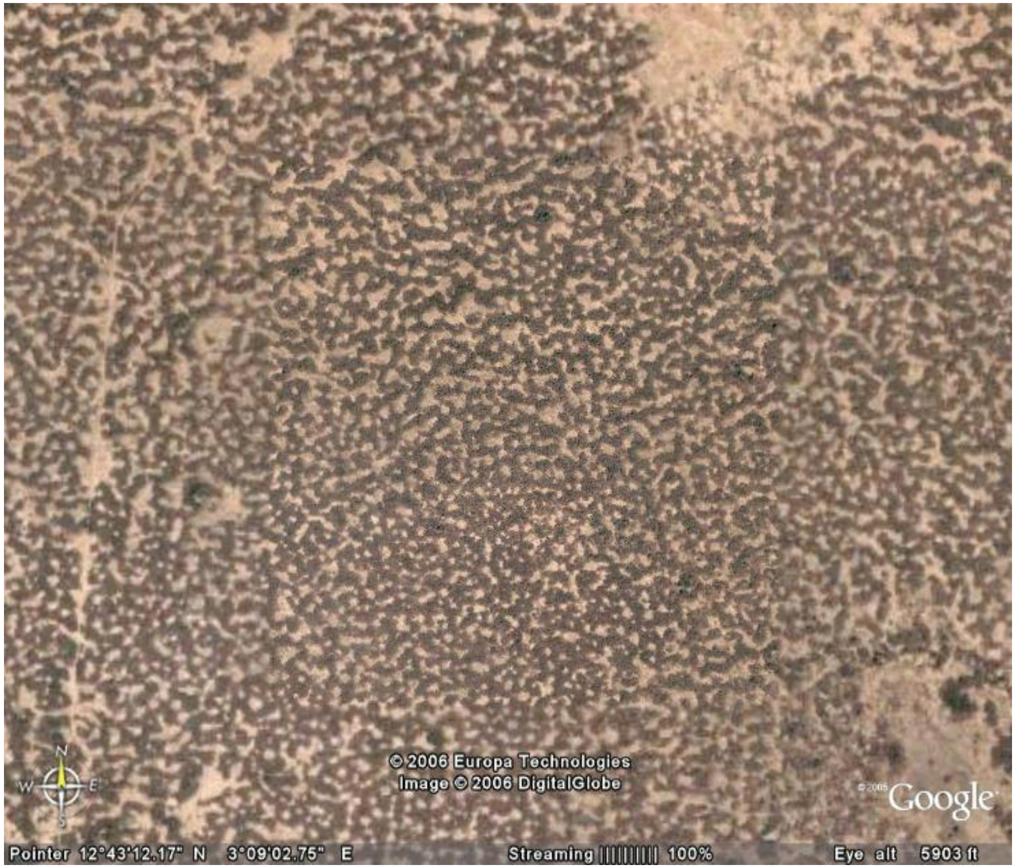
Examples:

FitzHugh-Nagumo (FH-N) - nerve conduction

Gierer-Meinhardt (GM) - 'morphogenesis'

………

EXAMPLE: Vegetation patterns



At the transition to `desertification' in Niger, Africa.

Interaction between plants, soil & (ground) water modelled by 2- or 3-component RDEs.

Some of these are remarkably familiar ...

The Klausmeier & Gray-Scott (GS) models

$$\begin{cases} W_t = CW_x - WP^2 + A(1-W) \\ P_t = D_p \Delta P + WP^2 - BP \end{cases} \quad \text{(Klausmeier)}$$

$W(x,y,t) \leftrightarrow$ water, $P(x,y,t) \leftrightarrow$ plant biomass

$$\begin{cases} U_t = D_u \Delta U - UV^2 + A(1-U) \\ V_t = D_v \Delta V + UV^2 - BV \end{cases} \quad \text{(Gray – Scott)}$$

$U(x,y,t), V(x,y,t) \leftrightarrow$ concentrations

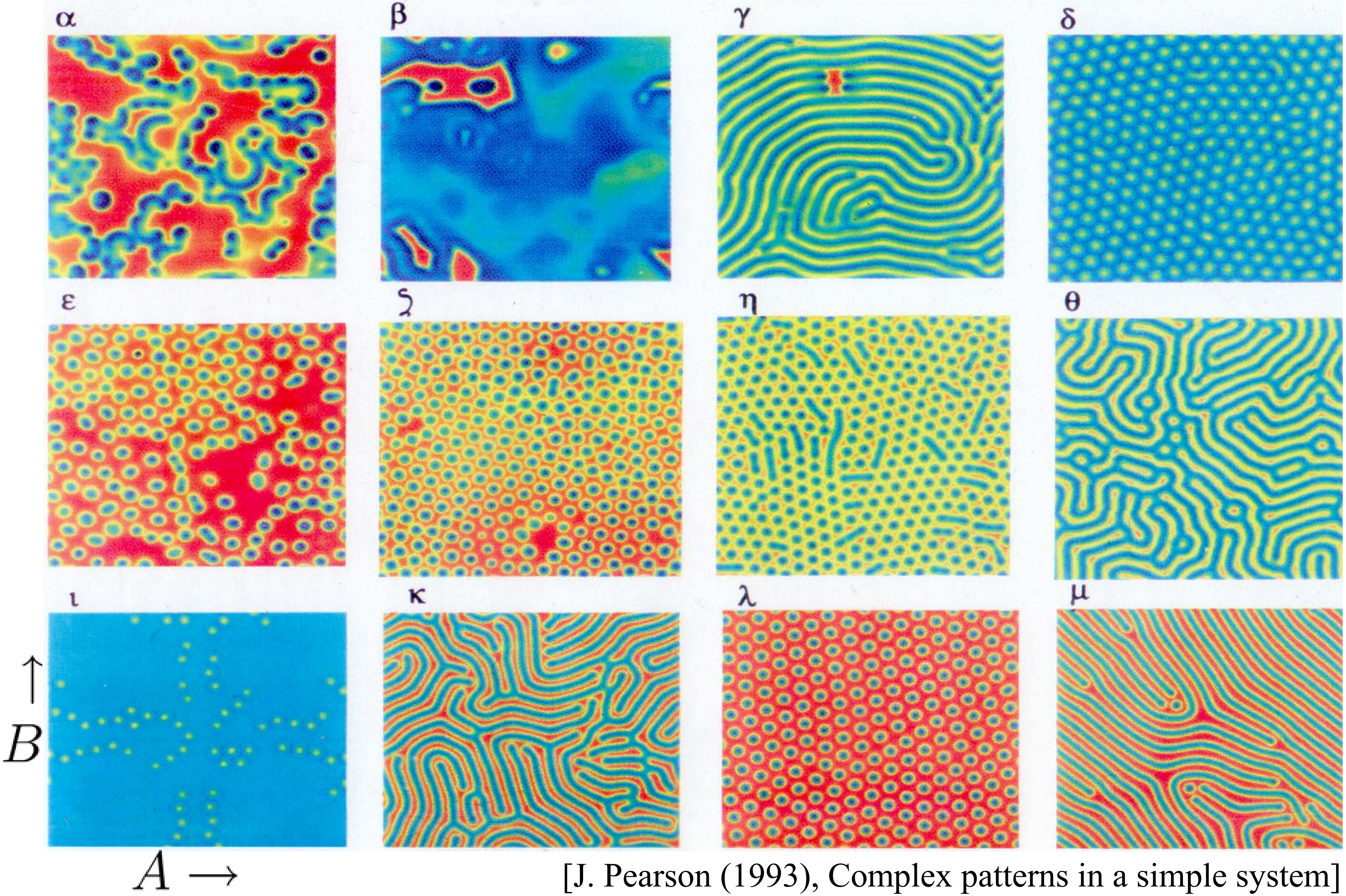
water flow on hill side $\leftrightarrow$ CW_x

horizontal water flow $\leftrightarrow$ $D_W \Delta W$ or $D_W \Delta W^\gamma$

$\Rightarrow$ Klausmeier $\leftrightsquigarrow$ GS/GS in porous media: GKGS

[Meron, Rietkerk, Sherratt, ...]

The dynamics of patterns in the GS equation



[J. Pearson (1993), Complex patterns in a simple system]

There is a (very) comparable richness in types of vegetation patterns ...

'spots'

'labyrinths'

'stripes'

EXAMPLE: Gas-discharge systems

From: http://www.uni-muenster.de/Physik.AP/Purwins/...

$$\partial_t u = d_u^2 \Delta u + f(u) - \kappa_3 v - \kappa_4 w + \kappa_1 - \kappa_2 \int_\Omega u \, d\Omega + \mu (\nabla u)(\nabla u),$$

$$\tau \partial_t v = d_v^2 \Delta v + u - v - \kappa_1' + \kappa_2' \int_\Omega v \, d\Omega,$$

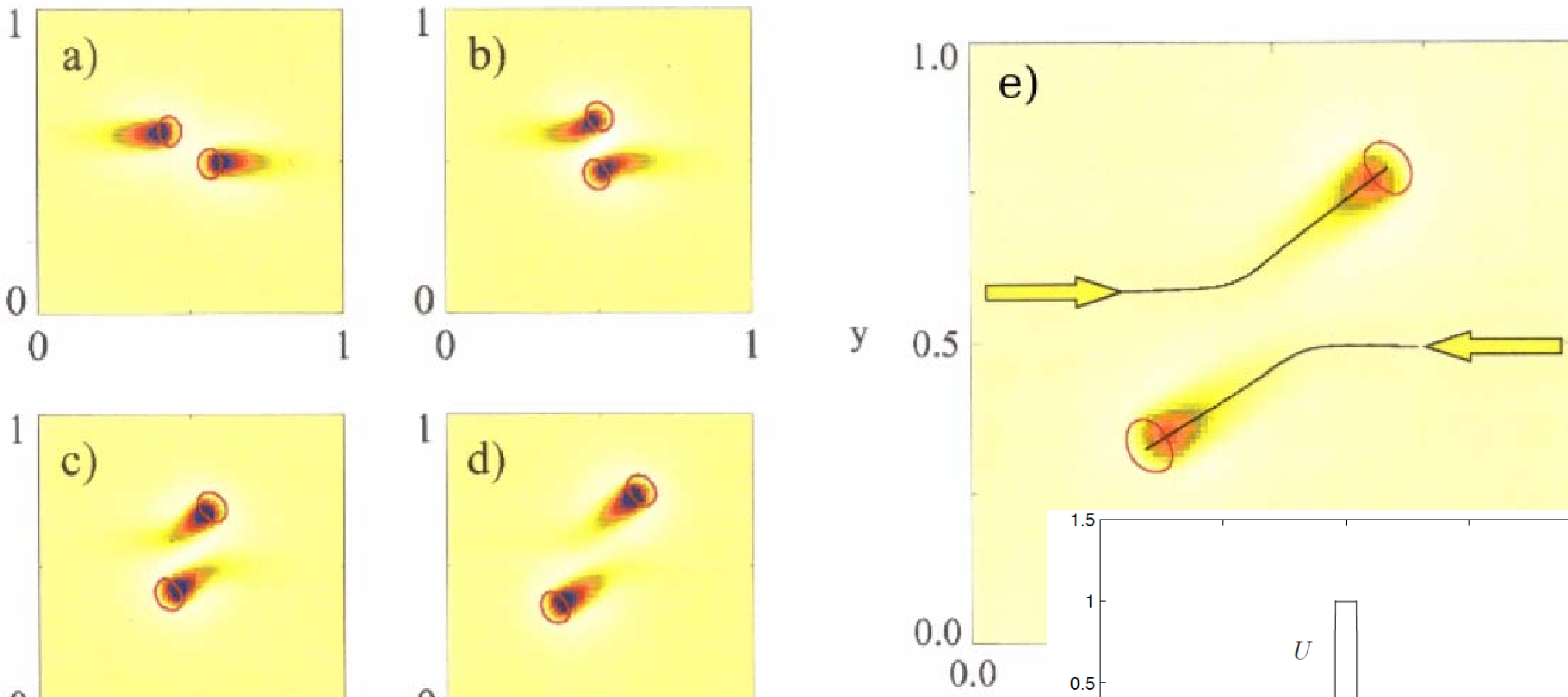
$$\Theta \partial_t w = d_v^2 \Delta w + u - w,$$

A PARADIGM MODEL $\Updownarrow$ (Nishiura et al.)

$$\begin{cases} U_t = U_{\xi\xi} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma), \\ \tau V_t = \frac{1}{\varepsilon^2} V_{\xi\xi} + U - V, \\ \theta W_t = \frac{D^2}{\varepsilon^2} W_{\xi\xi} + U - W, \end{cases}$$

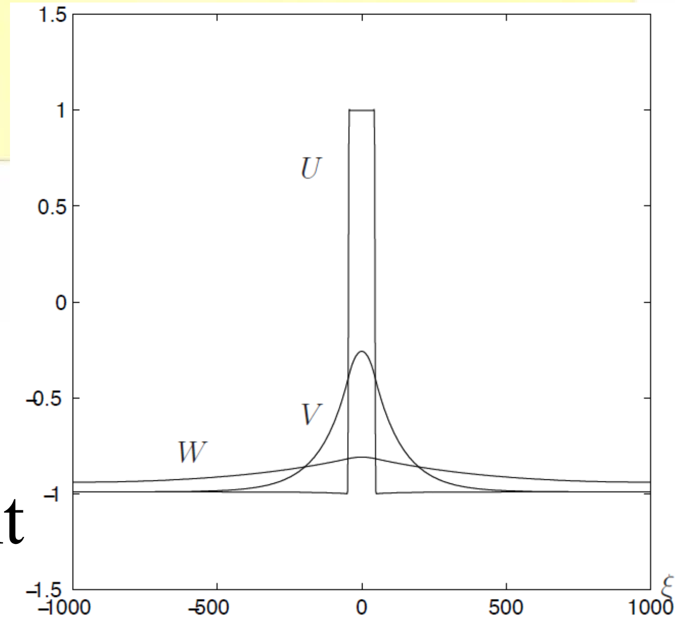
In 1D: van Heijster, D, Kaper, Promislow, in 2D: van Heijster, Sandstede

Again from the work (homepage) of the Münster group

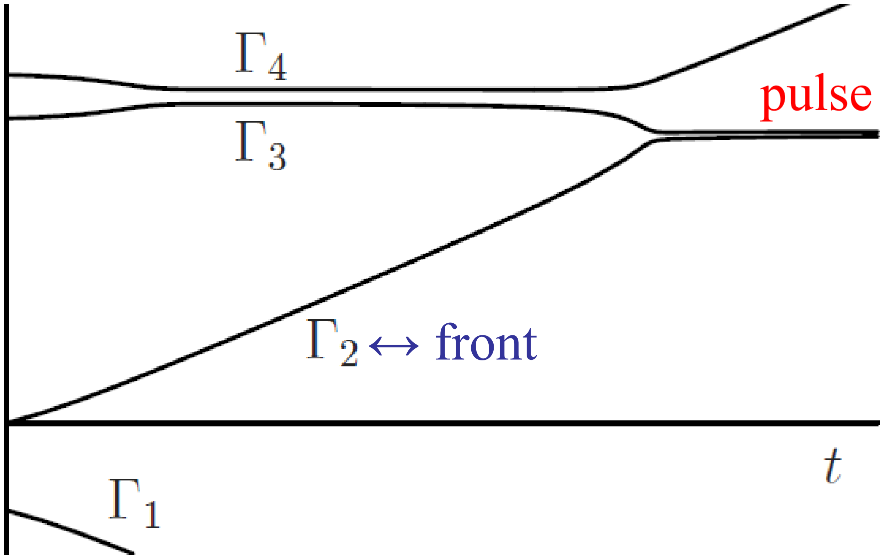


Spot interactions in 2 dimensions

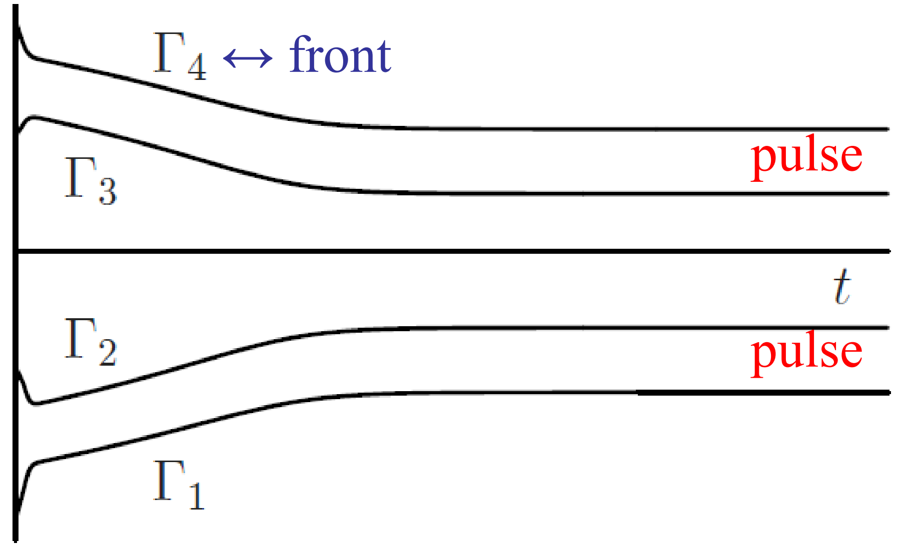
As 1D structure: pulse ↔2-front



From Peter van Heijster, AD, Tasso Kaper, Keith Promislow



$(\alpha, \beta, \gamma, D, \varepsilon) = (6, -3, -1, 5, 0.1)$



$(\alpha, \beta, \gamma, D, \varepsilon) = (2, -1, -0.25, 5, 0.01)$

SEMI-STRONG INTERACTIONS

1-dimensional pulses appearing from N-front dynamics.

PDE dynamics reduce to N-dim ODEs for front positions

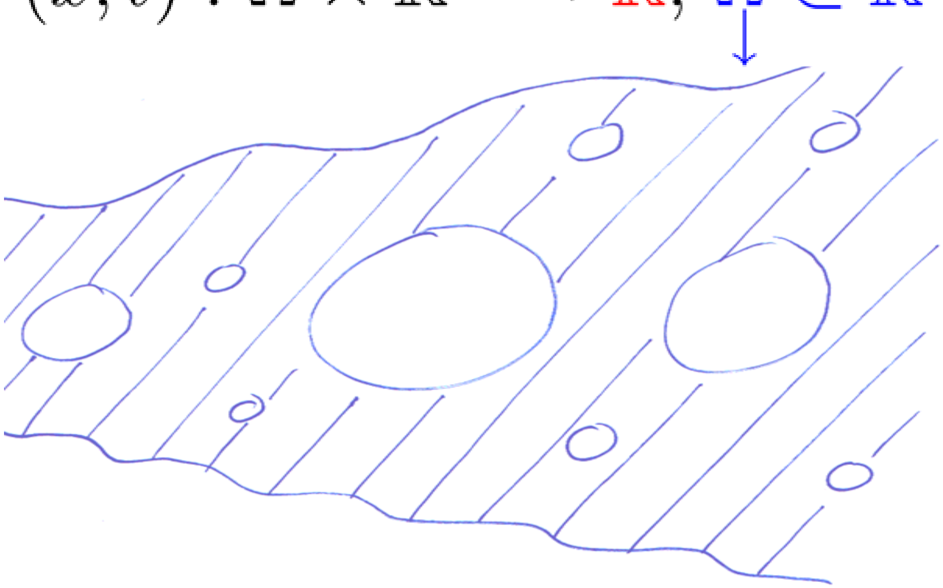
($\rightarrow$ attractivity of N-dim manifold + dynamics on manifold)

TOPICS THAT WON'T BE DISCUSSED:

- SCALAR EQUATIONS

$$U_t = \Delta U + F(U),$$

$$U(x,t) : \Omega \times \mathbb{R}^+ \to \mathbb{R},\ \Omega \subset \mathbb{R}^d.$$



'Tools':

- Maximum principles
- Gradient structure

'Waves in random media' [Berestycki, Hamel, Xin, ...]

- GRADIENT FLOWS, such as

$*$ the Cahn-Hillard equation ($\leftrightarrow$ interface dynamics),

$$U_t = -\Delta((\varepsilon^2)\Delta U + F(U)),$$

$U(x,t) : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^2$, and

$*$ the real Ginzburg-Landau equation ($\leftrightarrow$ defects),

$$(U_t =)\ \Delta U + U - |U|^2 U\ (= 0),$$

$U(x,t) : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{C}$, $\Omega \subset \mathbb{R}^2$.

[Fife, Brezis, Nishiura, Sternberg, ...]

- INTERFACE DYNAMICS in 2D (curvature!)

∗ in gradient systems ($\leftarrow\!\!\rightsquigarrow$ Cahn-Hilliard)

∗ in singularly perturbed 'excitable' systems

$$\begin{cases} U_t &= \Delta U + F(U,V) \\ V_t &= \delta\Delta V + \varepsilon G(U,V) \end{cases}$$

$U, V : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$, $0 < \varepsilon, \delta \ll 1$,

∗ in general

- BOUNDARY EFFECTS

[Fife, 'Japanese school' (Mimura, Nishiura, ...), Sandstede, Scheel, ...]

ANALYTICAL APPROACHES

Restriction/Condition: 'We' want explicit control on the nature/structure of the solutions/patterns

$\Downarrow$

- Study solutions 'near' simple patterns

$\rightarrow$ Modulated patterns & modulation equations.

- Study equations 'close to' simple equations (??)

$\rightarrow$ (Singularly) perturbed equations & near-gradient/ near-integrable systems

(nonlinear Schrödinger $\rightsquigarrow$ complex Ginzburg-Landau)

• SOLUTIONS NEAR SIMPLE PATTERNS

* 'Weakly nonlinear stability theory'

($\Leftrightarrow$ evolution of small patterns near a weakly unstable trivial state)

$\longrightarrow$ the complex Ginzburg-Landau equation (and more).

* Modulated wave trains

($\Leftrightarrow$ dynamics of almost spatially periodic patterns)

$\longrightarrow$ the Burgers equation, the Korteweg-de Vries equation, the Kuramoto-Sivashinsky equation, ...

* Modulated localized structures.

[Eckhaus, Newell, Schneider, Kopell, van Harten, D, Sandstede, Scheel, ...]

- EQUATIONS NEAR SIMPLE EQUATIONS

$*$ SINGULARLY PERTURBED RDEs

Natural assumption: (U, V) are bounded on $\mathbb{R}^d$. Then,

$$\begin{cases} U_t = & \Delta U + F(U,V) \\ V_t = & \varepsilon^2 \Delta V + G(U,V) \end{cases} \rightarrow \begin{cases} \varepsilon^2 U_t = & \tilde{\Delta} U + \varepsilon^2 F(U,V) \\ V_t = & \tilde{\Delta} V + G(U,V) \end{cases}$$

with $0 < \varepsilon^2 = \frac{D_V}{D_U} \ll 1 \rightsquigarrow U \approx U_0$, constant & V solves

$$V_t = \tilde{\Delta} V + G(U_0, V)$$

a scalar equation.

Nevertheless, SP-RDEs exhibit the dynamics of systems.

PATTERNS CLOSE TO EQUILIBRIUM

EXAMPLE: 2-component systems in $\mathbb{R}^1$,

$$\begin{cases} U_t &= & U_{xx} &+ & F(U,V) \\ V_t &= & DV_{xx} &+ & G(U,V) \end{cases}$$

A 'trivial pattern' $(U(x,t),V(x,t)) \equiv (U_0,V_0)$ solves

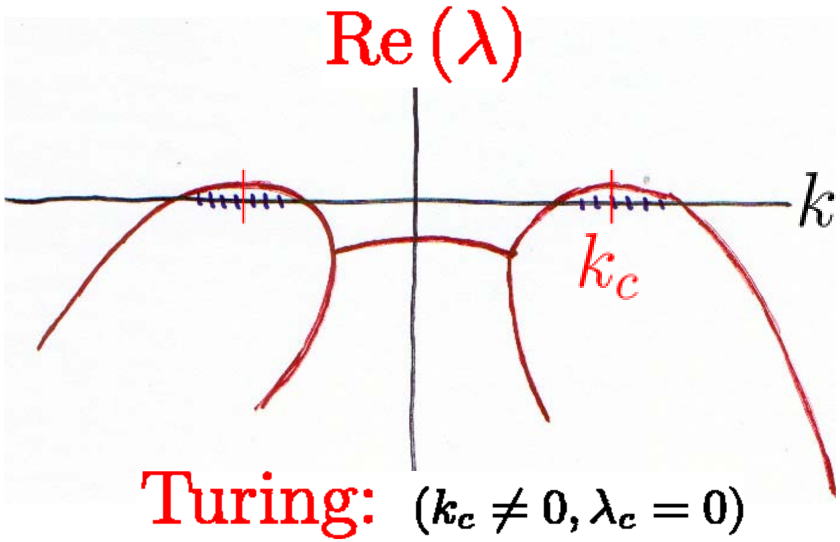
$$F(U_0,V_0) = G(U_0,V_0) = 0.$$

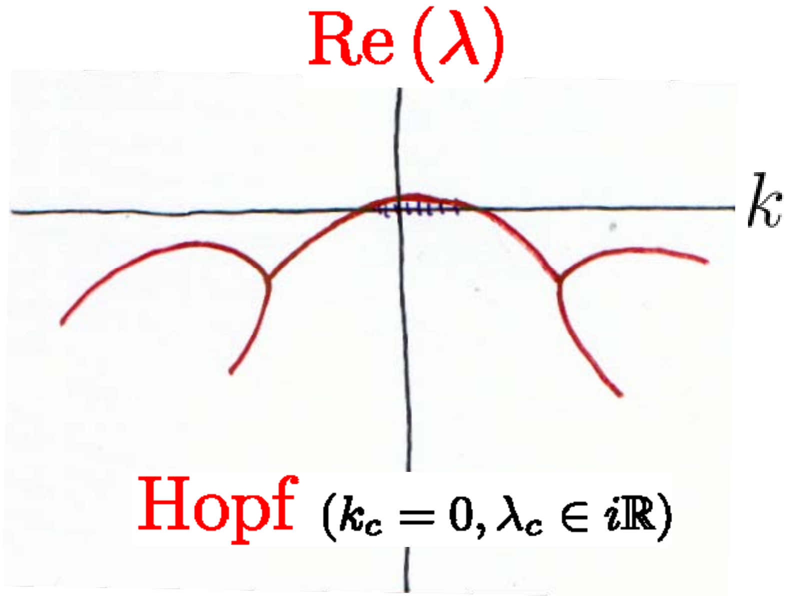
Its linear stability is determined by setting

$$(U(x,t),V(x,t)) = (U_0,V_0) + (\alpha,\beta)\, e^{ikx+\lambda(k^2)t}$$

with $k \in \mathbb{R}$, $(\alpha,\beta) \in \mathbb{R}^2$. The eigenvalues $\lambda_{1,2}(k^2) \in \mathbb{C}$ can be computed explicitly as functions of k^2.

Two typical pattern-generating bifurcations





Small amplitude patterns at near-criticality are described by a modulation equation for the complex amplitude A, where $A = A(\xi, \tau)$ is related to (U, V) by

$$(U(x,t), V(x,t)) = (U_0, V_0) + \varepsilon A e^{ik_c x + \lambda_c t}(\alpha_c, \beta_c) + \text{c.c.} + \text{h.o.t.}$$

[Note. Turing-Hopf: no reversibility (GKGS), $k_c, \lambda_c \neq 0$]

Turing: Evolution of A is described by the rGL,

$$A_\tau = A_{\xi\xi} + A \pm |A|^2 A.$$

(Turing-)Hopf: Evolution of A is described by the cGL,

$$A_\tau = (1 + ia)A_{\xi\xi} + A \pm (1 + ib)|A|^2 A.$$

[proofs of validity by Schneider]

Turing: Dynamics of patterns fully understood (near-criticality).

(Turing-)Hopf: Stable periodic patterns for $\pm \to -$ and

$$1 + ab > 0 \ \ (\text{Benjamin} - \text{Feir}/\text{Newell})$$

Q: Dynamics small amplitude patterns if $1 + ab < 0$??

cGL analysis in GKGS model

$$\begin{cases} U_t = U^\gamma_{xx} + CU_x + A(1-U) - UV^2 \\ V_t = \delta^{2\sigma} V_{xx} - BV + UV^2, \end{cases}$$

With

- $\delta^\sigma \ll 1$: ratio spreading speed plants:water
- nonlinear diffusion $\gamma \geq 1$ (mostly $\gamma = 1$ or 2)
- A main parameter $\leftrightsquigarrow$ yearly precipitation
- $C \leftrightsquigarrow$ slope, $B \leftrightsquigarrow$ mortality plants

For given B, C a Turing ($C = 0$)/Turing-Hopf ($C \neq 0$) bifurcation takes place at $A_{T(H)}$ (for decreasing A)

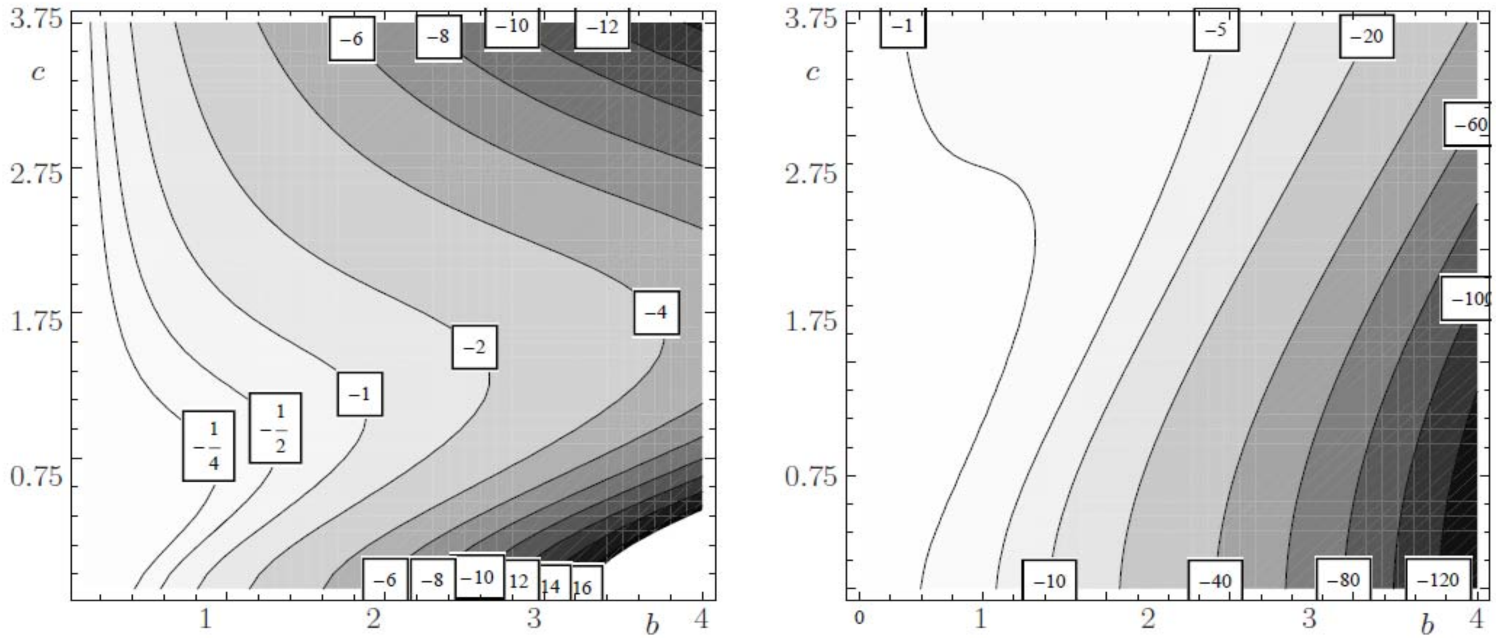
$\longrightarrow$ A cGL analysis near $A = A_{T(H)}(B, C)$

[van der Stelt, D., Hek, Rademacher]

$$\mathcal{A}_\tau = (a_1 + \mathrm{i}a_2)\mathcal{A}_{\xi\xi} + (b_1 + \mathrm{i}b_2)\mathcal{A} + (L_1 + \mathrm{i}L_2)|\mathcal{A}|^2\mathcal{A}$$

$\to L_1 = L_1(B,C)$ & $L_1(B,C)<0 \leftrightarrow \pm \to -$ (patterns)

$\gamma = 1$



$\gamma = 2$

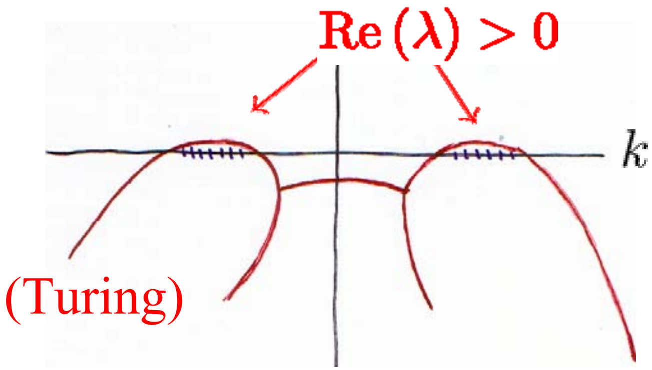
B-F/N also OK: always stable patterns at onset (!?)

Note $C = 0$: $\mathcal{A}_\tau = 2\sqrt{2}\,\mathcal{A}_{\xi\xi} + b_1(\gamma)\mathcal{A} + L_1(\gamma)|\mathcal{A}|^2\mathcal{A}$ with

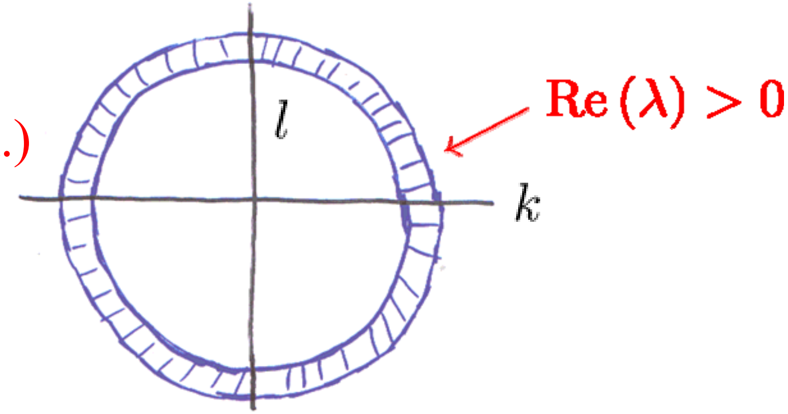
$$b_1(\gamma) = [-39 + 27\sqrt{2} + (41 - 29\sqrt{2})\gamma]\left(\frac{g\gamma}{b}\right)^{\frac{-1}{1+\gamma}}\frac{1}{b}$$

$$L_1(\gamma) = -\frac{1}{9}(2-\sqrt{2})\left[18(3+2\sqrt{2}) + 12(2+\sqrt{2})\gamma + (-8+3\sqrt{2})\gamma^2\right]\left(\frac{g\gamma}{b}\right)^{\frac{2}{\gamma+1}}b^3$$

Q: NEAR-CRITICAL PATTERN FORMATION IN $\mathbb{R}^2$??



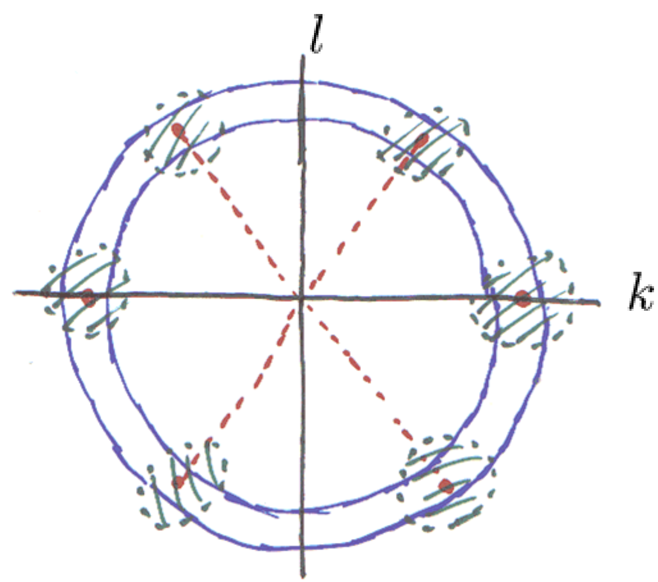
(spatial symm.) ⇒



CANNOT BE COVERED BY A 2-D cGL,

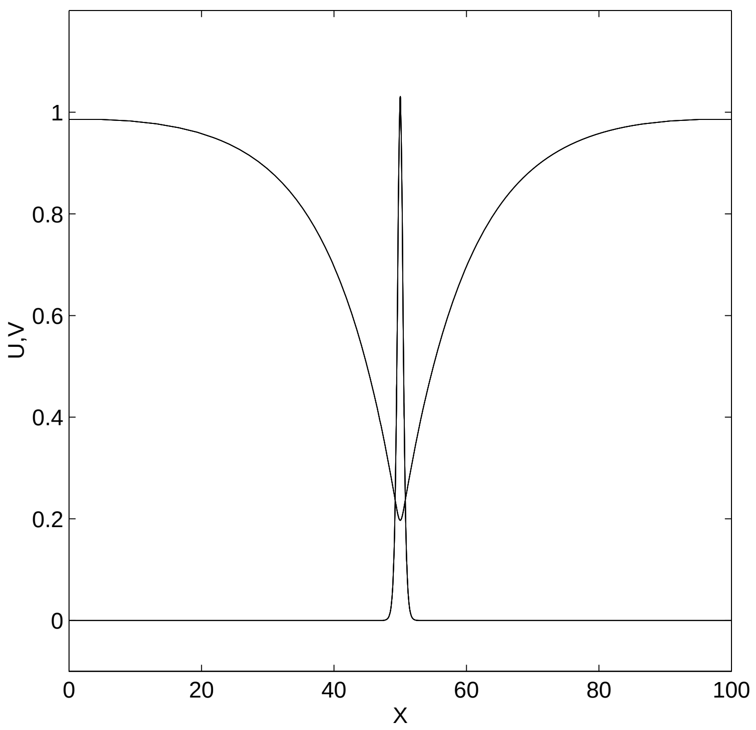
$$A_t = D_{11} A_{\xi\xi} + D_{12} A_{\xi\eta} + D_{22} A_{\eta\eta} + A \pm (1 + ib)|A|^2.$$

NOTE: Even the GL-extension of the system of coupled amplitude equations for hexagonal patterns only covers a small part of the ring of unstable 'modes'.

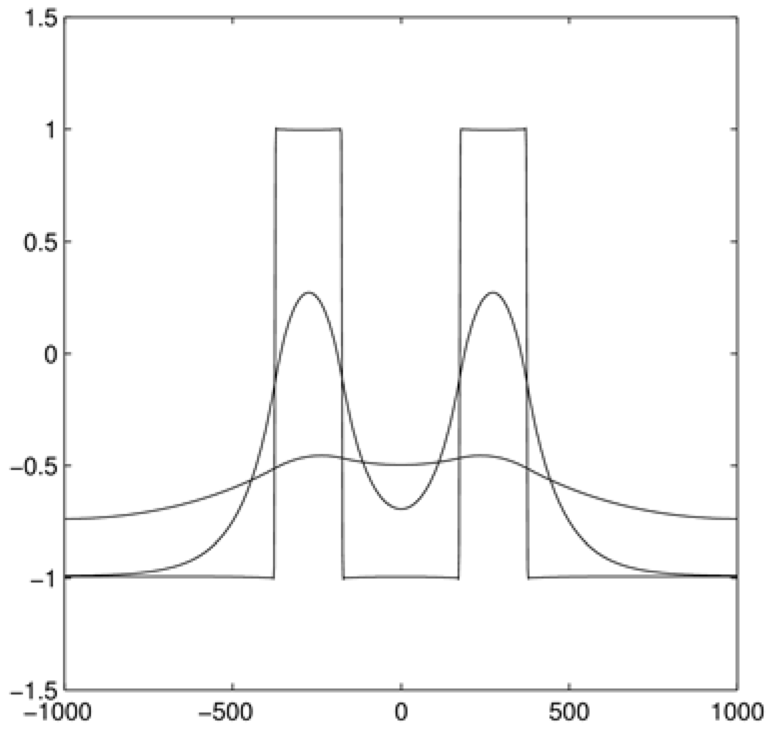


LOCALIZED STRUCTURES

Far-from-equilibrium patterns that are `close' to a trivial state, except for a small spatial region.



A (simple) pulse in GS



A 2-pulse or 4-front in a 3-component model

Pulses/fronts correspond to homo-/hetero-clinic orbits.

Prototypical example (that drove the development of 'geometric singular perturbation theory' [Fenichel,Jones, ...]):

FitzHugh-Nagumo

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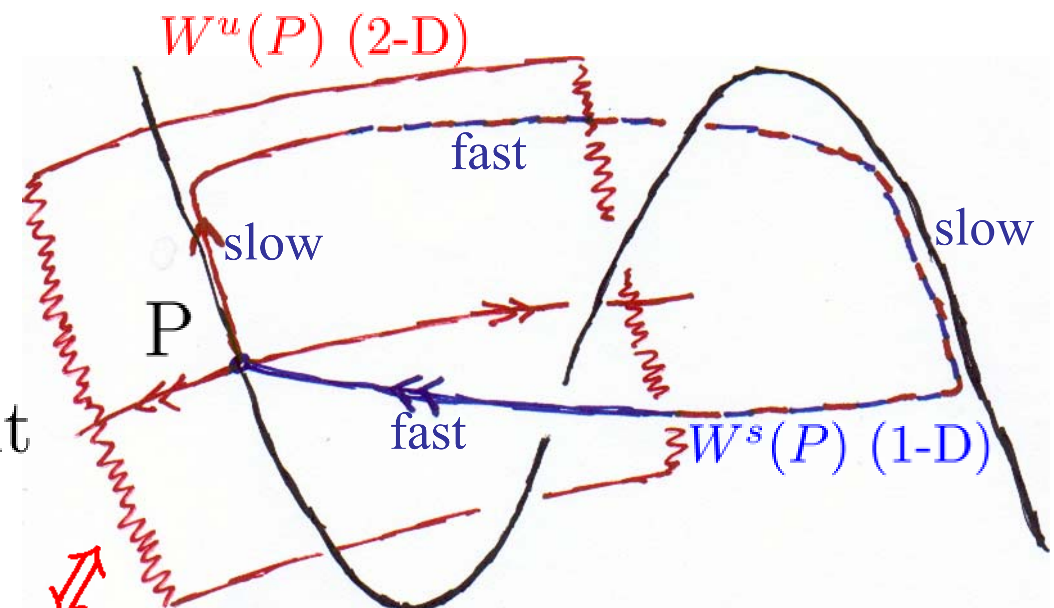
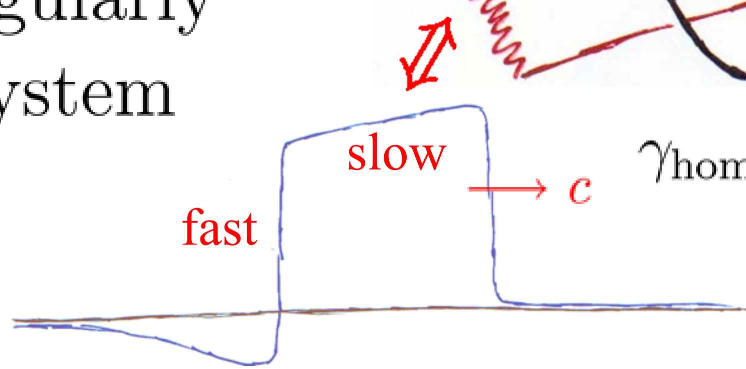
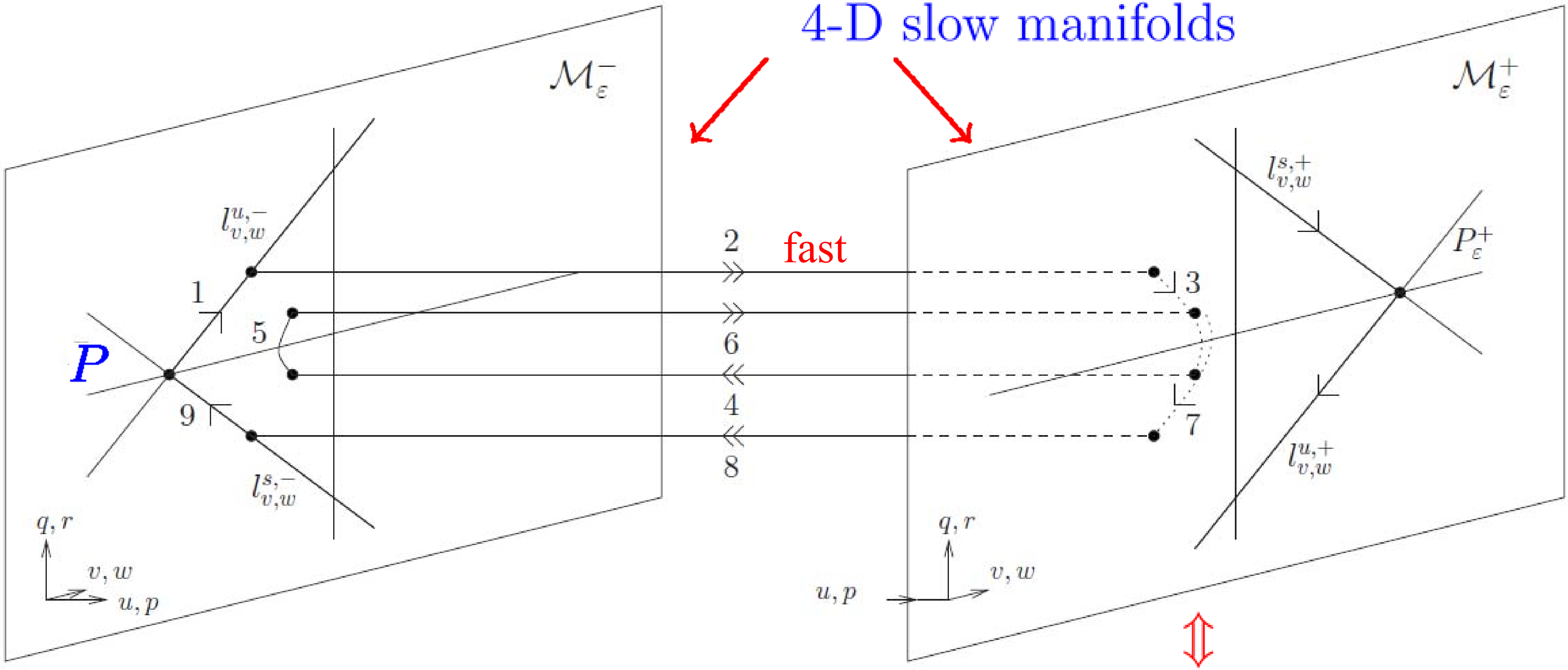
Construct 1-pulse, or 2-front homoclinic orbit in a 3-D singularly perturbed system

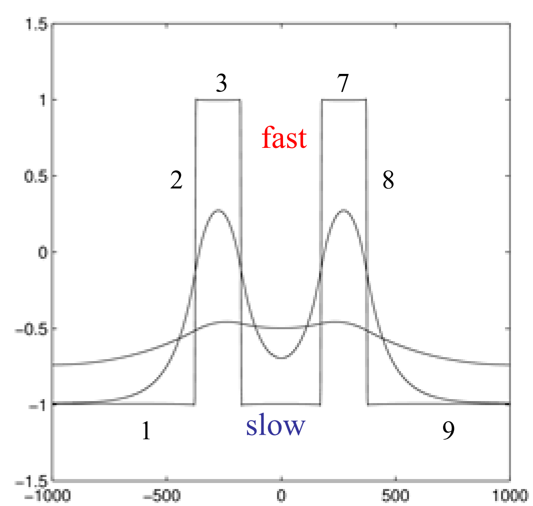




$$\gamma_{\text{hom}}(\xi) \subset W^u(P) \cap W^s(P)$$

4-D slow manifolds





$$\gamma_{\text{hom}}(\xi) \subset W^u(P) \cap W^s(P)$$

(= 3-D ∩ 3-D in 6-D space)

SPECTRAL STABILITY

EXAMPLE: 2-component system on $\mathbb{R}^1$.

$$(U(x,t), V(x,t)) = (U_{\text{hom}}(x), V_{\text{hom}}(x)) + (u(x), v(x))e^{\lambda t}$$

$$\Rightarrow \mathcal{L}(x)\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

Introduce $\Phi(x) = (u, u_x, v, v_x)$, then

$$\Phi_x = \mathcal{A}(x;\lambda)\Phi,$$

with $\mathcal{A}$ a 4×4 matrix with $\operatorname{Tr}\mathcal{A} = 0$, and

$$\lim_{x \to \pm\infty} \mathcal{A}(x;\lambda) = \mathcal{A}_\infty(\lambda)$$

Let $\{\Phi_1(x;\lambda), \Phi_2(x;\lambda), \Phi_3(x;\lambda), \Phi_4(x;\lambda)\}$ be 4 independent solutions so that

$$\lim_{x\to-\infty} \Phi_{1,2}(x;\lambda) = 0, \lim_{x\to+\infty} \Phi_{3,4}(x;\lambda) = 0$$

(this is possible for $\lambda \notin \sigma_{\text{ess}}$). The Evans function associated to this stability problem is defined by

$$\mathcal{D}(\lambda) = \det\left[\Phi_1(x;\lambda), \Phi_2(x;\lambda), \Phi_3(x;\lambda), \Phi_4(x;\lambda)\right]$$

- $\mathcal{D}$ does not depend on x
- $\mathcal{D}$ is analytic as function of λ for $\lambda \notin \sigma_{\text{ess}}$
- $\mathcal{D} = 0 \Leftrightarrow \lambda$ is an eigenvalue

[Evans, Alexander, Gardner, Jones, Pego, Weinstein]

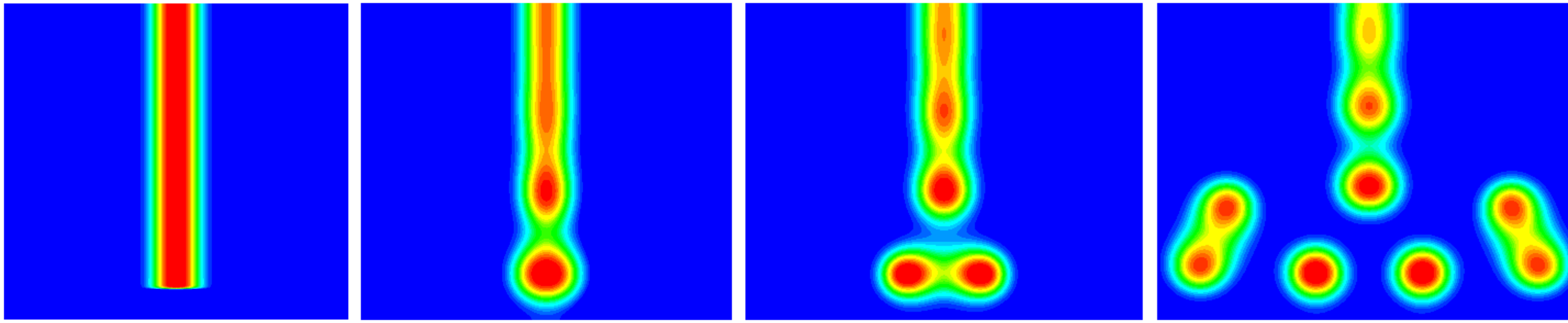
If the system is singularly perturbed, $\mathcal{D}(\lambda)$ can be decomposed,

$$\mathcal{D}(\lambda) = \mathcal{D}_{\text{fast}}(\lambda)\mathcal{D}_{\text{slow}}(\lambda)$$

• $\mathcal{D}_{\text{fast}}(\lambda)$ is analytic for $\lambda \notin \sigma_{\text{ess}}$;
• $\mathcal{D}_{\text{slow}}(\lambda)$ is meromorphic.
• the zeroes of $\mathcal{D}_{\text{fast}}(\lambda)$ are given by a scalar problem and can be determined; some of these correspond to poles of $\mathcal{D}_{\text{slow}}(\lambda)$
• the zeroes of $\mathcal{D}_{\text{slow}}(\lambda)$ can be determined by a Melnikov-like approach

[D,Gardner,Kaper, ..,Veerman]

What about localized 2-D patterns?



Spots, stripes, 'volcanoes', ..., most (all?) existence and stability analysis done for (or 'close to') 'symmetric' patterns

(Again) PDE $\rightsquigarrow$ ODE-analysis

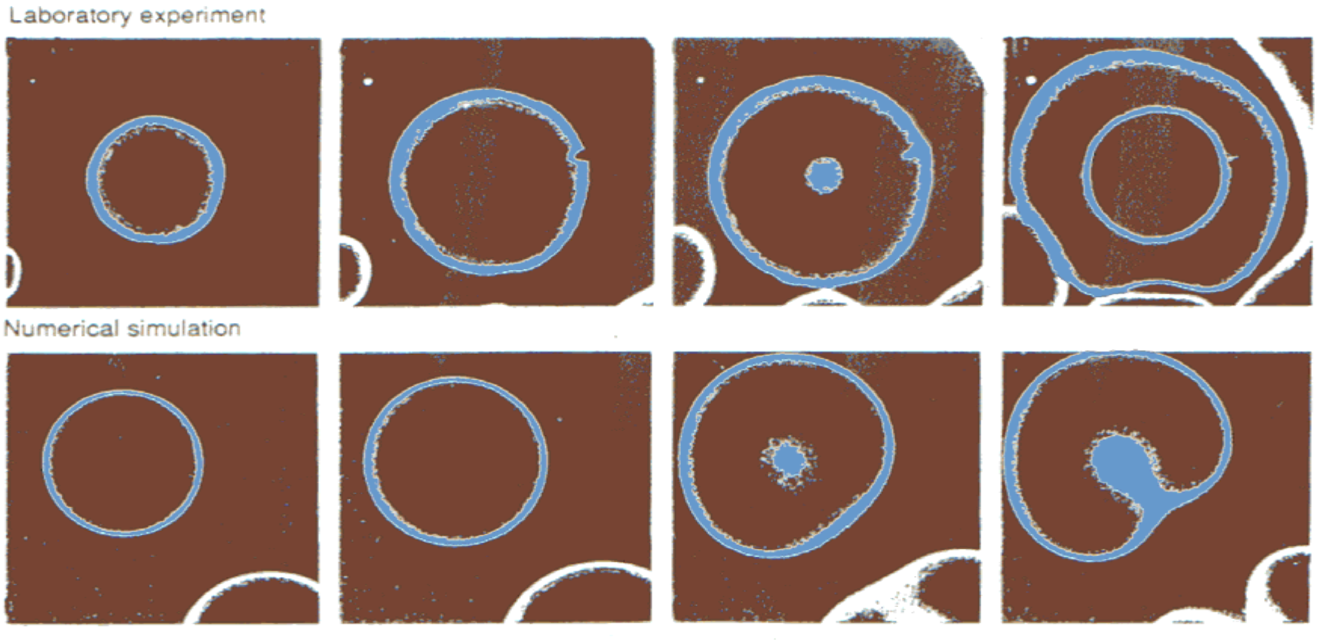
Note, however: polar/spherical symmetries,

$$\Delta \to \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r},$$

an inhomogeneous term with singularity at $r = 0$.

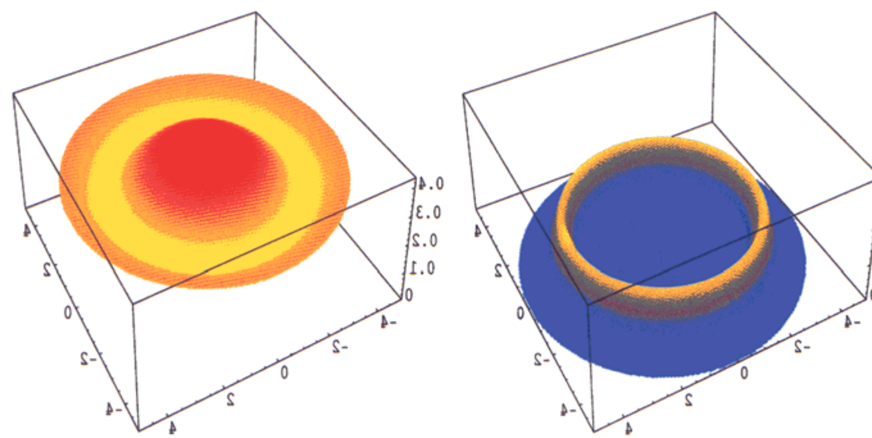
[Ward,Wei,Winter, van Heijster & Sandstede, ...]

‘Volcanoes’ and ‘Rings’ in Klausmeier/Gray-Scott



[Pearson, Swinney et al. 1994]

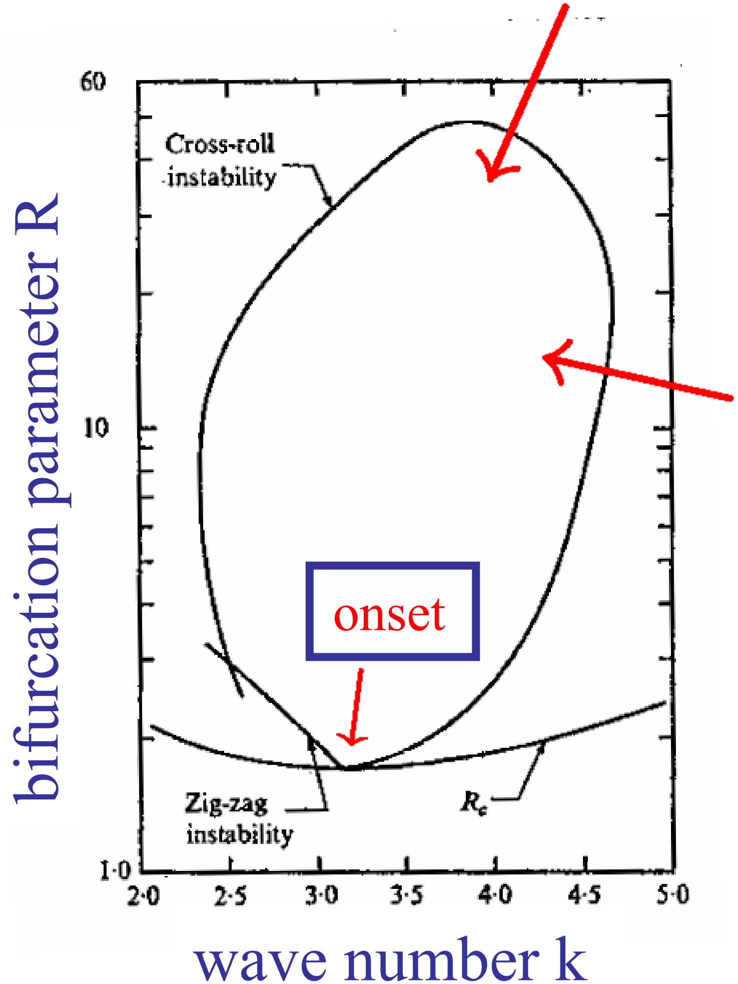
The ‘volcano/ring-patterns are (fairly) well-understood



[Morgan & Kaper, 2004]

PERIODIC PATTERNS & BUSSE BALLOONS

A natural connection between periodic patterns near criticality and far-from-equilibrium patterns

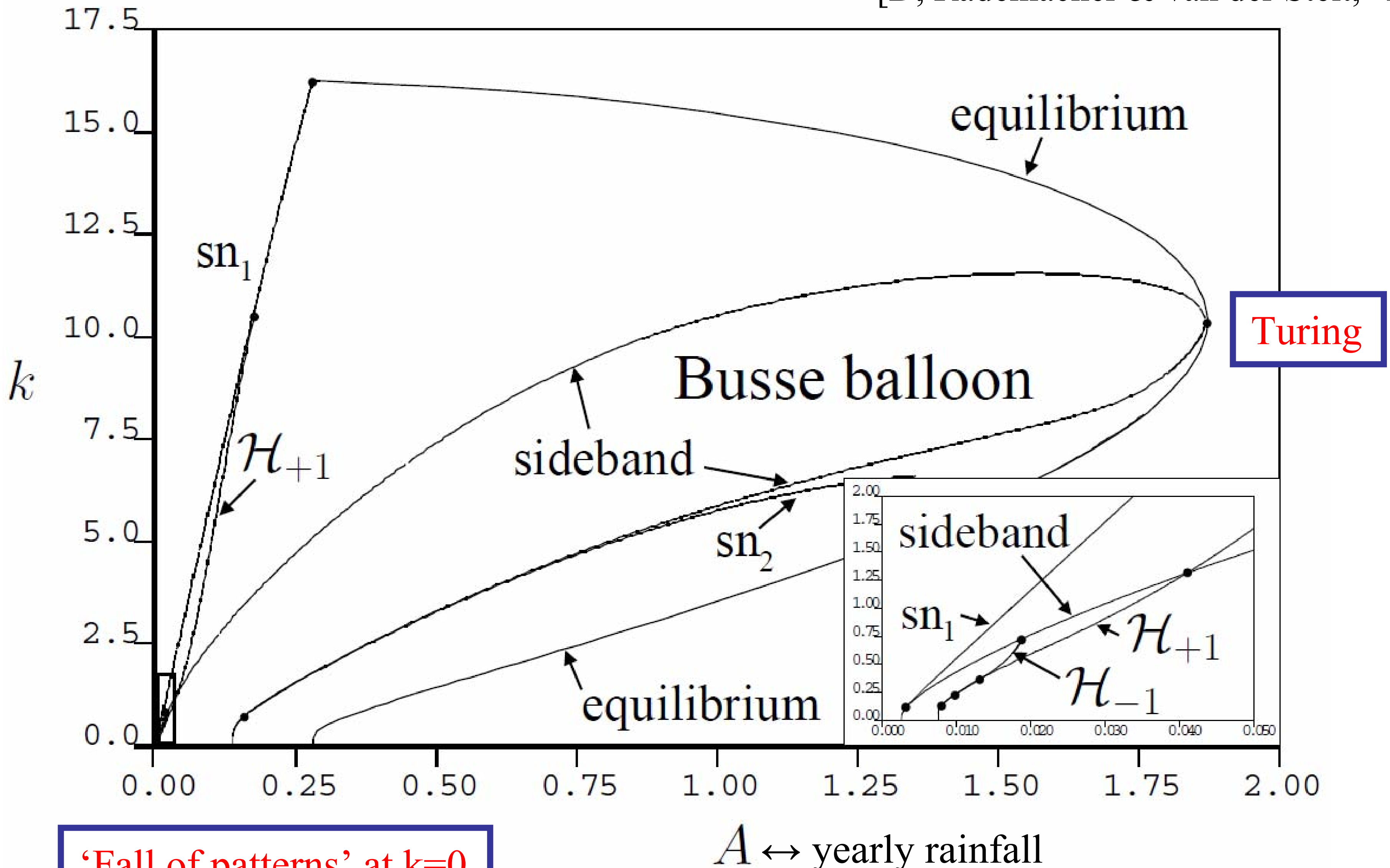


Region in (k,R)-space in which STABLE periodic patterns exist

[Busse, 1978] (convection)

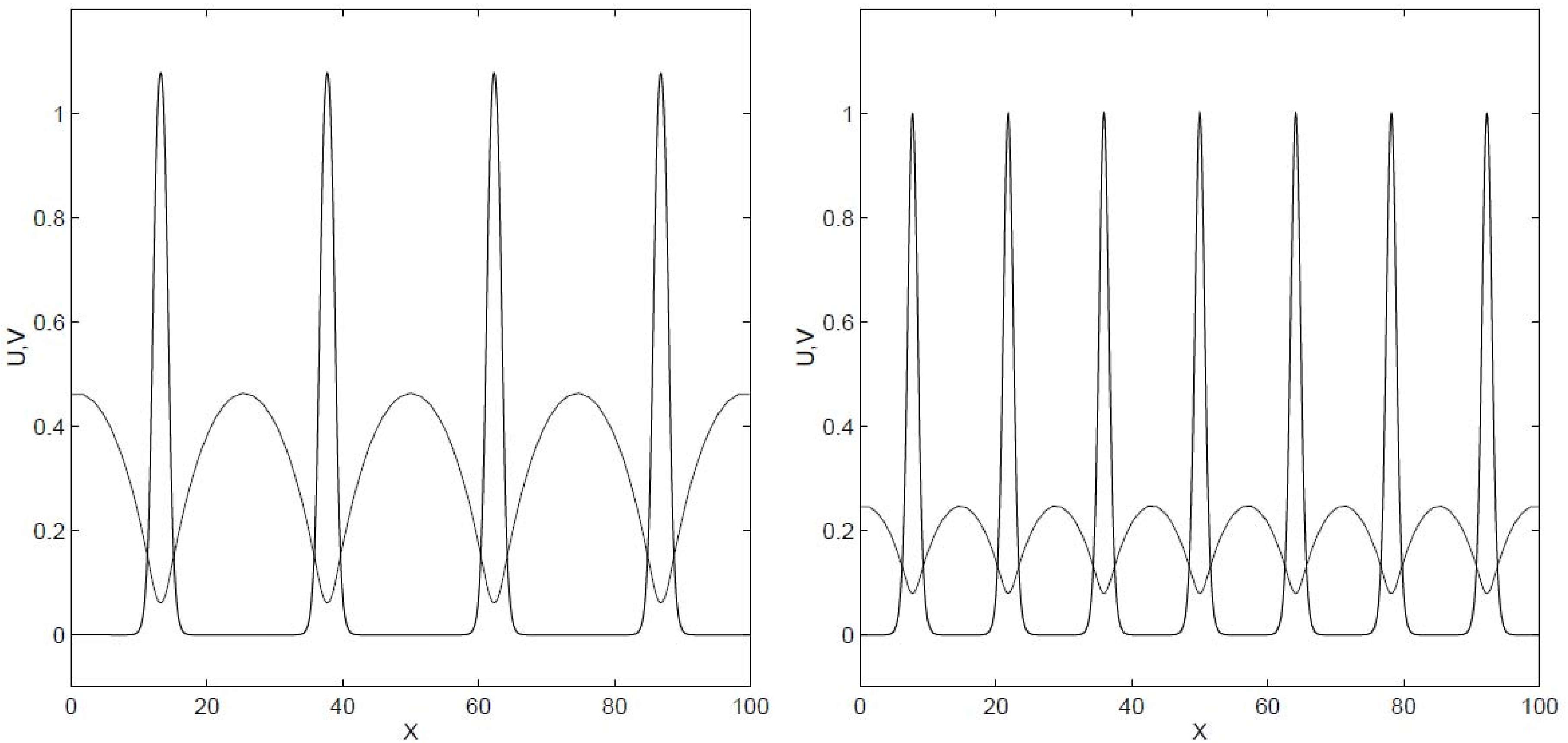
A Busse balloon for the GS model

[D, Rademacher & van der Stelt, '12]



'Fall of patterns' at k=0

Periodic patterns near *k=0*: singular localized pulses (of vegetation pattern kind)



Coexisting stable patterns (for the same parameter values)

What do we know analytically?

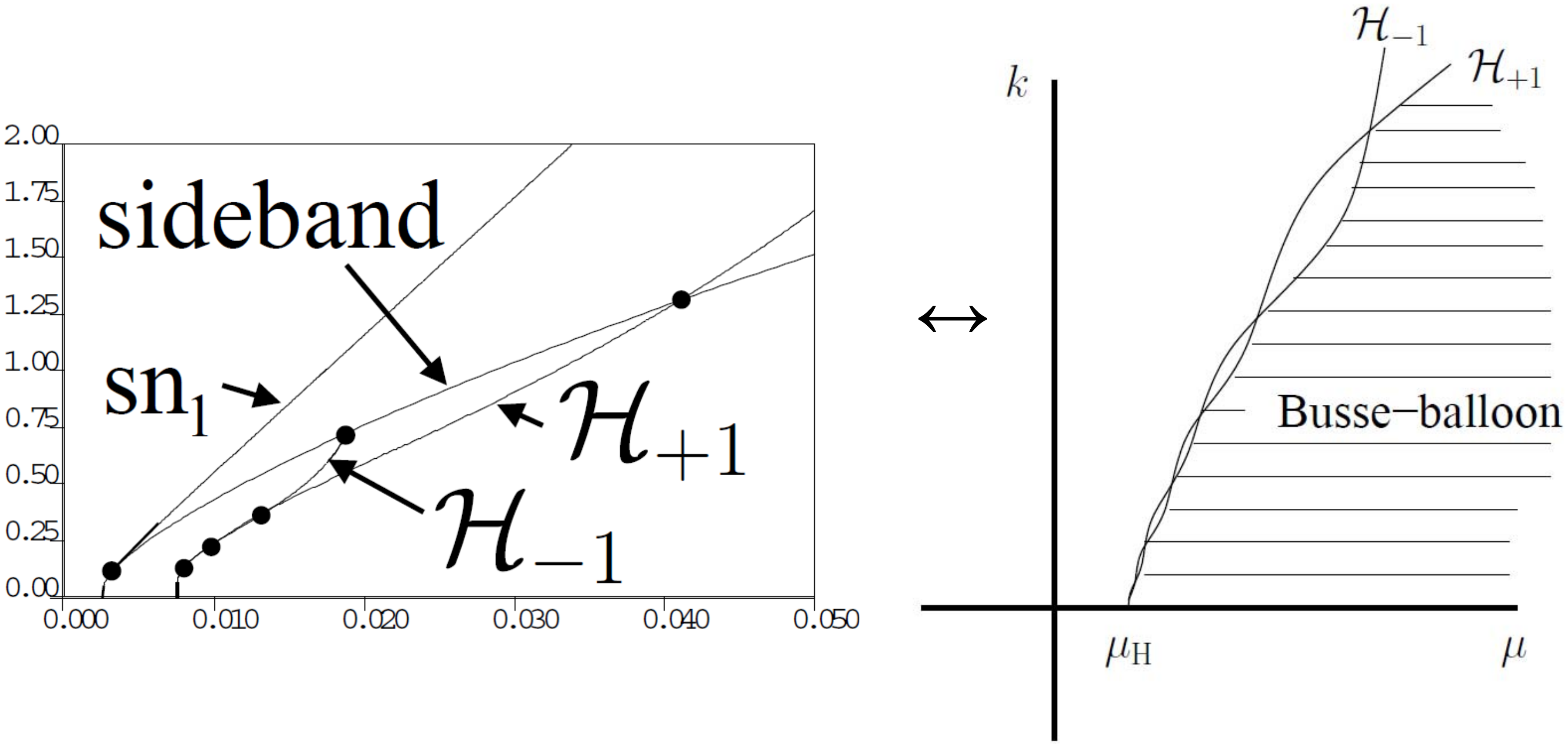
- Near onset/the Turing bifurcation: ‘full analytical control’ through Ginzburg-Landau theory.

- A complete classification of the generic character of the boundary of the Busse balloon [Rademacher & Scheel, ’07].

- Near the ‘fall of patterns’: existence and stability of singular patterns [D, Gardner & Kaper, ’01; van der Ploeg & D, ’05; D, Rademacher & van der Stelt ’12].

No further general insight in (the boundary of) the Busse balloon.

A spin-off: the Hopf dance, a novel fine-structure

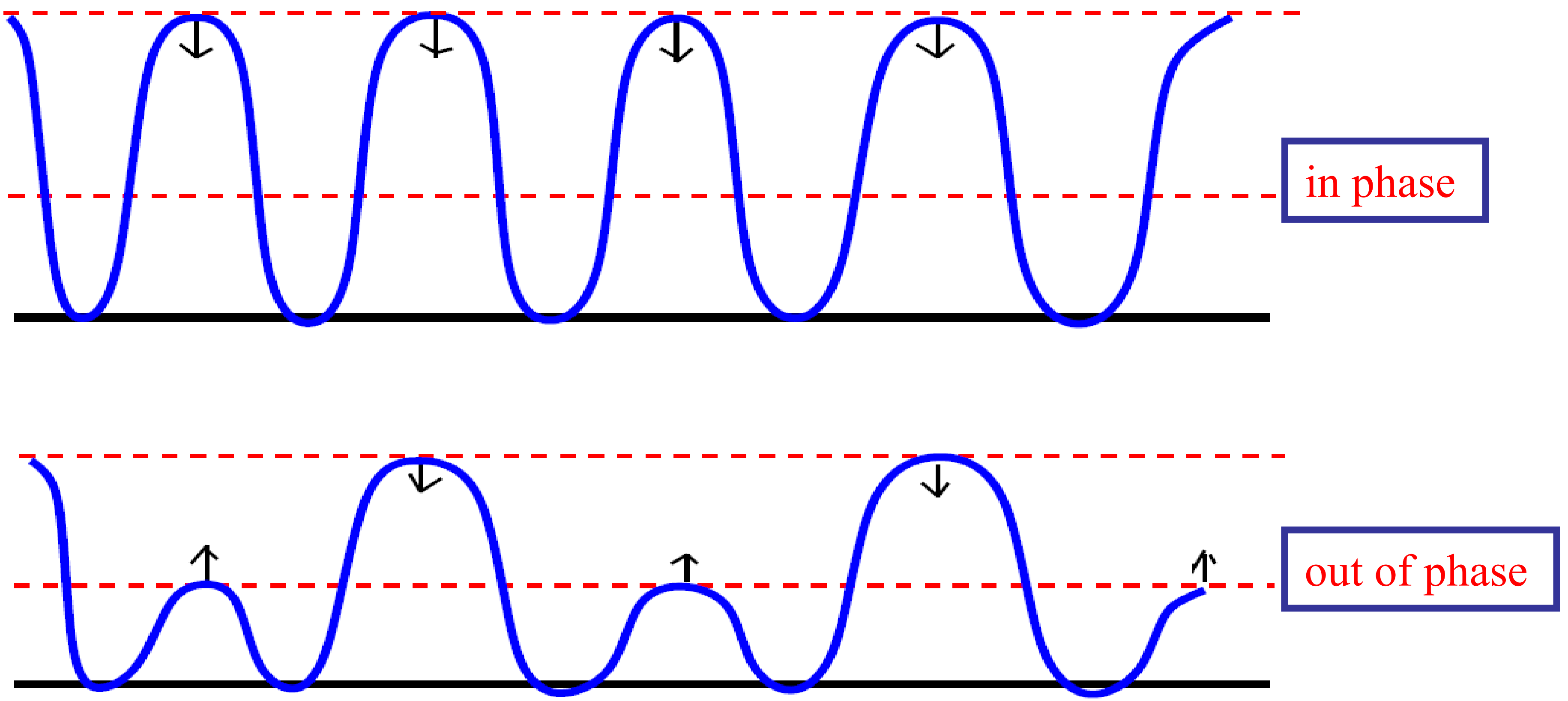


A 'dance' of intertwining Hopf bifurcations.

The homoclinic (*k=0*) 'oasis' pattern is the last to destabilize

(Ni's conjecture)

Two types of Hopf bifurcations?

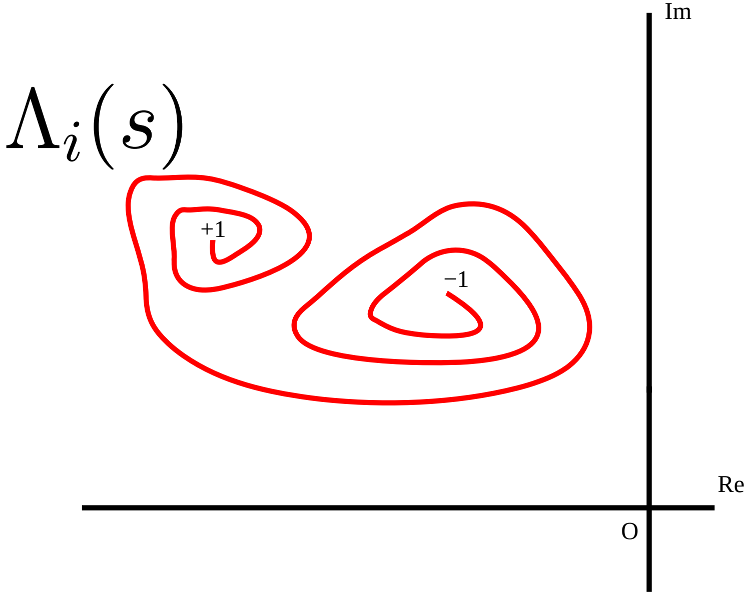


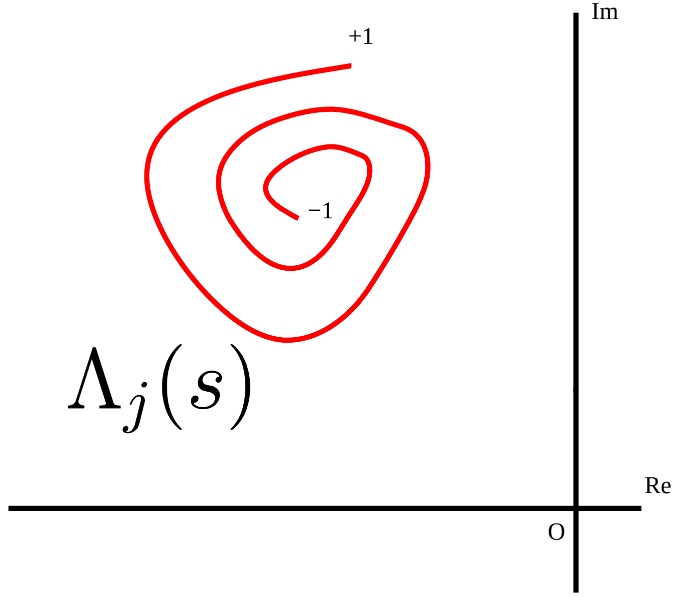
Why only these two?

Spectral analysis

STABILITY: 'Solution' = 'Pattern' + 'Perturbation'

- LINEARIZATION: 'Perturbation' $= P(x)e^{\lambda t}$, $\lambda \in \mathbb{C}$.
- INSTABILITY: There is a λ s.t. $\mathrm{Re}(\lambda) > 0$.
- FACT: $\lambda = \{\Lambda_i(s), s \in [-1, 1], i = 1, 2, ..., N/\infty\}$.

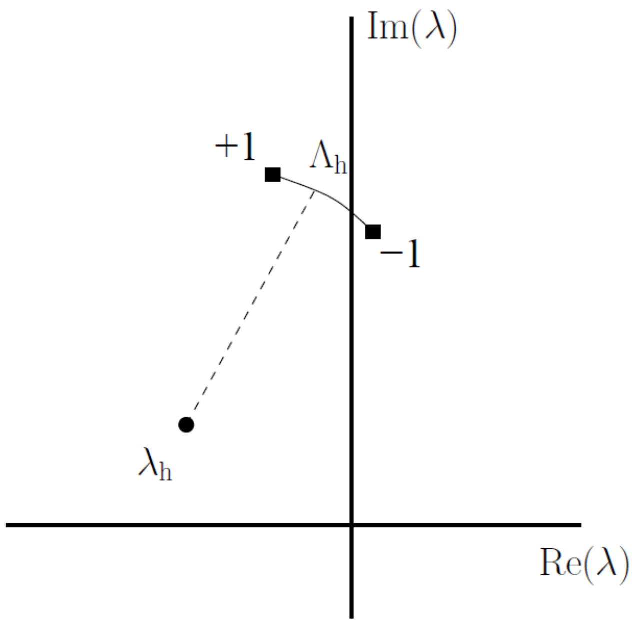


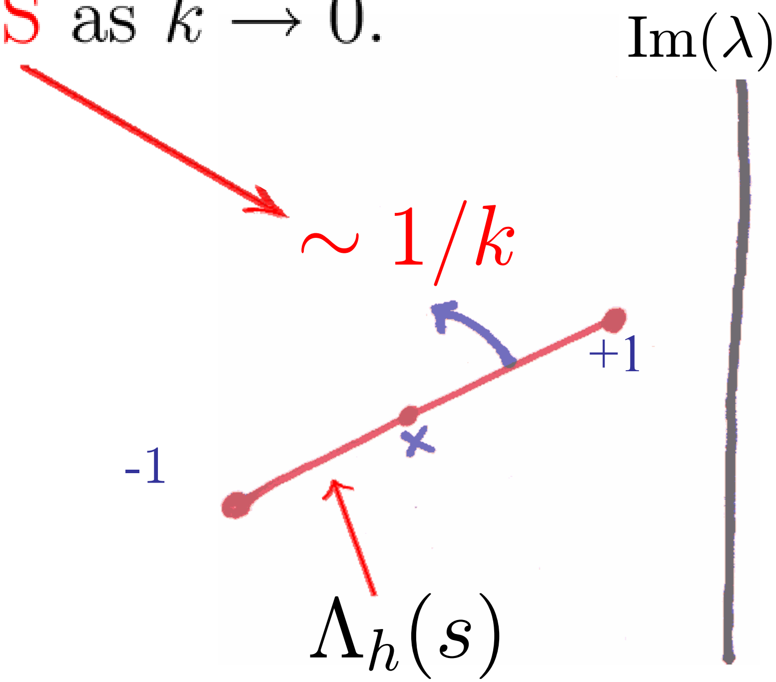


Note: ± 1 endpoints correspond to $\mathcal{H}_{\pm 1}$ Hopf bifurcations.

The long wavelength limit ($k \sim 0$)

- The critical spectral branch $\Lambda_h(s)$ 'unrolls'.
- The 'oasis' state is the last pattern to destabilize.
- $\Lambda_h(s)$ shrinks and ROTATES as $k \to 0$.





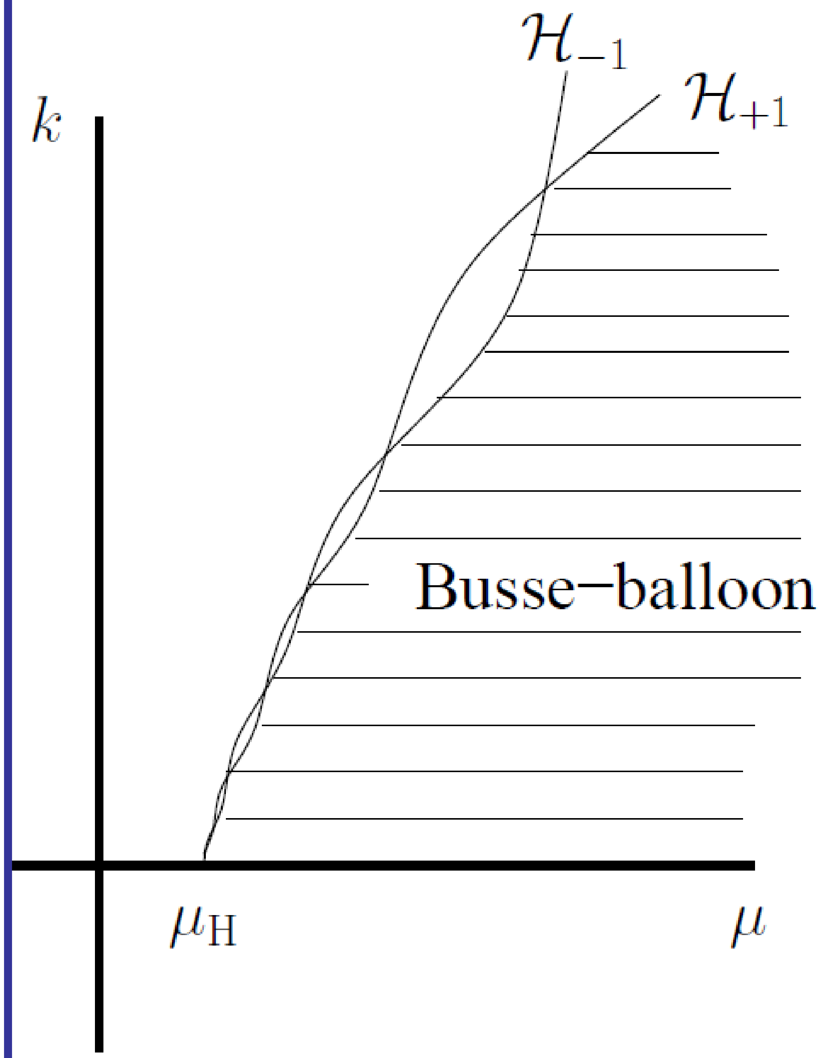
By: Evans function for periodic patterns [Gardner, Zumbrun, D & van der Ploeg]

A novel general insight in the ‘fall of patterns’

[D, Rademacher & van der Stelt, ’12]

In a general class – well, … – of reaction-diffusion models:

- The homoclinic ‘oasis’ pattern is the last pattern to become unstable (↔Ni’s conjecture).
- The Hopf dance: near the destabilization of the homoclinic pattern, the Busse balloon has a ‘fine structure’ of two intertwining curves of Hopf bifurcations.

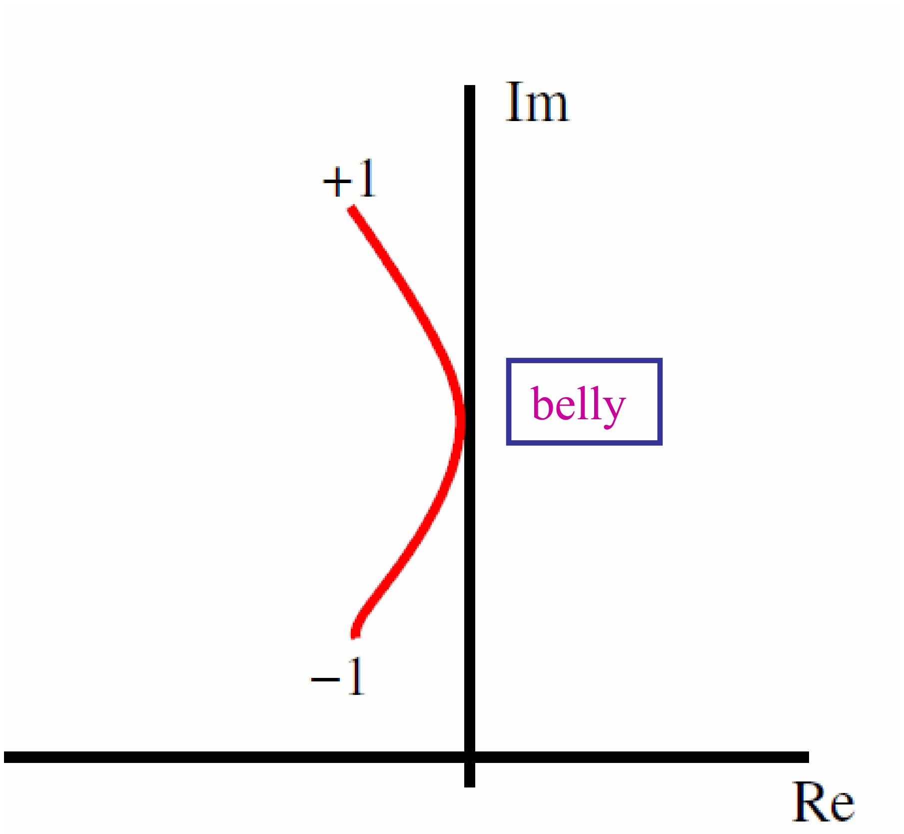


THE BELLY DANCE

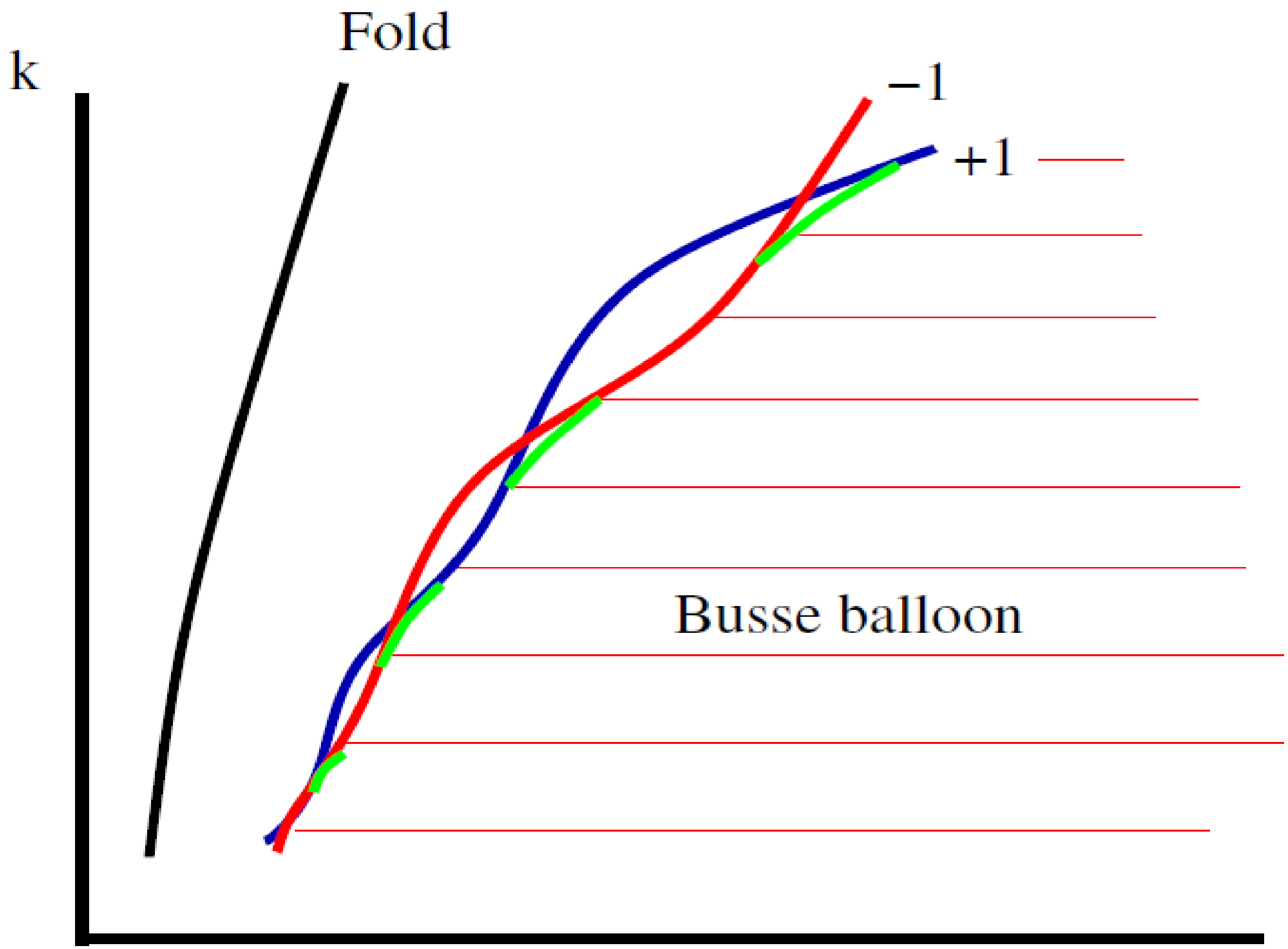
The spectral branch is only to leading order a straight line/an interval.

In general it will be (slightly) bent.

This may yield small regions of 'internal Hopf destabilizations' and the corners in the boundary of the BB will disappear ↔ the orientation of the belly.



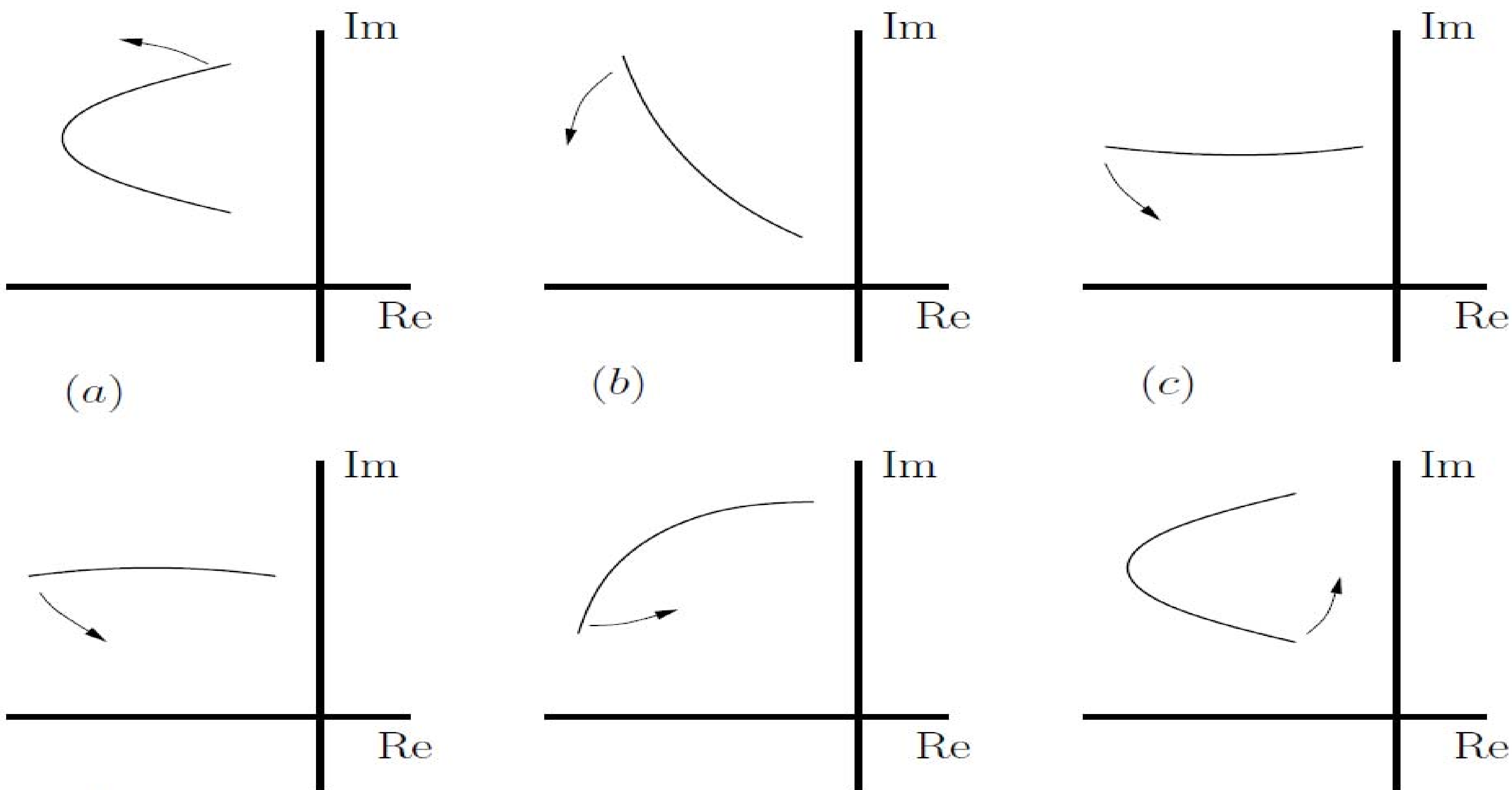
A more typical Busse balloon?



Or more generic (?): sometimes a co-dimension 2 intersection, sometimes an ‘internal Hopf bridge’?

This is however not the case. In the class of considered model systems, a BELLY DANCE takes place.

The belly always points away from the *Im*-axis near the ‘corner’ at which +1 and -1 cross at the same time.



WHY??

The theory includes in essence ‘all explicit models in the literature’

(↔ Gray-Scott/Klausmeier, Gierer-Meinhard, Schnakenberg, gas-discharge, ….)

HOWEVER, if one looks carefully it’s clear these models are in fact very special.

All these prototypical systems exhibit patterns that are only ‘locally nonlinear’ (?!)

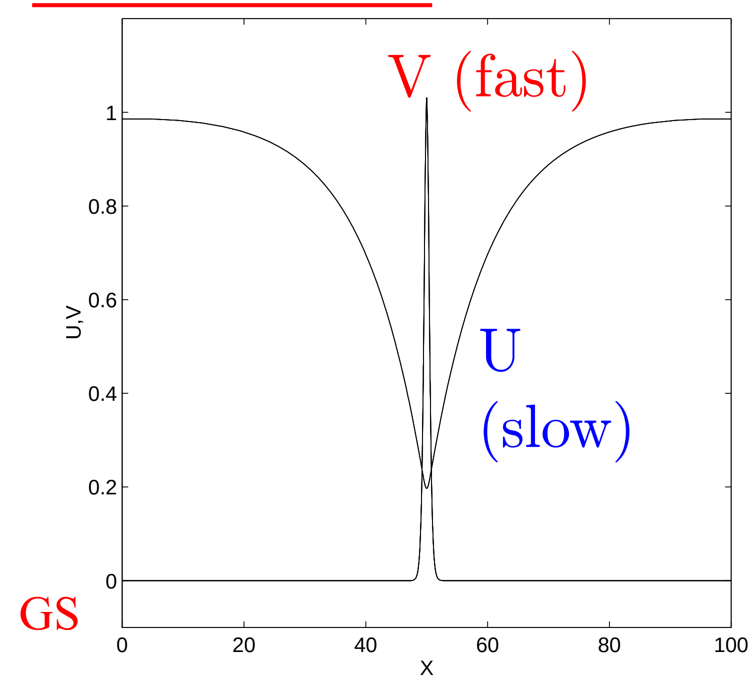
WHAT?

$$(\mathrm{GM}) \quad \begin{cases} U_t = U_{xx} - \mu U + V^2 \\ V_t = \varepsilon^2 V_{xx} - V + \frac{V^2}{U} \end{cases}$$

$$(\mathrm{GS}) \quad \begin{cases} U_t = U_{xx} + A(1-U) - UV^2 \\ V_t = \varepsilon^2 V_{xx} - BV + UV^2 \end{cases}$$

These equations share special non-generic features.

$\Longrightarrow$ Consider the 'slow' and 'fast' reduced limits.



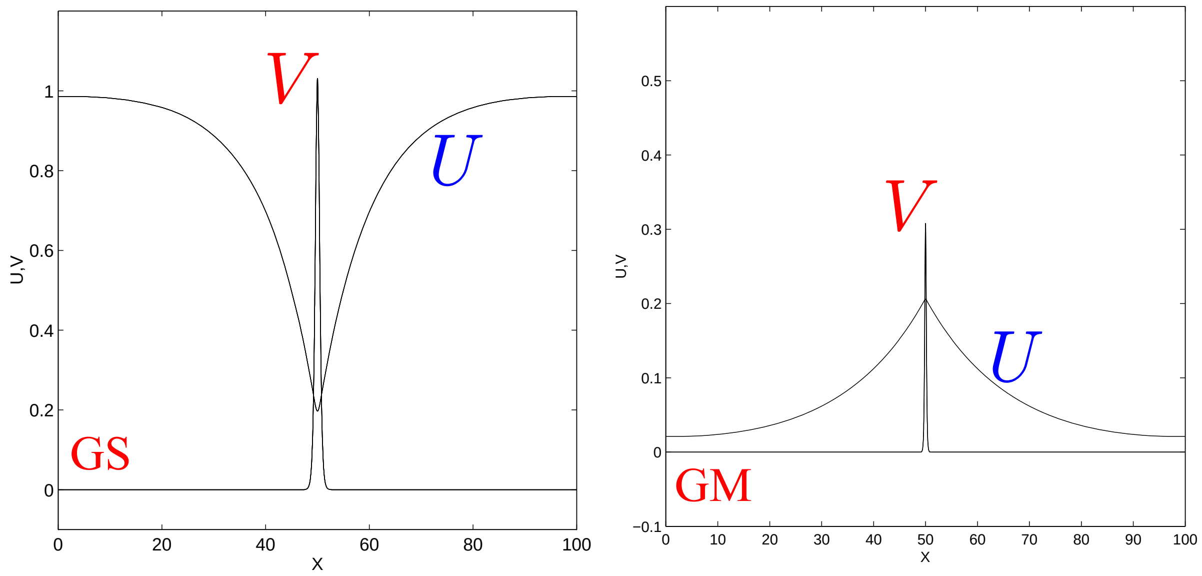
THE MOST GENERAL MODEL:

- Reaction-diffusion equation.
- Two-components, $U(x,t)$ & $V(x,t)$.
- On the unbounded domain: $x \in \mathbb{R}^1$.
- A stable background state $(U,V) \equiv (0,0)$.
- Singularly perturbed: $U(x,t)$ 'slow', $V(x,t)$ 'fast'.

$$\begin{cases} U_t = U_{xx} + \mu_{11}U + \mu_{12}V + F(U,V;\varepsilon) \\ V_t = \varepsilon^2 V_{xx} + \mu_{21}U + \mu_{22}V + G(U,V;\varepsilon) \end{cases}$$

* with $\mu_{11} + \mu_{22} < 0$ and $\mu_{11}\mu_{22} - \mu_{12}\mu_{21} > 0$.
* some technical conditions on $F(U,V)$ and $G(U,V)$.

THE SLOW REDUCED LIMIT: $\varepsilon = 0,\ V(x,t) \equiv 0.$



The slow fields:

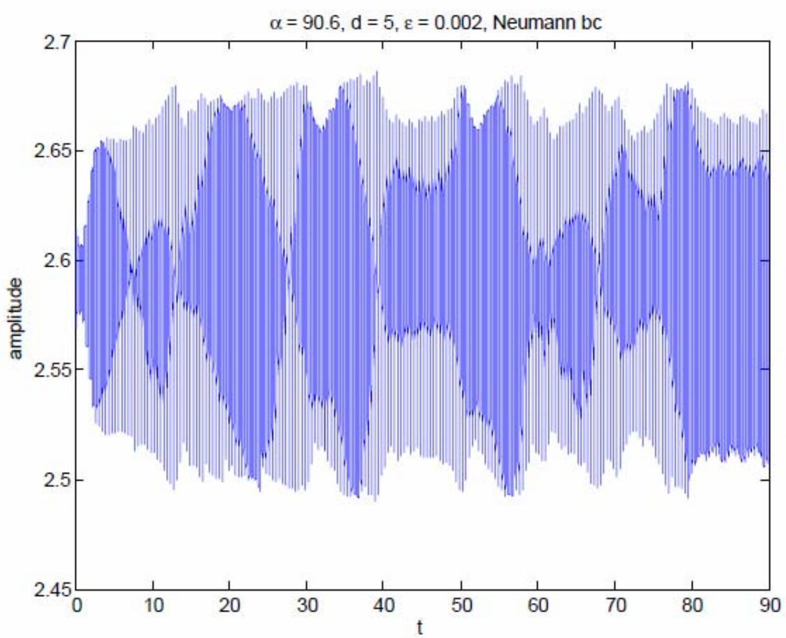
$$\begin{array}{lrcl} \text{(GM)} & U_t & = & U_{xx} - \mu U \\ \text{(GS)} & U_t & = & U_{xx} + A(1-U) \\ \text{GENERAL} & U_t & = & U_{xx} + \mu_{11}U + F(U,0;0) \end{array}$$

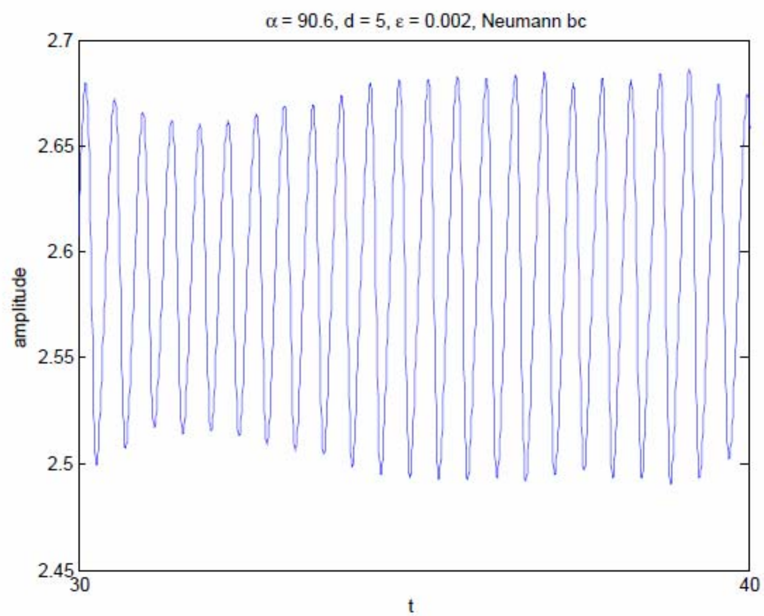
GM/GS: LINEAR, $F(U,0;0) \equiv 0!!$

Crucial for stability analysis & for Hopf/belly dance

Consider existence and stability of pulses in generic singularly perturbed systems (i.e. systems that are also nonlinear outside the localized fast pulses)

→ Significant extension Evans function approach (↔ Frits Veerman)





A Gierer-Meinhardt equation with a 'slow non-linearity'

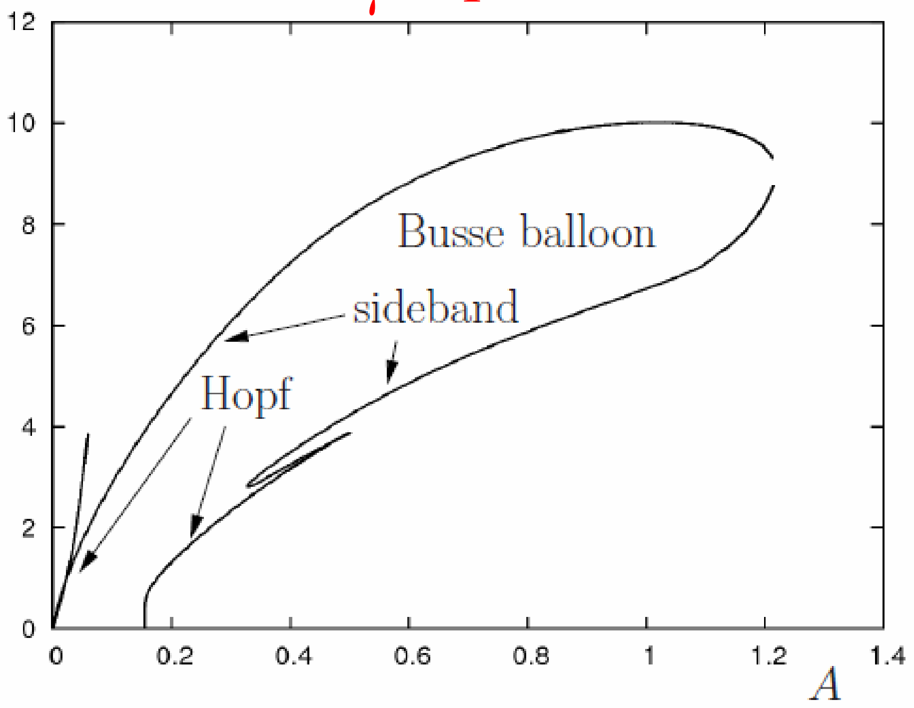
A chaotically oscillating standing pulse??

Busse balloons in the GKGS model

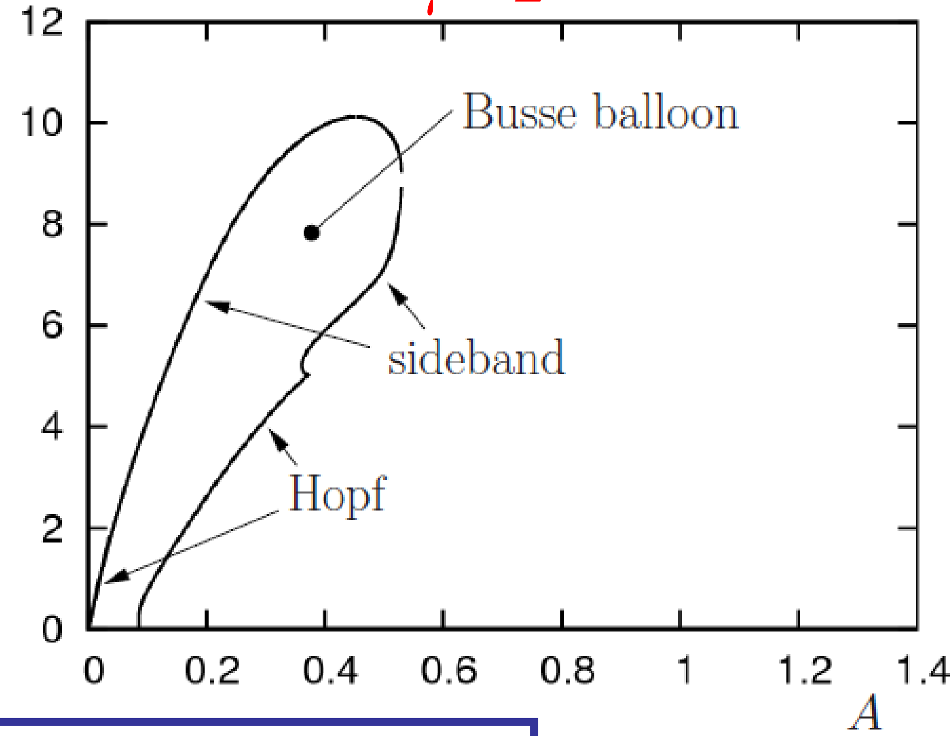
$$\begin{cases} U_t = U^\gamma_{xx} + CU_x + A(1-U) - UV^2 \\ V_t = \delta^{2\sigma} V_{xx} - BV + UV^2, \end{cases}$$

$B = C = 0.2$

$\gamma = 1$



$\gamma = 2$



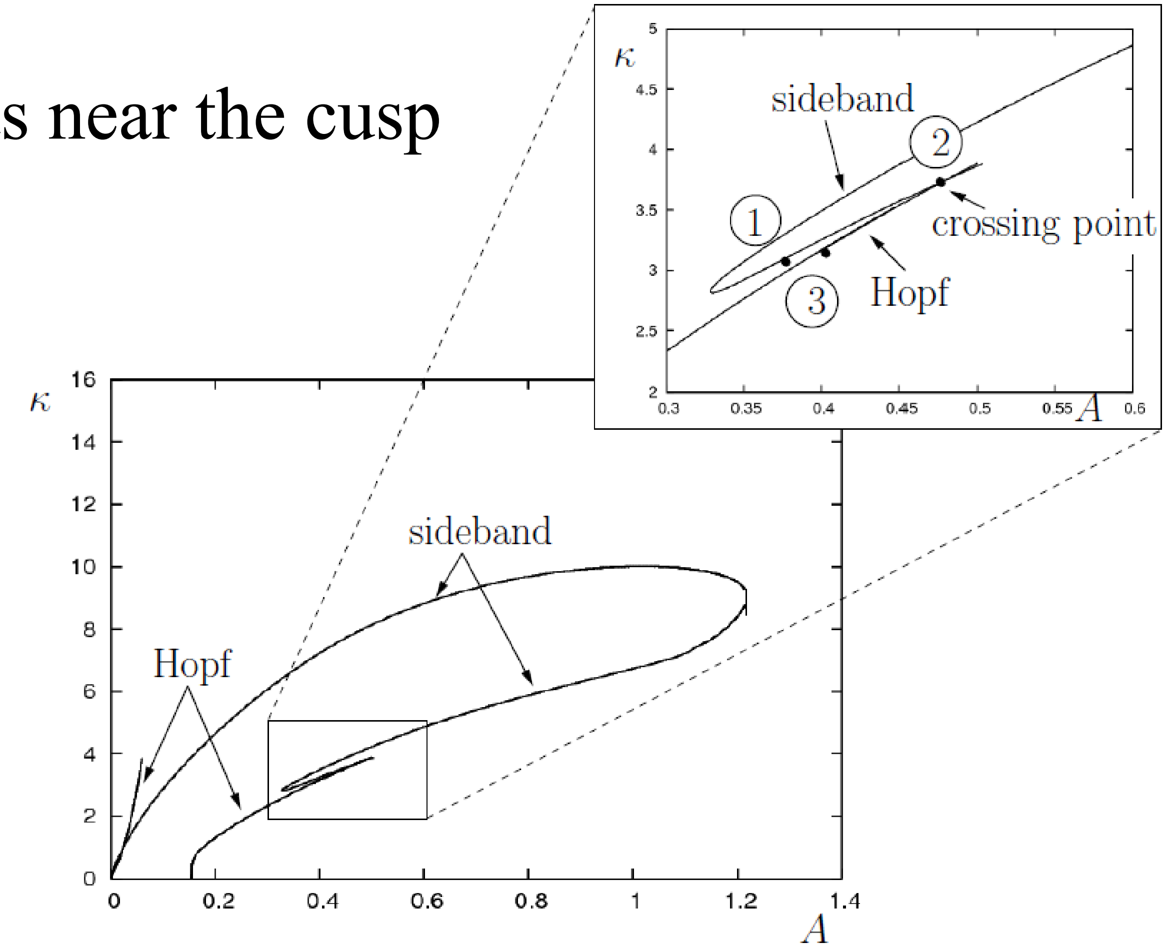
REMARKABLY SIMILAR!

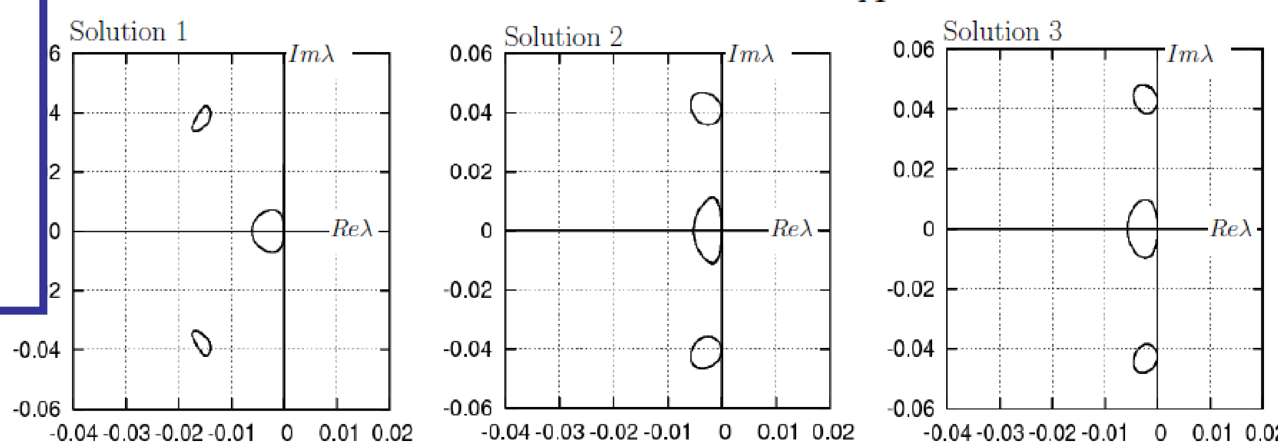
[van der Stelt, D., Hek, Rademacher]

Some spectral plots near the cusp

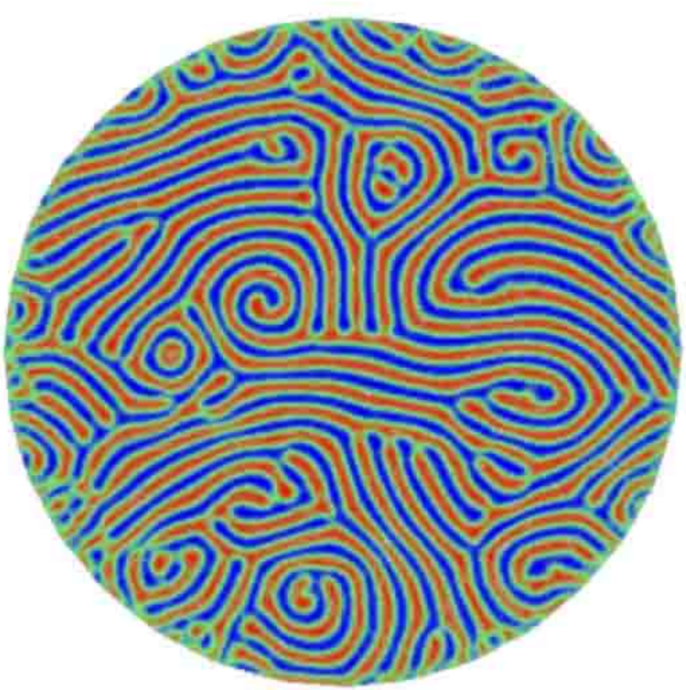
Many, many open questions about structure & nature of Busse balloons in RD-systems

→ new project
D & Rademacher





What about (almost) periodic 2-D patterns?



A defect pattern in a convection experiment

DEFECT PATTERNS

⇕

Slow modulations of (parallel) stripe patterns + localized defects

⇕

Phase-diffusion equations with defects as singularities

[Cross, Newell, Ercolani, ...]

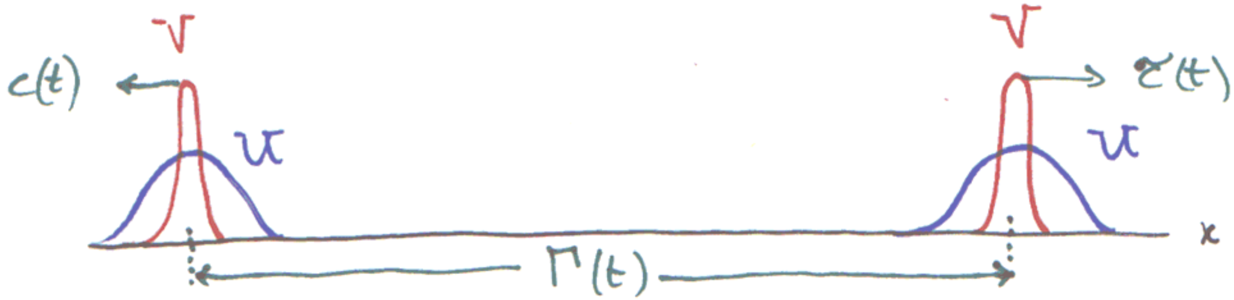
INTERACTIONS (OF LOCALIZED PATTERNS)

A hierarchy of problems

- Existence of stationary (or uniformly traveling) solutions
- The stability of the localized patterns
- The INTERACTIONS

Note: It’s no longer possible to reduce the PDE to an ODE

WEAK INTERACTIONS



General theory for exponentially small tail-tail interactions

[Ei, Promislow, Sandstede]

$\frac{d}{dt}\Gamma = C_1 e^{-C_2\Gamma}$ at leading order, for Γ large enough

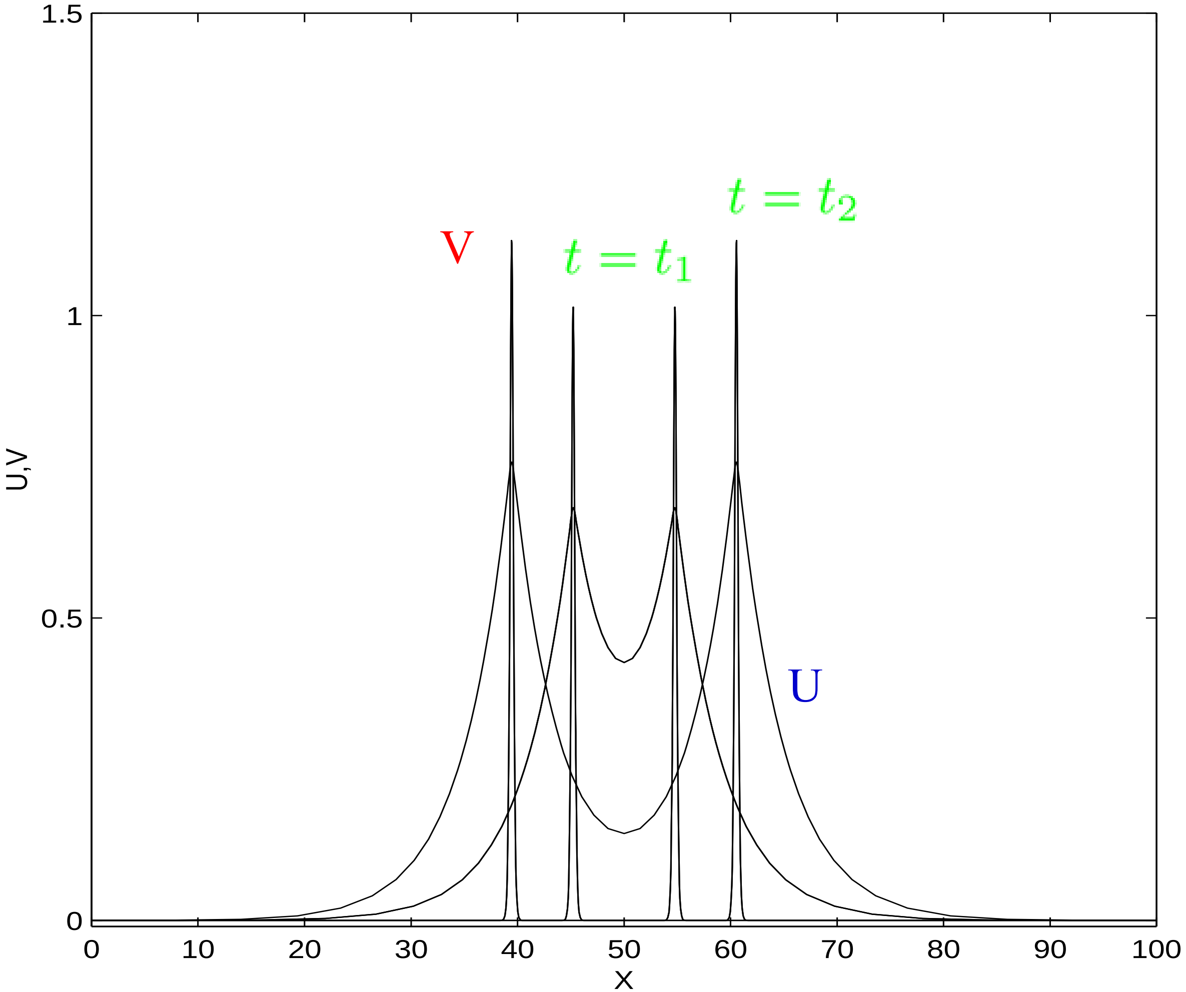
Essential: components can be treated as ‘particles’

$$\vec{U}(x,t) = \vec{U}_h(x + \frac{1}{2}\Gamma) + \vec{U}_h(x - \frac{1}{2}\Gamma)$$

is solution of the PDE up to exponentially small terms

SEMI-STRONG INTERACTIONS

- Pulses evolve and change in magnitude and shape.
- Only O(1) interactions through one component, the other components have negligible interactions

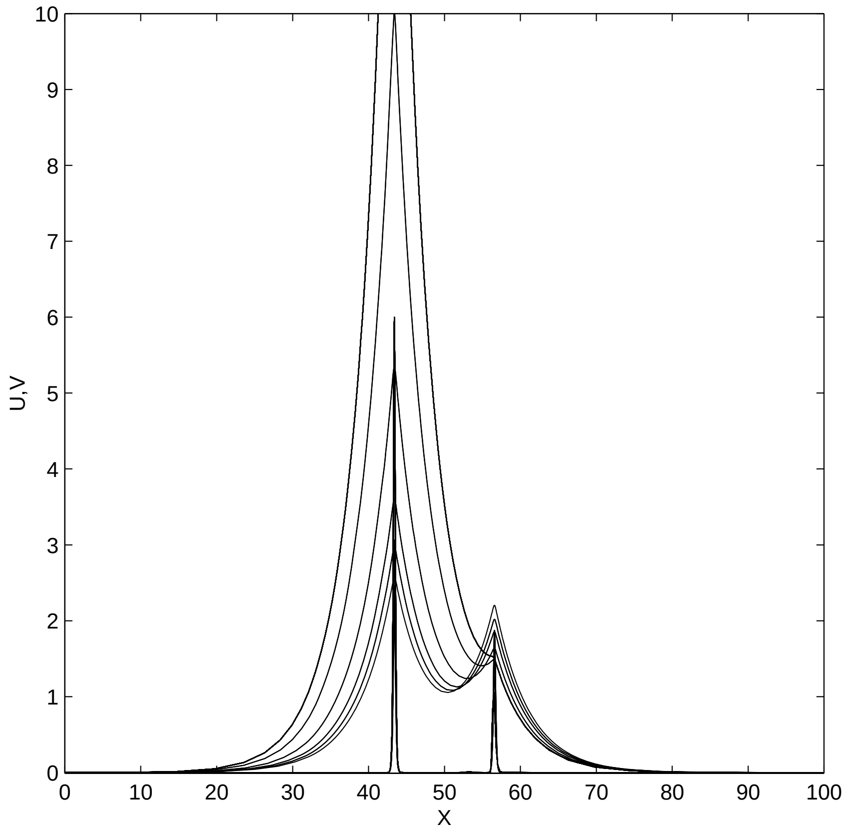


$\Updownarrow$

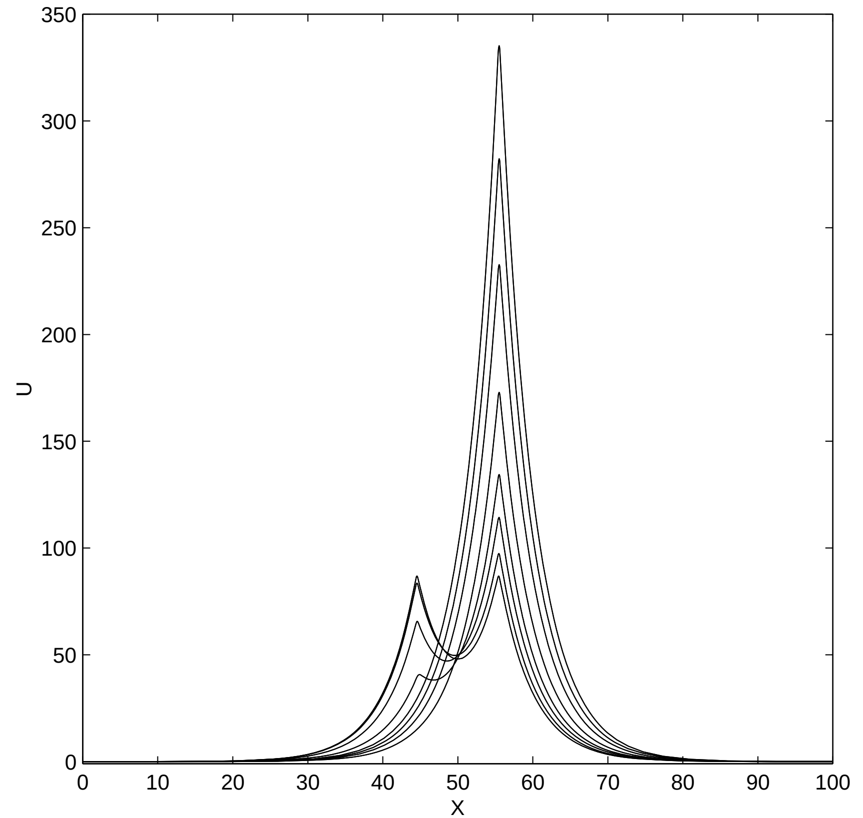
'Gap' in decay rates $\Leftrightarrow$ PDE is singularly perturbed

Pulses are no ‘particles’ and may ‘push’ each other through a `bifurcation’.

Semi-strong dynamics in two (different) modified GM models



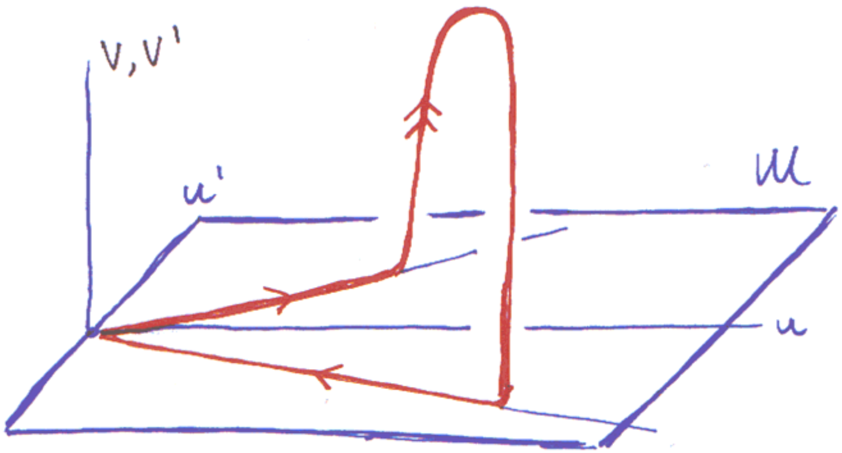
finite-time blow-up

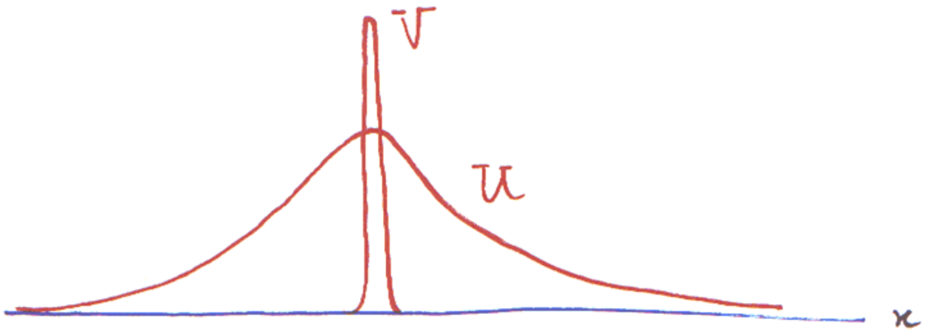


a symmetry breaking bifurcation

[D. & Kaper ’03]

Example: Pulse-interactions in (regularized) GM



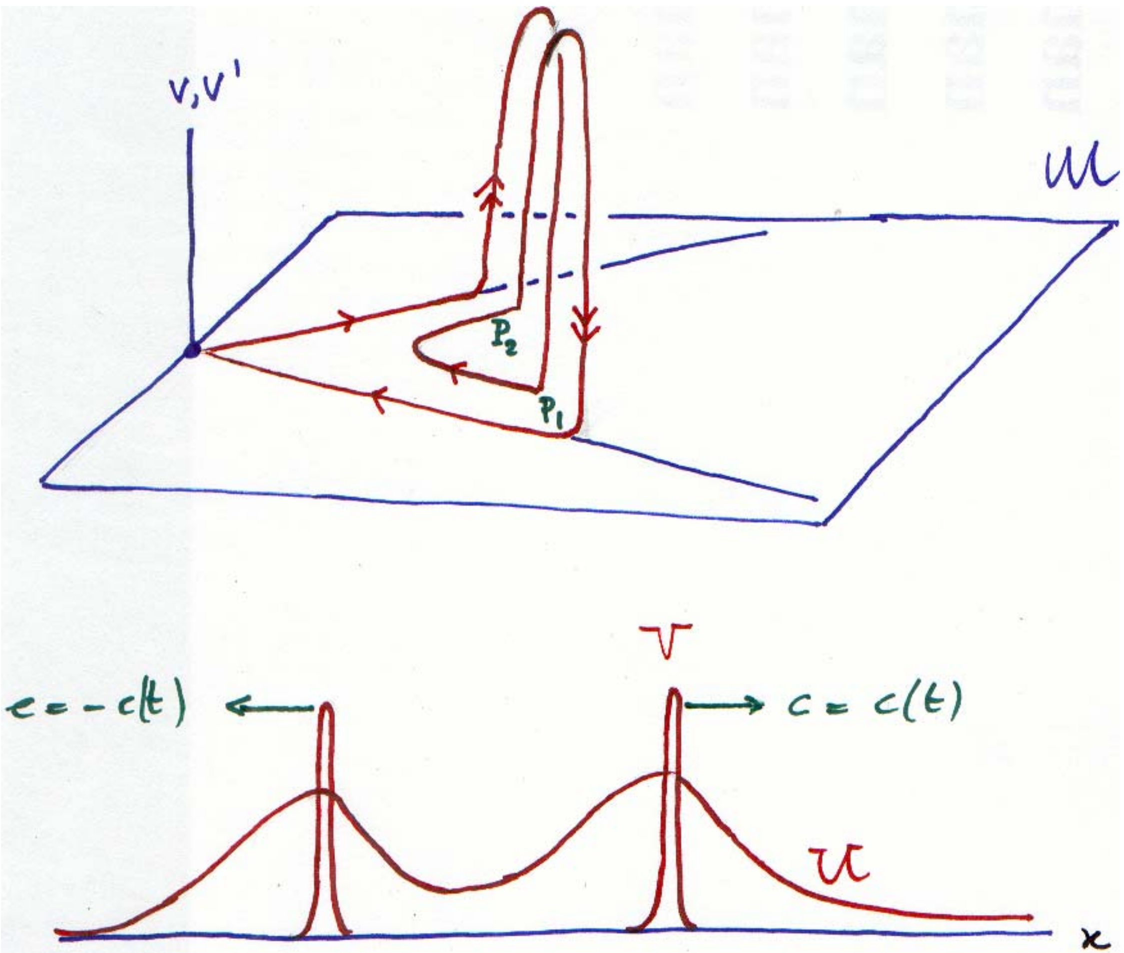


Existence and Stability

Theorem. [Doelman, Gardner, Kaper]
Let ε be small enough.

- For $0 < \mu \ll \frac{1}{\varepsilon^4}$ there is a homoclinic pulse solution $(U_h(x), V_h(x)) = \Phi_h(x)$.
- For $\mu > \mu_{\text{Hopf}}$ the pulse is spectrally stable.

The construction of the 2-pulse $\Phi_\Gamma(x)$



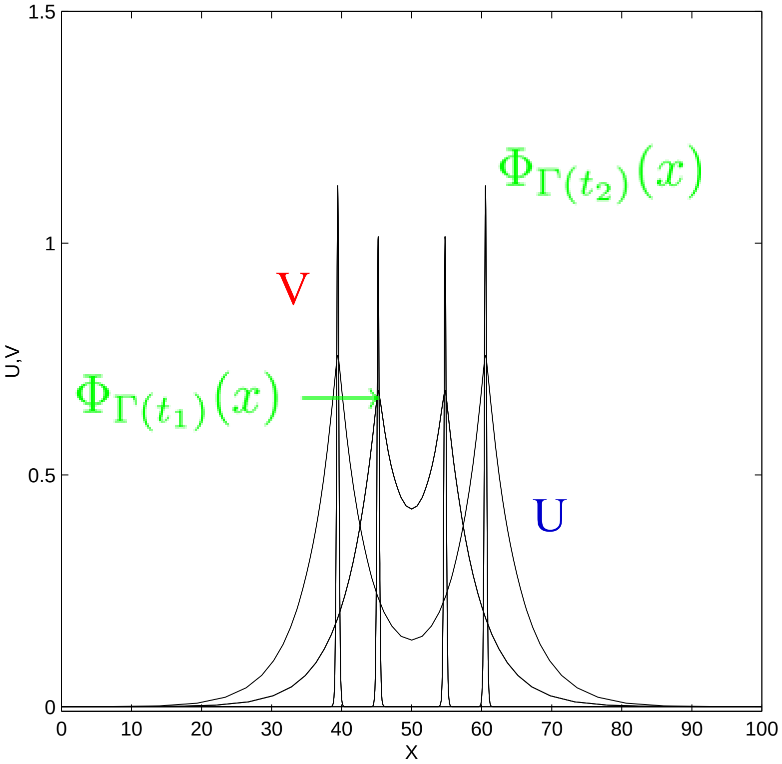
- 2 different ODE reductions with (unknown) speeds $\pm c$: one at each 'fast' V-pulse;
- outside 'fast' regions c is negligible $\mathcal{O}(\varepsilon^6)$ effect: solve the 'slow' U-eqs.

- distance between pulses $= 2\Gamma(t) = 2\int_{t_0}^{t} c(s)ds =$ 'time-of-flight' $P_1 \to P_2 = F(c) = F(\frac{d}{dt}\Gamma(t))$

$$\Rightarrow \frac{d}{dt}\Gamma = \frac{1}{2}\varepsilon^2\sqrt{\mu}\frac{e^{-2\varepsilon^2\Gamma\sqrt{\mu}}}{1+e^{-2\varepsilon^2\Gamma\sqrt{\mu}}}$$

$$\sup U_\Gamma = A(\Gamma)$$
$$\sup V_\Gamma = \frac{3}{2}A(\Gamma)$$

$$A(\Gamma) = \frac{\sqrt{\mu}}{3}\frac{1}{1+e^{-2\varepsilon^2\Gamma\sqrt{\mu}}}$$



Intrinsically formal result [Doelman, Kaper, Ward]

Note: $\Gamma \gg 1/\varepsilon^2 \rightarrow$ the weak interaction limit:

$\frac{d}{dt}\Gamma = \frac{1}{2}\varepsilon^2\sqrt{\mu}e^{-2\varepsilon^2\Gamma\sqrt{\mu}}$ and $A(\Gamma) = \frac{\sqrt{\mu}}{3}$, constant

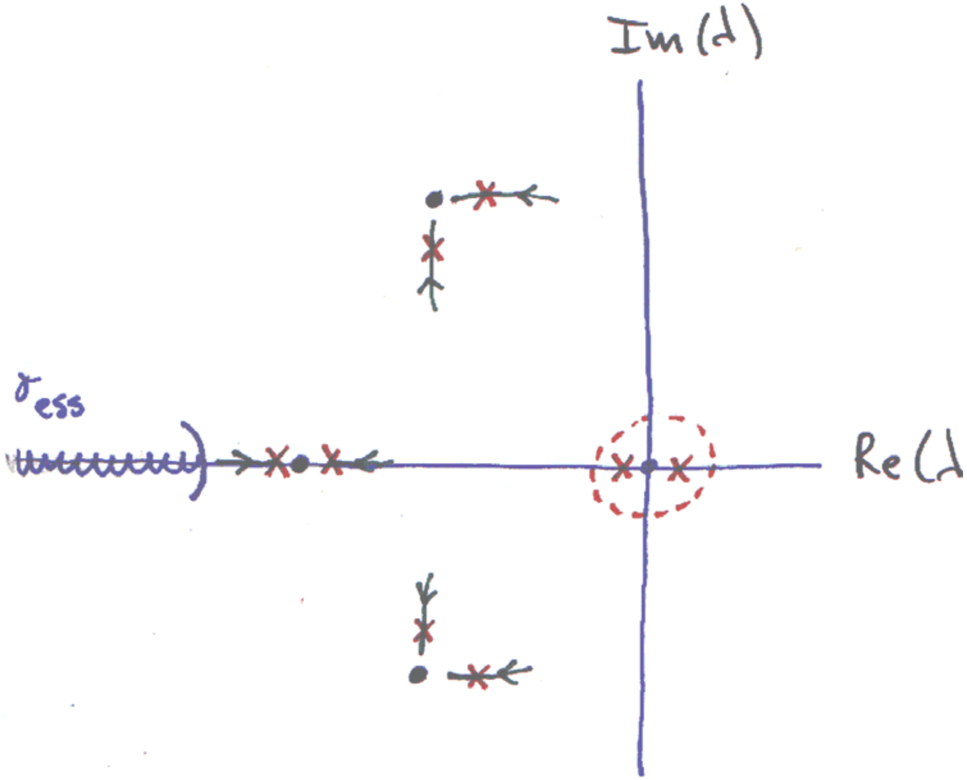
(2 `copies' of the stationary pulses)

Stability of the 2-pulse solution:

Q: What is ‘linearized stability’?

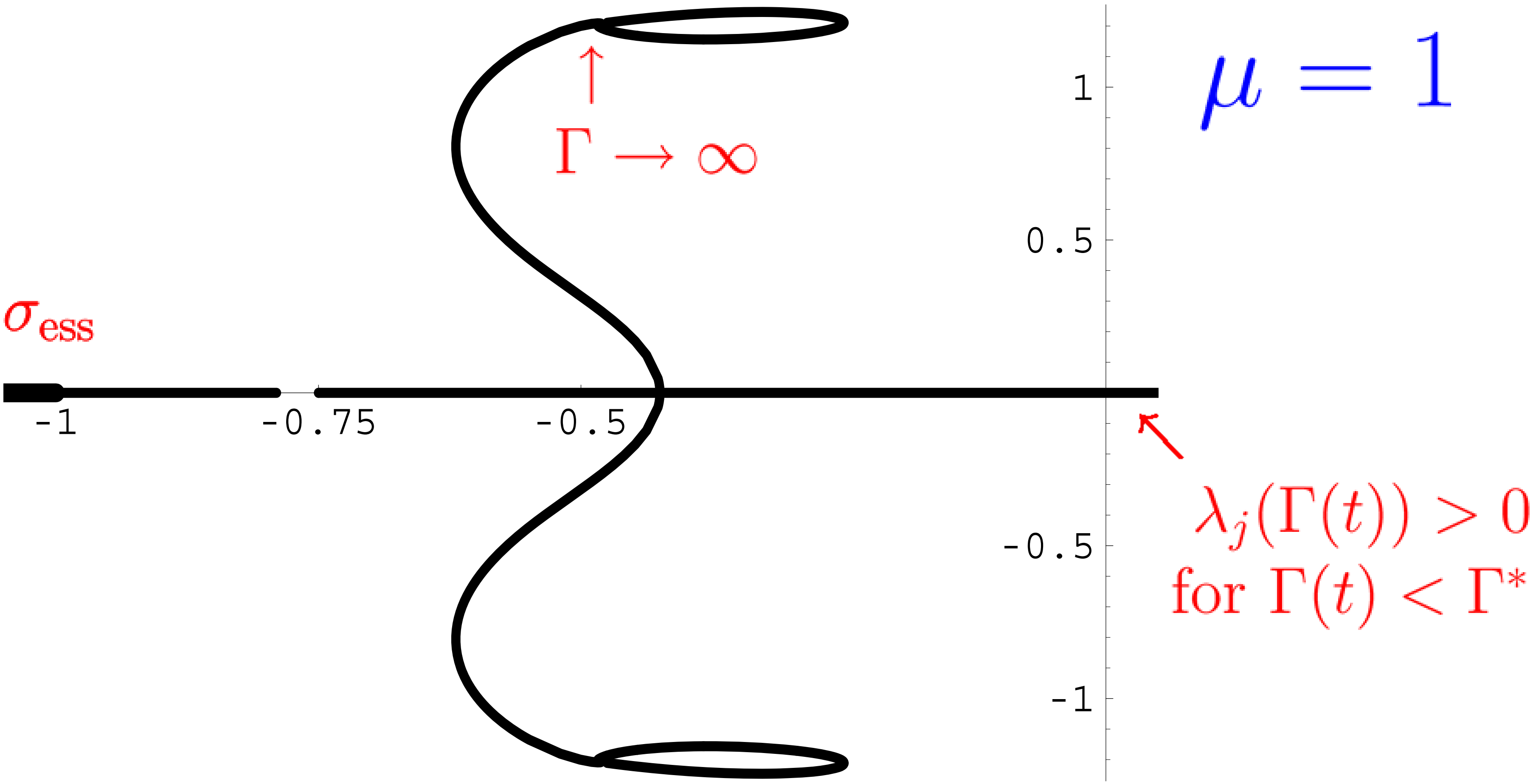
A: ‘Freeze’ solution and determine ‘quasi-steady eigenvalues’

Note: ‘not unrealistic’, since 2-pulse evolves slowly



Two pairs of eigenvalues ‘travel’ through $\mathbb{C}$ as function of the distance Γ between the pulses, and approach the eigenvalues of the stationary 1-pulse solutions as $\Gamma \to \infty$.

The Evans function approach can be used to explicitly determine the paths of the eigenvalues



Note: $\Gamma^* = \frac{\log 3}{\varepsilon^2 \sqrt{\mu}}$ for $\mu > \mu_t > \mu_{\text{Hopf}}$

Nonlinear Asymptotic Stability & Validity

Theorem [Doelman, Kaper, Promislow]

Define $W(x,t)$ by

$$(U(x,t), V(x,t)) = \Phi_{\Gamma(t)}(x) + W(x,t).$$

Let $\varepsilon > 0$ be sufficiently small, $\mu > \mu_{\text{Hopf}}$, and assume that $(U_0(x), V_0(x))$ is sufficiently close to $\Phi_{\Gamma(0)}(x)$ with $\Gamma(0) > \Gamma^*$. Then there exist $M, \nu > 0$ such that

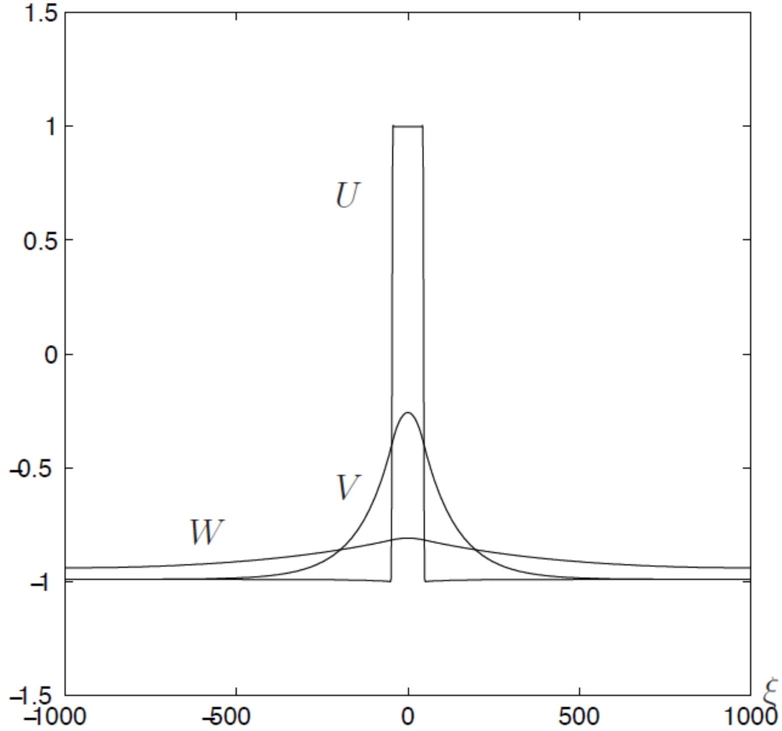
$$\|W\|_X \leq M(e^{-\nu t}\|W_0\|_X + \varepsilon^3)$$

with $\|W\|_X = \varepsilon\|W_1\|_{L^2} + \frac{1}{\varepsilon}\|\partial_x W_1\|_{L^2} + \|W_2\|_{H^1}$

Proof: Renormalization Group Method

THE 3-COMPONENT (gas-discharge) MODEL

$$\begin{cases} U_t &= U_{\xi\xi} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma)\,, \\ \tau V_t &= \frac{1}{\varepsilon^2} V_{\xi\xi} + U - V\,, \\ \theta W_t &= \frac{D^2}{\varepsilon^2} W_{\xi\xi} + U - W\,, \end{cases}$$



[Peter van Heijster, AD, Tasso Kaper, Keith Promislow '08,'09,'10]

Simple and explicit results on existence and stability

Theorem

Our system possesses a standing pulse if there exists an $A \in (0,1)$ *which solves*

$$\alpha A^2 + \beta A^{\frac{2}{D}} = \gamma \, .$$

Moreover, if $|\alpha D| > |\beta|$ *and* $\mathrm{sgn}(\alpha) \neq \mathrm{sgn}(\beta)$, *then there is a saddle-node bifurcation of homoclinic orbits at* $\gamma = \gamma_{SN}$.

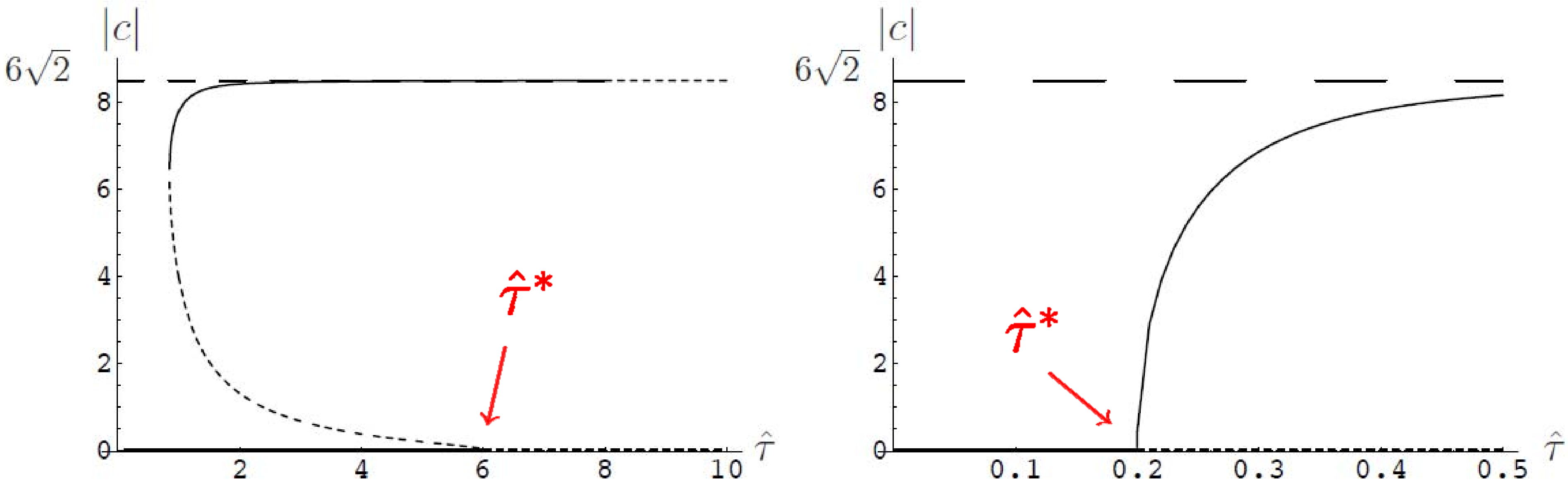
Theorem

The standing pulse with $\mathcal{O}(1)$*-parameters is stable if and only if*

$$\alpha A^2 + \frac{\beta}{D} A^{\frac{2}{D}} > 0 \, .$$

Sub- and supercritical bifurcations into traveling pulses

$$\tau = \mathcal{O}(1/\varepsilon^2) = \hat{\tau}/\varepsilon^2, \text{ speed} = \mathcal{O}(\varepsilon^2) = \varepsilon^2 c$$

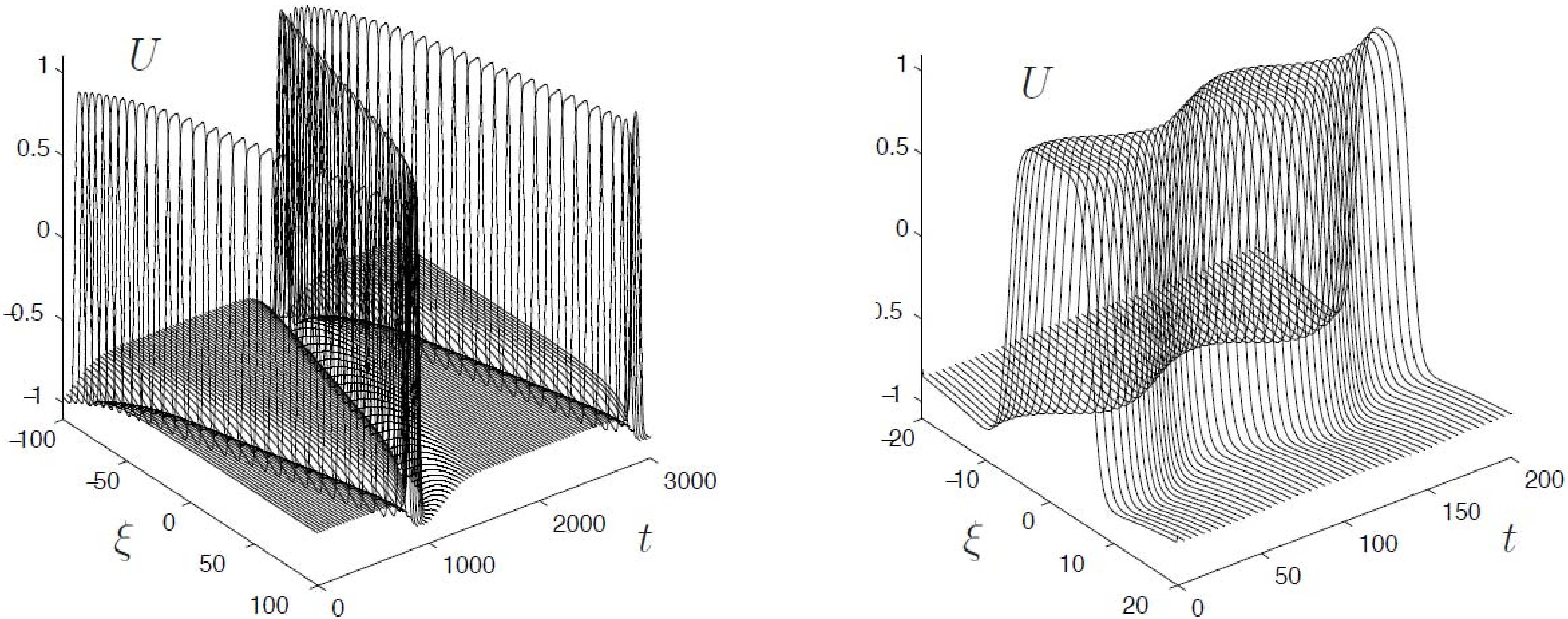


Bifurcation diagrams for two typical parameter combinations

(There is an explicit analytical expression for $\hat{\tau}^*$, etc)

Interaction between Hopf and bifurcation into traveling pulse

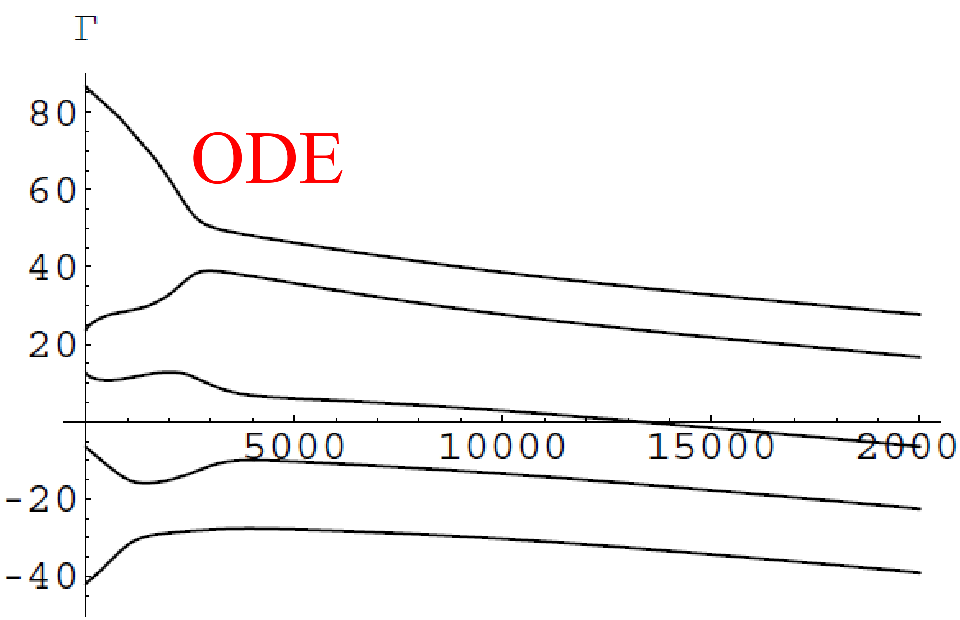
$$\tau = \mathcal{O}(1/\varepsilon^2)$$

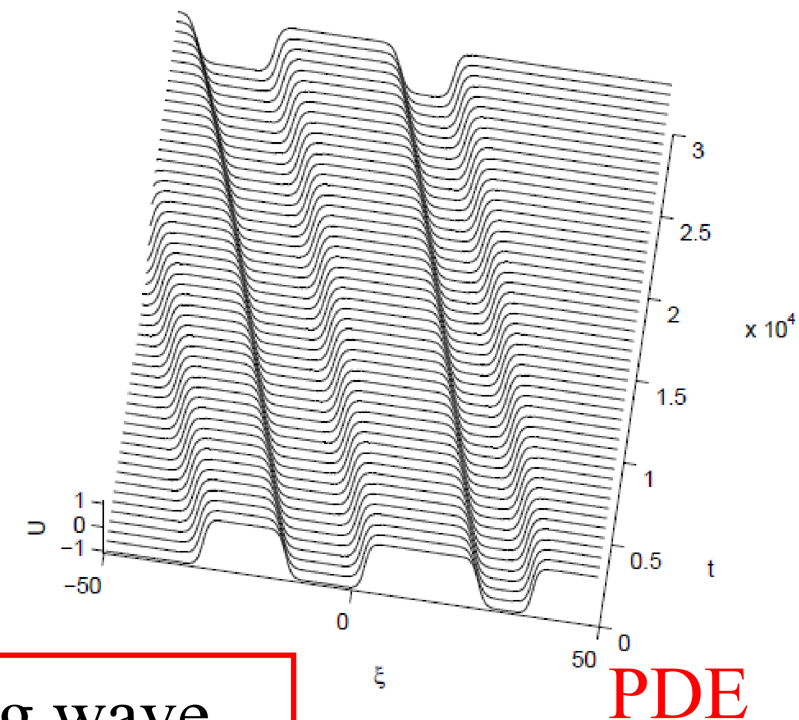


Simulations for two typical parameter combinations

Front interactions: similar validity/reduction results

$$
\begin{aligned}
\dot{\Gamma}_i(t) \quad = \quad & (-1)^{i+1}\frac{3}{2}\sqrt{2}\varepsilon\left[\gamma + \alpha\left(-e^{\varepsilon(\Gamma_1-\Gamma_i)} + \ldots + (-1)^{i-1}e^{\varepsilon(\Gamma_{i-1}-\Gamma_i)}\right.\right. \\
& +(-1)^i e^{\varepsilon(\Gamma_i-\Gamma_{i+1})} + \ldots + (-1)^{N-1}e^{\varepsilon(\Gamma_i-\Gamma_N)}\Big) + \beta\Big(-e^{\frac{\varepsilon}{D}(\Gamma_1-\Gamma_i)} \\
& + \ldots + (-1)^{i-1}e^{\frac{\varepsilon}{D}(\Gamma_{i-1}-\Gamma_i)} + (-1)^i e^{\frac{\varepsilon}{D}(\Gamma_i-\Gamma_{i+1})} + \ldots \\
& \left.\left. + (-1)^{N-1}e^{\frac{\varepsilon}{D}(\Gamma_i-\Gamma_N)}\right)\right] \quad \text{for} \quad i = 1\ldots N.
\end{aligned}
$$

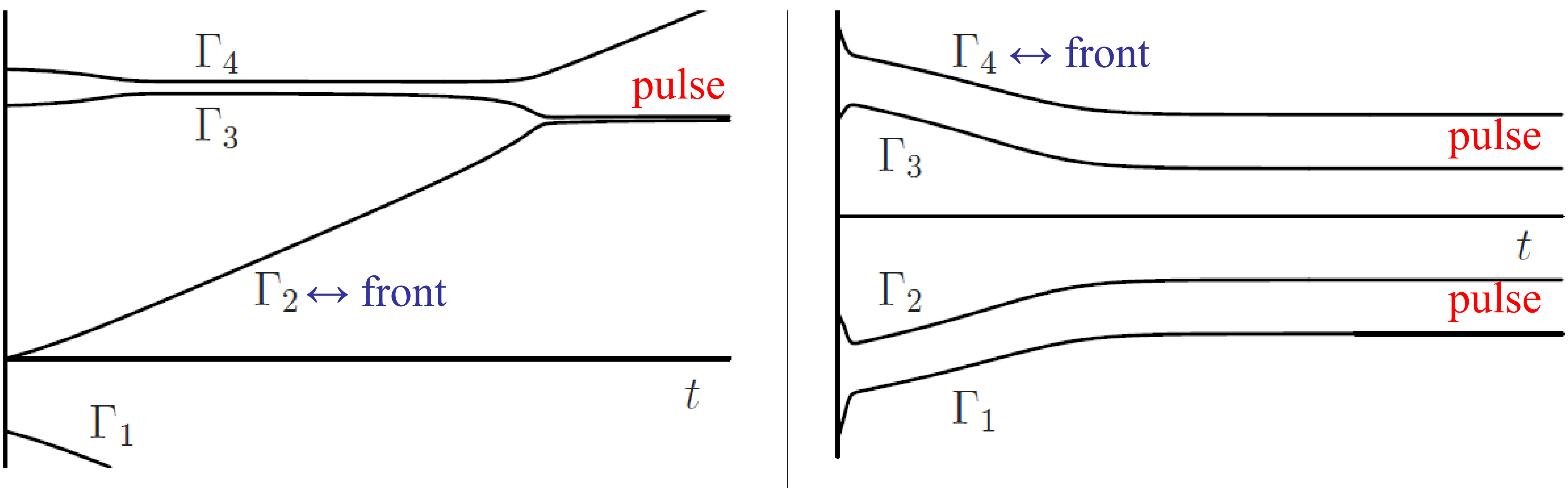




The formation of a 5-front traveling wave

N = 4

$$
\left\{
\begin{aligned}
\dot{\Gamma}_1(t) &= \tfrac{3}{2}\sqrt{2}\varepsilon\left(\gamma + \alpha\left(-e^{\varepsilon(\Gamma_1-\Gamma_2)} + e^{\varepsilon(\Gamma_1-\Gamma_3)} - e^{\varepsilon(\Gamma_1-\Gamma_4)}\right)\right.\\
&\quad \left. + \beta\left(-e^{\frac{\varepsilon}{D}(\Gamma_1-\Gamma_2)} + e^{\frac{\varepsilon}{D}(\Gamma_1-\Gamma_3)} - e^{\frac{\varepsilon}{D}(\Gamma_1-\Gamma_4)}\right)\right),\\
\dot{\Gamma}_2(t) &= -\tfrac{3}{2}\sqrt{2}\varepsilon\left(\gamma + \alpha\left(-e^{\varepsilon(\Gamma_1-\Gamma_2)} + e^{\varepsilon(\Gamma_2-\Gamma_3)} - e^{\varepsilon(\Gamma_2-\Gamma_4)}\right)\right.\\
&\quad \left. + \beta\left(-e^{\frac{\varepsilon}{D}(\Gamma_1-\Gamma_2)} + e^{\frac{\varepsilon}{D}(\Gamma_2-\Gamma_3)} - e^{\frac{\varepsilon}{D}(\Gamma_2-\Gamma_4)}\right)\right),\\
\dot{\Gamma}_3(t) &= \tfrac{3}{2}\sqrt{2}\varepsilon\left(\gamma + \alpha\left(-e^{\varepsilon(\Gamma_1-\Gamma_3)} + e^{\varepsilon(\Gamma_2-\Gamma_3)} - e^{\varepsilon(\Gamma_3-\Gamma_4)}\right)\right.\\
&\quad \left. + \beta\left(-e^{\frac{\varepsilon}{D}(\Gamma_1-\Gamma_3)} + e^{\frac{\varepsilon}{D}(\Gamma_2-\Gamma_3)} - e^{\frac{\varepsilon}{D}(\Gamma_3-\Gamma_4)}\right)\right),\\
\dot{\Gamma}_4(t) &= -\tfrac{3}{2}\sqrt{2}\varepsilon\left(\gamma + \alpha\left(-e^{\varepsilon(\Gamma_1-\Gamma_4)} + e^{\varepsilon(\Gamma_2-\Gamma_4)} - e^{\varepsilon(\Gamma_3-\Gamma_4)}\right)\right.\\
&\quad \left. + \beta\left(-e^{\frac{\varepsilon}{D}(\Gamma_1-\Gamma_4)} + e^{\frac{\varepsilon}{D}(\Gamma_2-\Gamma_4)} - e^{\frac{\varepsilon}{D}(\Gamma_3-\Gamma_4)}\right)\right).
\end{aligned}
\right.
$$



DISCUSSION AND MORE ....

There is a well-developed theory for ‘simple’ patterns (localized, spatially periodic, radially symmetric, ...) in ‘simple’ equations.

In 1 spatial dimension ‘quite some’ analytical insight can be obtained, but more complex dynamics are still beyond our grasp ...

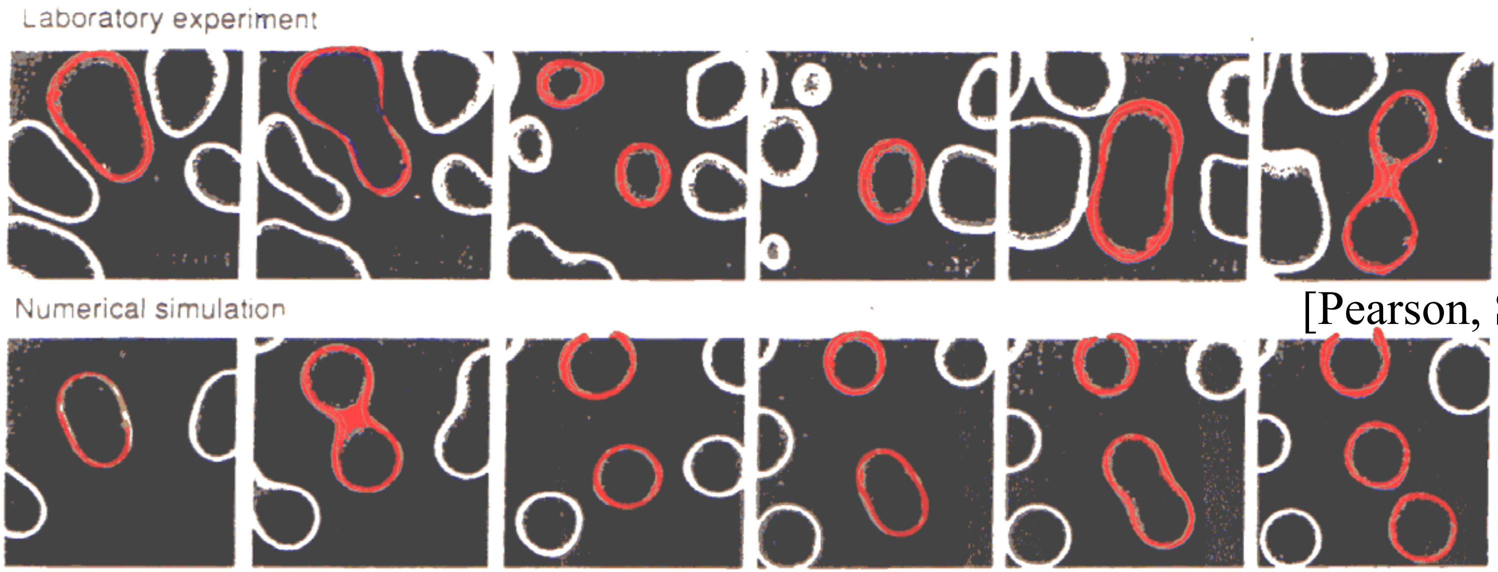
Challenges:

- Defects in 2-dimensional stripe patterns

- Strong pulse interactions (1 D!)

- ....

The GS equation perhaps is one of the most well-studied reaction-diffusion equations of the last decades.

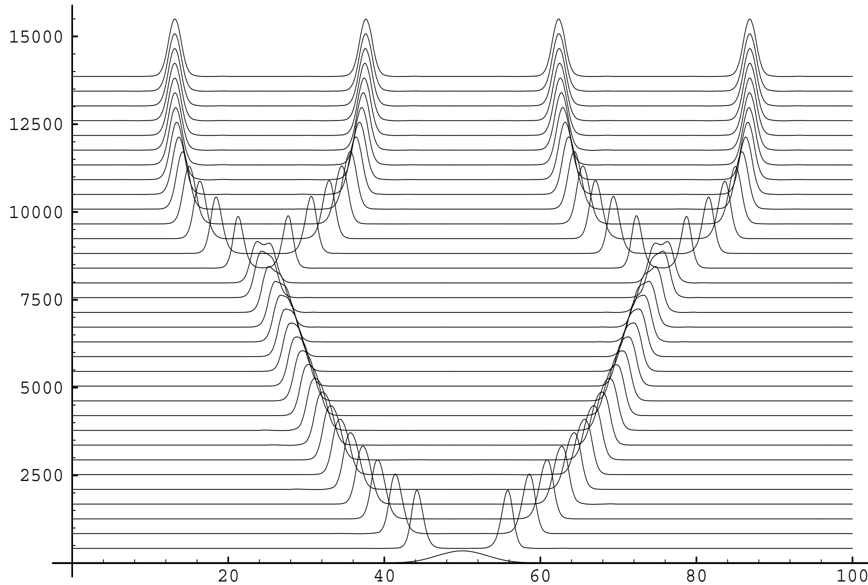


chemical reaction

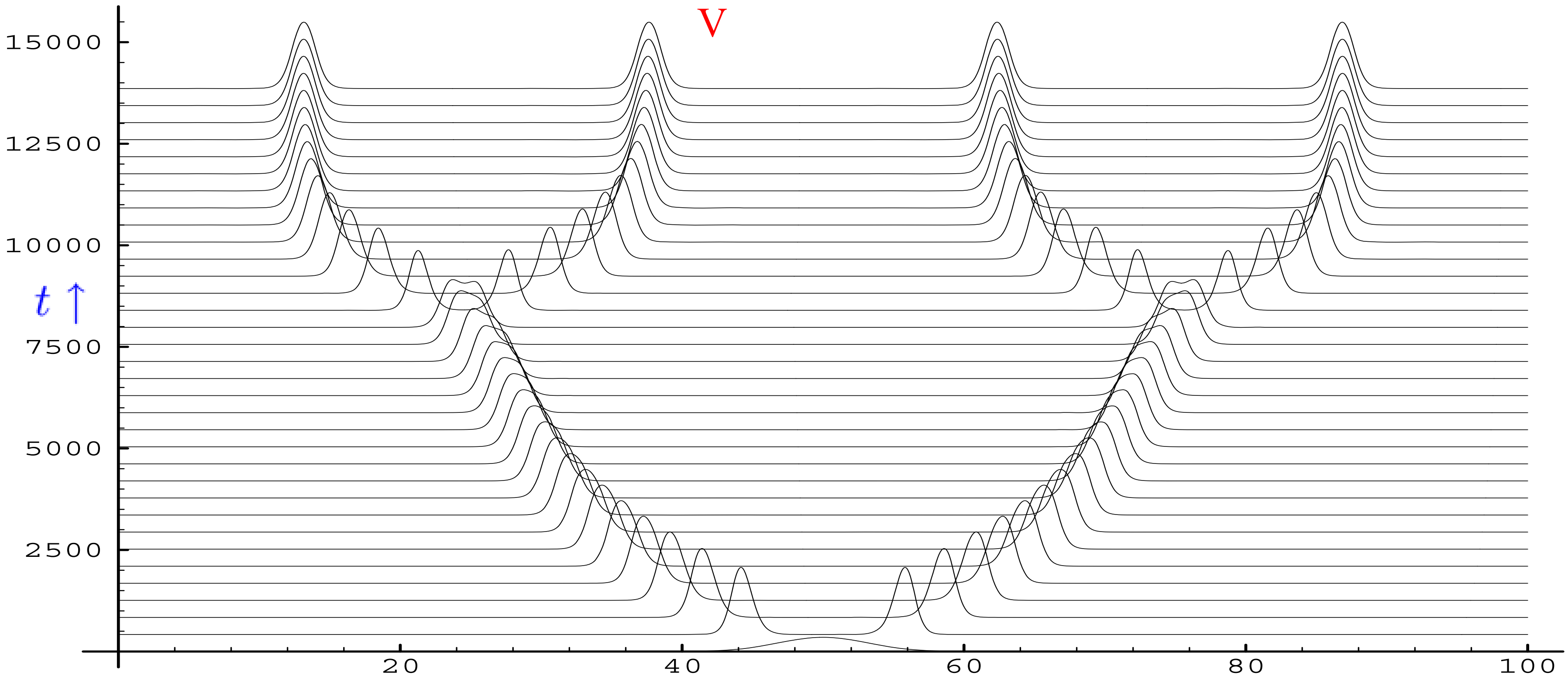
[Pearson, Swinney et al. 1994]

numerical simulation

It's mostly famous' for exhibiting 'self-replication dynamics'



Strong interactions ...



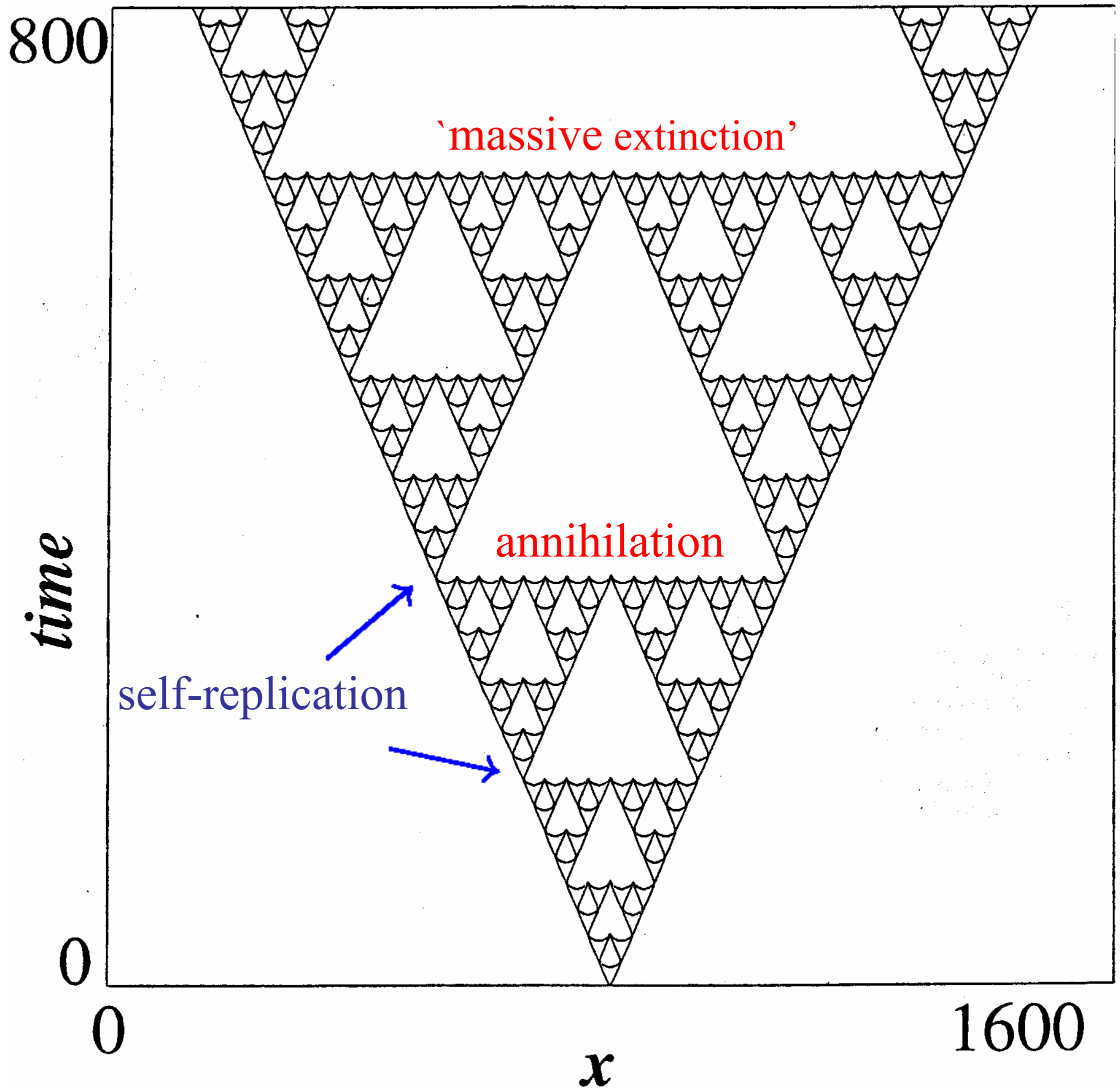
(simulations in GS)

<u>The pulse self-replication mechanism</u>

A generic phenomenon, originally discovered by Pearson et al in '93 in GS. Studied extensively, but still not understood.

[Pearson, Doelman, Kaper, Nishiura, Muratov,Ward, ...]

And there is more, much more ...



[Ohta, in GS & other systems]

A structurally stable Sierpinsky gasket ...

Various kinds of spot-, front-, stripe- interactions in 2D

[van Heijster, Sandstede]