Codim 1 and 2 bifurcations of planar ODEs

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USS Applied Bifurcation Theory
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Contents

Literature

1. A.A. Andronov, E.A. Leontovich, I.I. Gordon, and A.G. Maier *Qualitative Theory of Second-Order Dynamic Systems*, Willey & Sons, London, 1973

2. A.A. Andronov, E.A. Leontovich, I.I. Gordon, and A.G. Maier *Theory of Bifurcations of Dynamic Systems on a Plane*, Willey & Sons, London, 1973

3. F. Dumortier, J. Llibre, and J.C. Artés *Qualitative Theory of Planar Differential Systems*, Universitext, Springer-Verlag, Berlin, 2006

4. Yu.A. Kuznetsov *Elements of Applied Bifurcation Theory*, 3rd ed. Applied Mathematical Sciences 112, Springer-Verlag, New York, 2004

1. SOLUTIONS, ORBITS, AND PHASE PORTRAITS

General planar system:

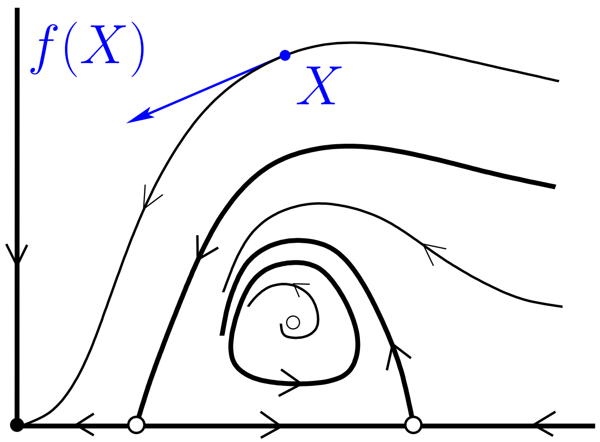
$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y) \end{cases} \quad \text{or} \quad \dot{X} = f(X), \quad X \in \mathbb{R}^2,$$

where

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(X) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}.$$

Theorem 1 *If f is smooth than for any inital point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ there exists a unique locally defined solution $t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ such that $x(0) = x_0$ and $y(0) = y_0$.*

Definition 1 *Let I be the maximal definition interval of a solution $t \mapsto X(t),\ t \in I$. The oriented by the advance of time image $X(I) \subset \mathbb{R}^2$ is called the* **orbit**.



Vector field: $X \mapsto f(X)$

$f(X) \neq 0$ is tangent to the orbit through X
$\Rightarrow$ orbits do not cross.

Definition 2 Phase portrait *of a planar system is the collection of all its orbits in $\mathbb{R}^2$.*

We draw only key orbits, which determine the topology of the phase portrait.

Types of orbits:

1. **Equilibria**: $X(t) = X_0$ so that $f(X_0) = 0$.

2. **Periodic orbits (cycles):** $X(t) \not\equiv X_0, \ X(t+T) = X(t), t \in \mathbb{R}$
   The minimal $T > 0$ is called the **period** of the cycle.

3. **Connecting orbits:** $\lim_{t \to \pm\infty} X(t) = X_\pm$ with $f(X_\pm) = 0$.

   If $X_- = X_+$ the connecting orbit is called **homoclinic**
   If $X_- \neq X_+$ the connecting orbit is called **heteroclinic**.

4. **All other orbits**

2. EQUILIBRIA $f(X) = 0 \quad \Leftrightarrow \quad \begin{cases} P(x,y) &= 0, \\ Q(x,y) &= 0. \end{cases}$

Jacobian matrix of the equilibrium $X_0 = (x_0, y_0)$:

$$A = f_X(X_0) = \left. \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \right|_{x=x_0, y=y_0}$$

Eigenvalues of the equilibrium X_0 are the eigenvalues of A, i.e. the solutions of

$$\lambda^2 - \sigma\lambda + \Delta = 0,$$

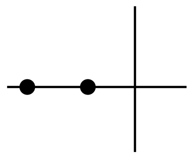
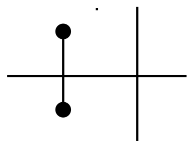
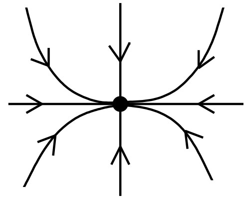
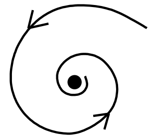
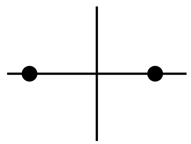
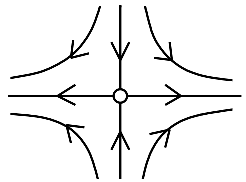
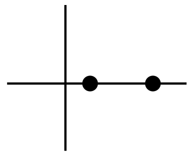
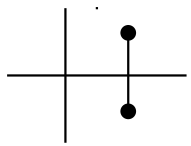
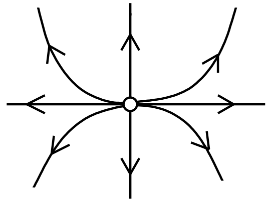
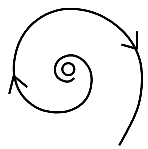
where

$$\begin{aligned} \sigma &= \lambda_1 + \lambda_2 = \mathrm{Tr} A = P_x(x_0, y_0) + Q_y(x_0, y_0), \\ \Delta &= \lambda_1 \lambda_2 = \det A = P_x(x_0, y_0) Q_y(x_0, y_0) - P_y(x_0, y_0) Q_x(x_0, y_0). \end{aligned}$$

$$\lambda_{1,2} = -\frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} - \Delta}$$

Definition 3 *An equilibrium X_0 is* **hyperbolic** *if $\Re(\lambda) \neq 0$.*

Phase portraits of generic planar systems $\dot{Y} = AY$

(n_u, n_s)	Eigenvalues	Phase portrait	Stability
(0, 2)	 	 node  focus	stable
(1, 1)		 saddle	unstable
(2, 0)	 	 node  focus	unstable

Definition 4 *Two systems are called* **topologically equivalent** *if their phase portraits are homeomorphic, i.e. there is a continuous invertible transformation that maps orbits of one system onto orbits of the other, preserving their orientation.*

Theorem 2 (Grobman-Hartman) *Consider a smooth nonlinear system*

$$\dot{X} = AX + F(X), \quad F = \mathcal{O}(\|X\|^2) \equiv O(2),$$

and its **linearization**

$$\dot{Y} = AY.$$

If $\Re(\lambda) \neq 0$ for all eigenvalues of A, then these systems are locally topologically equivalent near the origin.

Warning: A stable/unstable node is locally topologically equivalent to a stable/unstable focus.

Trivial topological equivalences

1. **Orbital equivalence**:

$$\dot{X} = f(X) \ \sim \ \dot{Y} = g(Y)f(Y)$$

where $g : \mathbb{R}^2 \mapsto \mathbb{R}$ is smooth positive function; $Y = h(X) = X$ preserves orbits.

2. **Smooth equivalence**:

$$\dot{X} = f(X) \ \sim \ \dot{Y} = [h_Y(Y)]^{-1} f(h(Y)),$$

where $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth diffeomorphism; the substitution $X = h(Y)$ transforms solutions onto solutions:

$$\dot{X} = h_Y(Y)\dot{Y} = f(h(Y)) = f(X).$$

3. **Smooth orbital equivalence**: 1. + 2.

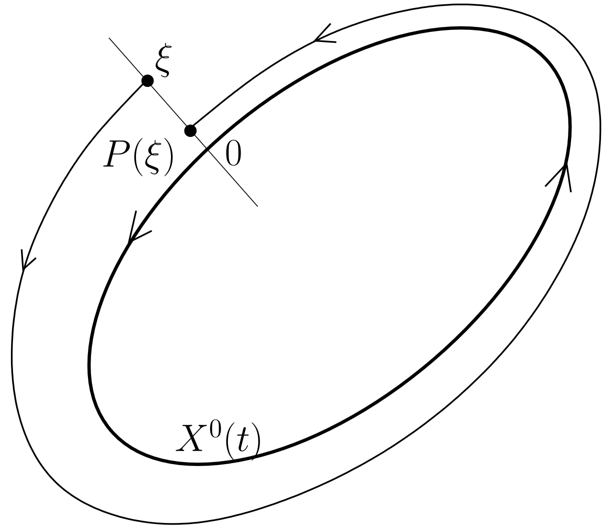
PERIODIC ORBITS AND LIMIT CYCLES

Poincaré map:

$$\xi \mapsto P(\xi) = \mu\xi + O(2),$$

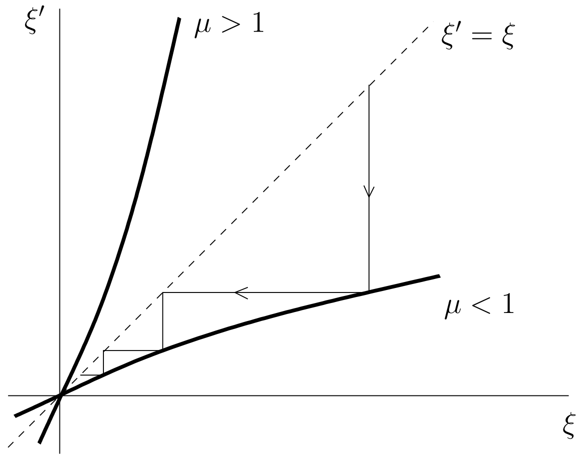
where the **multiplier**

$$\mu = \exp\left(\int_0^T (\text{div } f)(X^0(t))\, dt\right) > 0$$



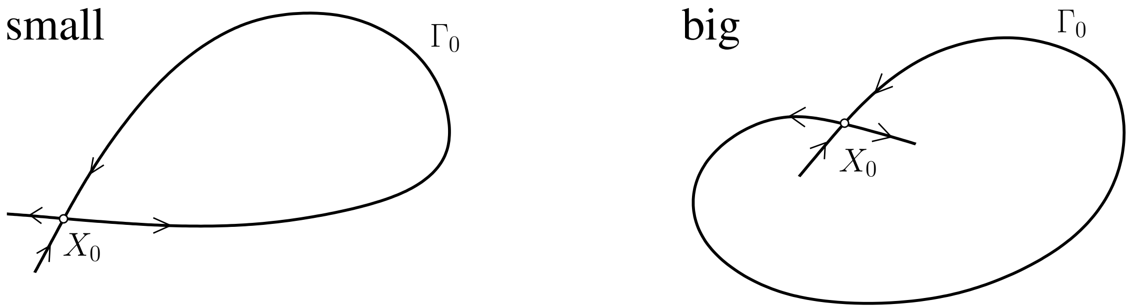
Definition 5 *A cycle of the planar system is* **hyperbolic** *if* $\mu \neq 1$.

The cycle is stable if $\mu < 1$ and is unstable if $\mu > 1$.

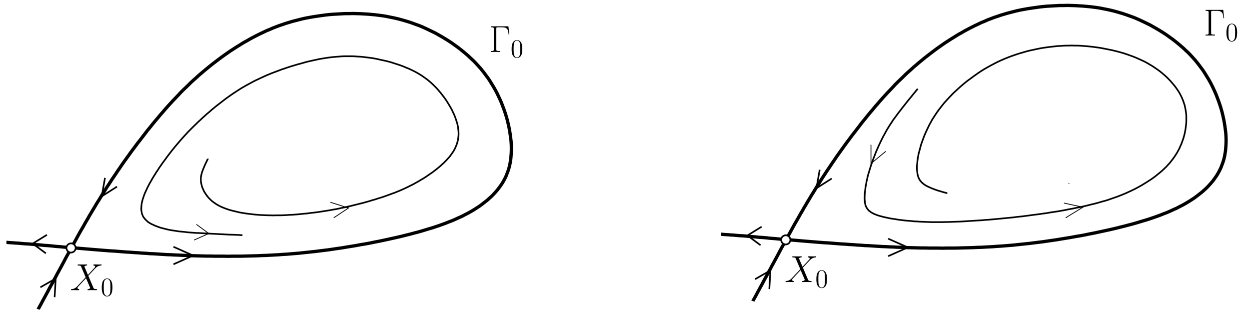


HOMOCLINIC ORBITS

Homoclinic orbits to saddles:



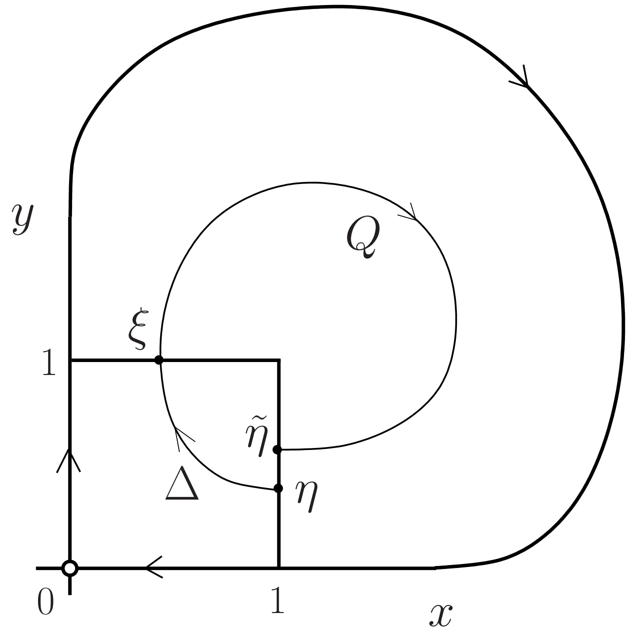
Definition 6 *The real number* $\sigma = \lambda_1 + \lambda_2 = (\text{div } f)(X_0)$ *is called the* **saddle quantity** *of* X_0.



$\sigma < 0$ $\sigma > 0$

Near the saddle, any planar system is C^1-equivalent to its linearization.

Singular map:



$$\begin{cases} \dot{x} & = & \lambda_1 x \\ \dot{y} & = & \lambda_2 y \end{cases}$$

$$\xi = \Delta(\eta) = \eta^{-\frac{\lambda_1}{\lambda_2}}$$

Regular map:

$$\tilde{\eta} = Q(\xi) = A\xi + O(2), \quad A > 0.$$

Poincaré map:

$$\eta \mapsto \tilde{\eta} = Q(\Delta(\eta)) = A\eta^{-\frac{\lambda_1}{\lambda_2}} + \dots$$

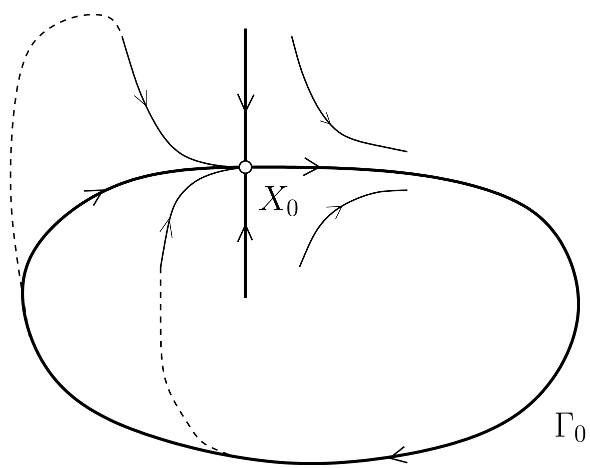
The homoclinic orbit is stable if $\sigma < 0$ and is unstable if $\sigma > 0$.

If $\sigma = \lambda_1 + \lambda_2 = 0$, then

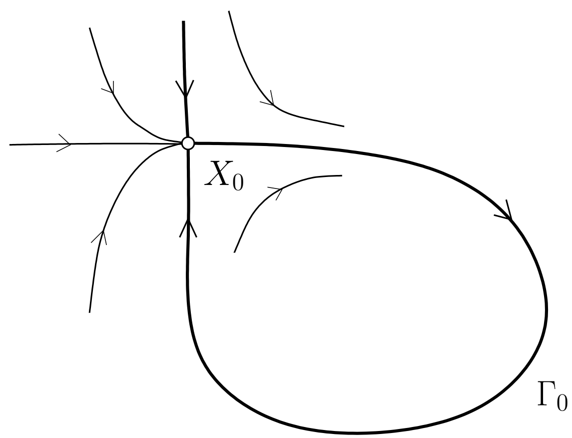
if $\int_{-\infty}^{\infty} (\text{div } f)(X^0(t))\; dt < 0$ the homoclinic orbit is stable;

if $\int_{-\infty}^{\infty} (\text{div } f)(X^0(t))\; dt > 0$ the homoclinic orbit is unstable.

Homoclinic orbits to saddle-nodes:



codim 1



codim 2

3. BIFURCATIONS AND THEIR CLASSIFICATION

Consider a smooth 2D system depending on one parameter

$$\dot{X} = f(X, \alpha), \quad X \in \mathbb{R}^2, \ \alpha \in \mathbb{R}.$$

Definition 7 *A point α_0 is called a* **bifurcation point** *if in any neighborhood of α_0 there is a point α for which*

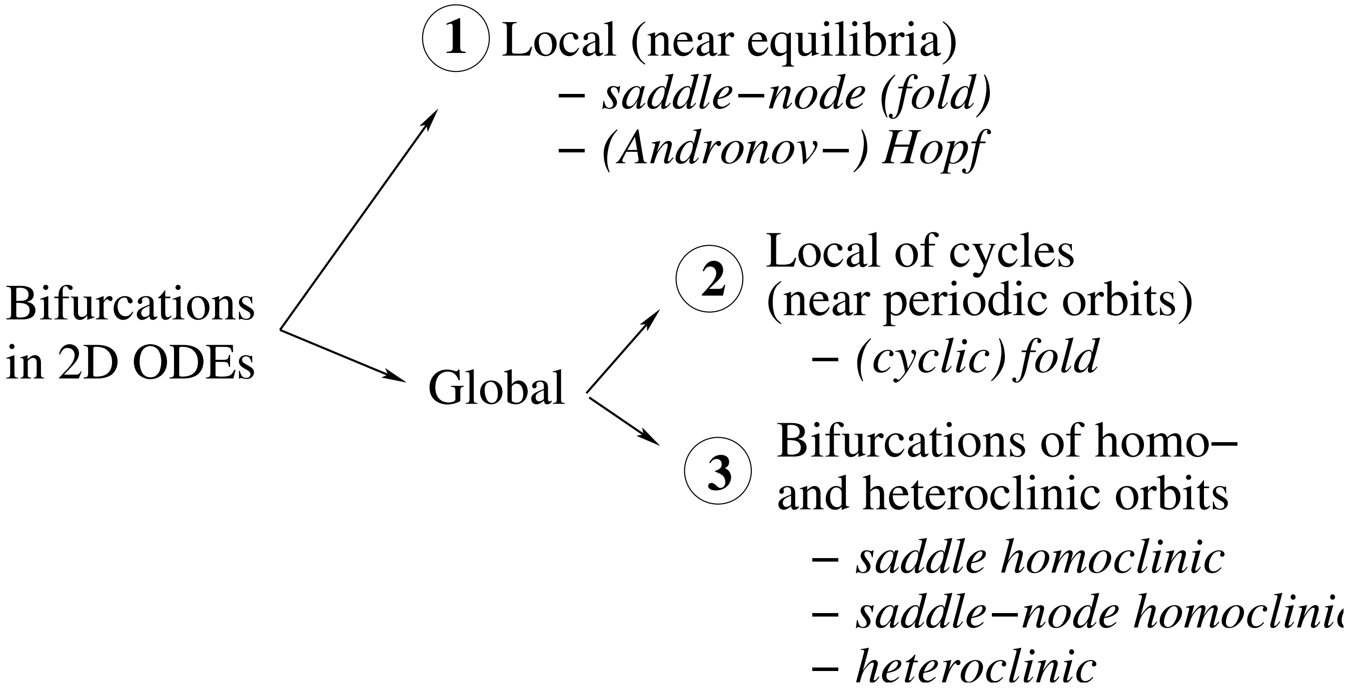
$$\dot{X} = f(X, \alpha) \quad \not\sim \quad \dot{X} = f(X, \alpha_0).$$

The appearance of a topologically non-equivalent system is called a **bifurcation***.*

Since the number of equilibria, the number of periodic orbits, and their stability, as well as the presence of connecting orbits, are topological invariants, a bifurcation of the 2D-system means a change of (some of) these properties.

Definition 8 *A* **codimension** *of a bifurcation is the number of conditions on which the bifurcating phase object has to satisfy.*

Classification of codimension-one bifurcations:



Only codim 1 bifurcations occur in generic one-parameter systems.

4. LOCAL CODIM 1 BIFURCATIONS

- If X_0 is a hyperbolic equilibrium of $\dot{X} = f(X, \alpha_0)$, then it remains hyperbolic for all α sufficiently close to α_0 (but can slightly shift).

- A local bifurcation can happen only to a non-hyperbolic equilibrium with $\Re(\lambda) = 0$.

- Codimension-1 critical cases:

  1. **Fold (saddle-node)**: $\lambda_1 = 0$

  2. **Andronov-Hopf (weak focus)**: $\lambda_{1,2} = \pm i\omega$

Fold bifurcation: $\lambda_1 = 0,\ \lambda_2 \neq 0$

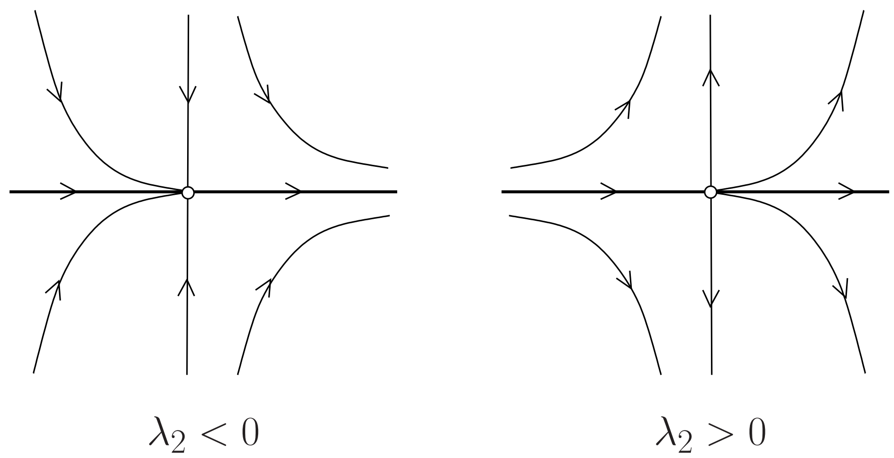
By a linear diffeomorphism, $\dot{X} = f(X, 0)$ can be transformed into

$$\begin{cases} \dot{x} &= ax^2 + bxy + cy^2 + O(3), \\ \dot{y} &= \lambda_2 y + O(2). \end{cases}$$

If $a \neq 0$ then $\dot{X} = f(X)$ is locally topologically equivalent near the origin to

$$\begin{cases} \dot{x} &= ax^2, \\ \dot{y} &= \lambda_2 y. \end{cases}$$

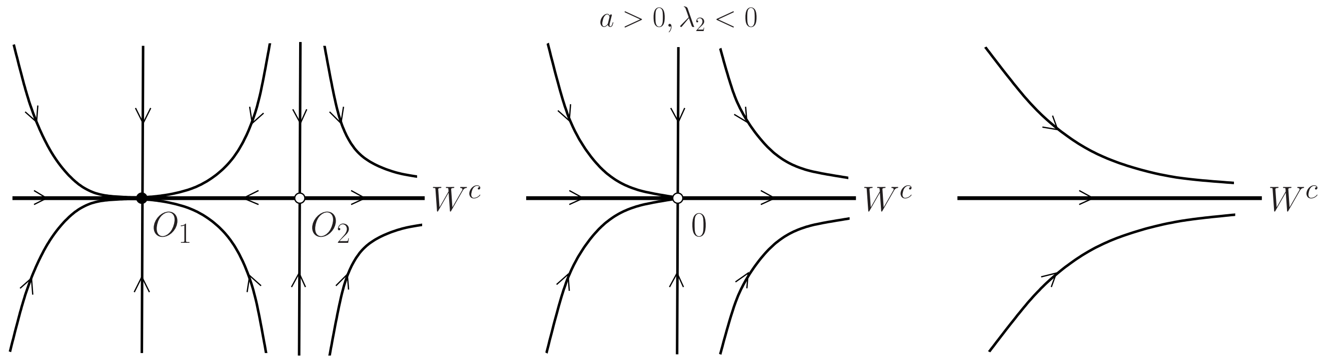
Saddle-node ($a > 0$):



Theorem 3 (Fold normal form) *If $a \neq 0$ and $\lambda_2 \neq 0$, then $\dot{X} = f(X, \alpha)$ is locally topologically equivalent near the saddle-node to*

$$\begin{cases} \dot{x} &= \beta(\alpha) + ax^2, \\ \dot{y} &= \lambda_2 y, \end{cases}$$

where $\beta(0) = 0$.



$\beta < 0$ $\beta = 0$ $\beta > 0$

Two equilibria $O_{1,2} = \left(\mp\sqrt{\frac{-\beta}{a}}, 0\right)$ collide and disappear in the 1D center manifold $W^c = \{y = 0\}$, provided $\beta'(0) \neq 0$.

Andronov-Hopf bifurcation: $\lambda_{1,2} = \pm i\omega,\ \omega > 0$

By a linear diffeomorphism, $\dot{X} = f(X, 0)$ can be transformed into

$$\begin{cases} \dot{x} &= -\omega y + R(x,y), \quad R = O(2), \\ \dot{y} &= \omega x + S(x,y), \quad S = O(2). \end{cases}$$

Introduce $z = x + iy \in \mathbb{C}$. Then this system becomes

$$\dot{z} = i\omega z + g(z, \bar{z}),$$

where

$$g(z,\bar{z}) = R\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iS\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

Write its Taylor expansion in $z, \bar{z}$:

$$g(z,\bar{z}) = \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \dots$$

Definition 9 *The* **first Lyapunov coefficient** *is*

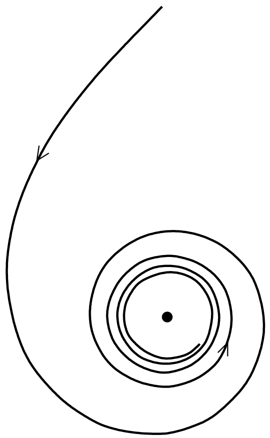
$$l_1 = \frac{1}{2\omega^2}\Re(ig_{20}g_{11} + \omega g_{21}).$$

If $l_1 \neq 0$ then $\dot{X} = f(X)$ is locally topologically equivalent near the origin to

$$\begin{cases} \dot{\rho} = l_1 \rho^3, \\ \dot{\varphi} = 1, \end{cases}$$

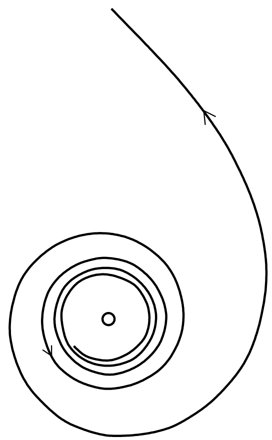
where (ρ, φ) are polar coordinates: $z = \rho e^{i\varphi}$.

Weak focus:



stable

$l_1 < 0$



unstable

$l_1 > 0$

Theorem 4 (Andronov-Hopf normal form) *If $l_1 \neq 0$ and $\omega > 0$, then $\dot{X} = f(X, \alpha)$ is locally topologically equivalent near the weak focus to*

$$\begin{cases} \dot{\rho} &= \rho(\beta(\alpha) + l_1\rho^2), \\ \dot{\varphi} &= 1. \end{cases}$$

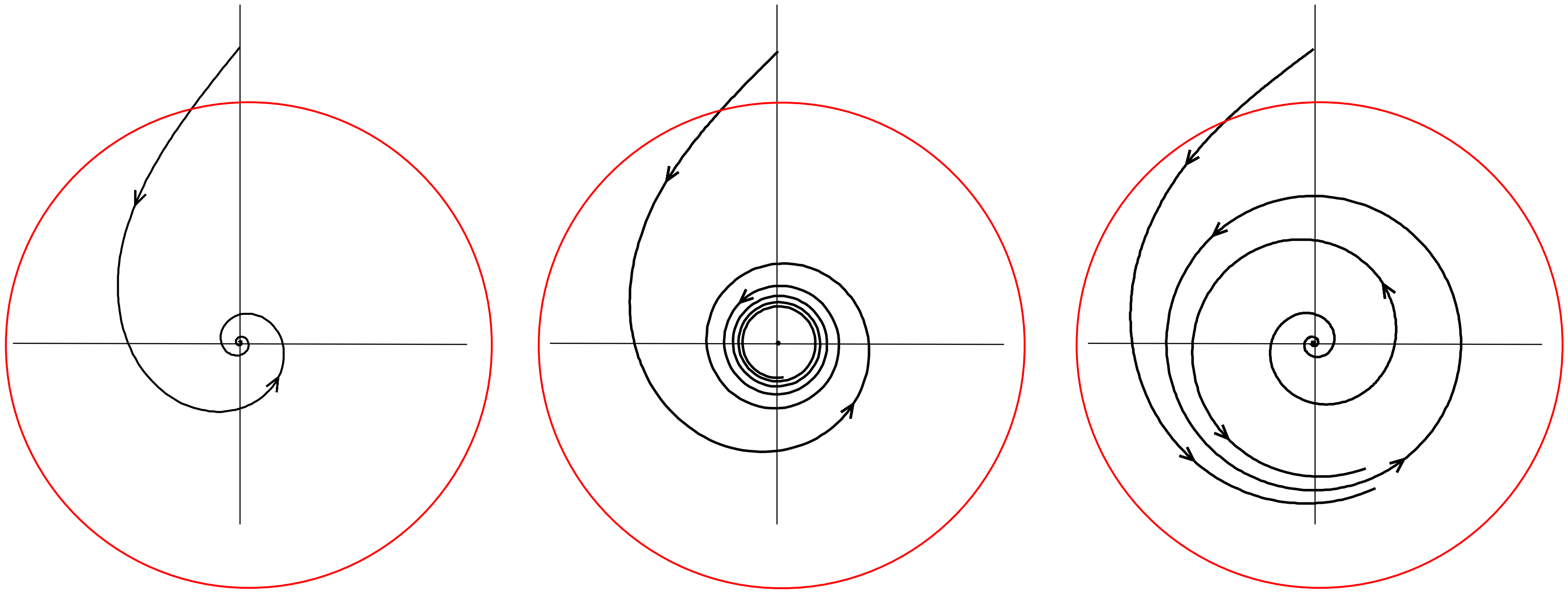
where $\beta(0) = 0$.

A **limit cycle** $\rho_0 = \sqrt{\frac{-\beta}{l_1}} > 0$ appears while the focus changes stability.

The direction of the cycle bifurcation is determined by the **first Lyapunov coefficient** l_1 of the weak focus:

- **supercritical** (soft, non-catastrophic) Andronov-Hopf bifurcation ($l_1 < 0$);
- **subcritical** (hard, catastrophic) Andronov-Hopf bifurcation ($l_1 > 0$).

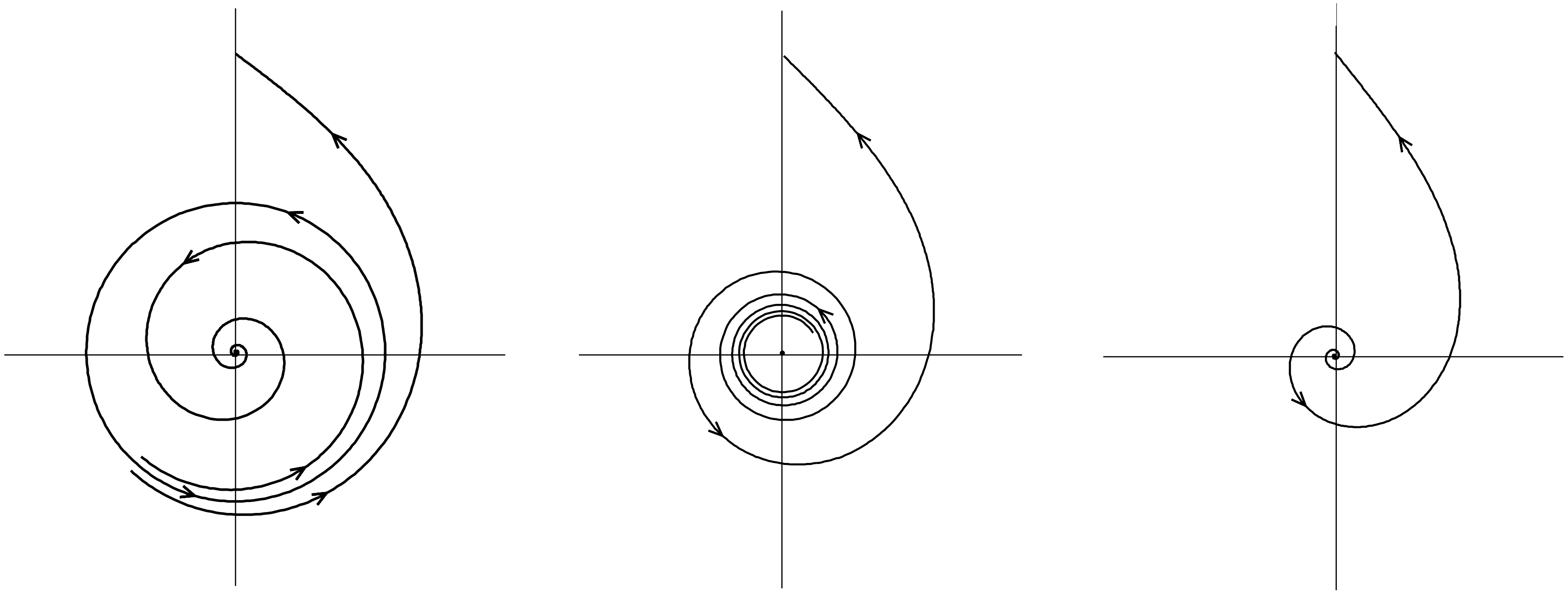
Supercritical Andronov-Hopf bifurcation: $l_1 < 0$



$\beta < 0$ $\beta = 0$ $\beta > 0$

The stable equilibrium is replaced by small-amplitude oscillations within an attracting domain.

Subcritical Andronov-Hopf bifurcation: $l_1 > 0$



$\beta < 0$ $\beta = 0$ $\beta > 0$

The domain of attraction of the stable focus shrinks, while it becomes unstable.

Example: $\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + \alpha y + x^2 + xy + y^2. \end{cases}$

At $\alpha = 0$ the equilibrium $x = y = 0$ of the **reversed system**

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= x - x^2 - xy - y^2, \end{cases}$$

has eigenvalues $\lambda_{1,2} = \pm i$ $(\omega = 1)$.

Introduce $z = x + iy$, then $x^2 + y^2 = |z|^2 = z\bar{z}$ and

$$\begin{aligned} \dot{z} &= \dot{x} + i\dot{y} = -y + ix - ix^2 - ixy - iy^2 \\ &= iz - iz\bar{z} - \frac{1}{4}(z^2 - \bar{z}^2) = iz - \frac{1}{4}z^2 - iz\bar{z} + \frac{1}{4}\bar{z}^2 \end{aligned}$$

so that $\omega = 1,\ g_{20} = -\frac{1}{2},\ g_{11} = -i,\ g_{02} = \frac{1}{2},\ g_{21} = 0.$

$$\tilde{l}_1 = \frac{1}{2\omega^2}\Re(ig_{20}g_{11} + \omega g_{21}) = \frac{1}{2}\left(i\ \frac{1}{2}\ i + 1 \cdot 0\right) = -\frac{1}{4}.$$

For the original system, $l_1 = \frac{1}{4} > 0 \ \Rightarrow$ subcritical Hopf bifurcation (an **unstable cycle** exists for small $\alpha < 0$ but disappears for $\alpha > 0$)

Practical computation of a and l_1 in $\mathbb{R}^2$ $(n = 2)$

Suppose $X_0 = 0,\ \alpha_0 = 0$ and write the Taylor expansion in the original coordinates:

$$f(X, 0) = AX + \frac{1}{2}B(X, X) + \frac{1}{6}C(X, X, X) + O(4)$$

where

$$\begin{aligned}
(AX)_i &= \sum_{j=1}^{n} \left.\frac{\partial f_i(U, 0)}{\partial U_j}\right|_{U=0} X_j, \\
B_i(X, Y) &= \sum_{j,k=1}^{n} \left.\frac{\partial^2 f_i(U, 0)}{\partial U_j \partial U_k}\right|_{U=0} X_j Y_k, \\
C_i(X, Y, Z) &= \sum_{j,k,l=1}^{n} \left.\frac{\partial^3 f_i(U, 0)}{\partial U_j \partial U_k \partial U_l}\right|_{U=0} X_j Y_k Z_l,
\end{aligned}$$

for $i = 1, \ldots, n$.

Theorem 5 *The fold normal form coefficient can be computed as*

$$a = \frac{1}{2}\langle p, B(q,q)\rangle$$

where $p, q \in \mathbb{R}^2$ *satisfy*

$$Aq = A^{\mathsf{T}} p = 0$$

and $p^{\mathsf{T}} q \equiv \langle p, q\rangle = 1$*.*

Theorem 6 *The first Lyapunov coefficient can be computed in 2D as*

$$l_1 = \frac{1}{2\omega^2}\Re\left[i\langle p, B(q,q)\rangle\langle p, B(q,\bar{q})\rangle + \omega\langle p, C(q,q,\bar{q})\rangle\right]$$

where $p, q \in \mathbb{C}^2$ *satisfy*

$$Aq = i\omega q, \quad A^{\mathsf{T}} p = -i\omega p$$

and $\bar{p}^{\mathsf{T}} q \equiv \langle p, q\rangle = 1$*.*

Example: Hopf bifurcation in a prey-predator system

Consider the following system

$$\begin{cases} \dot{x}_1 &= rx_1(1-x_1) - \dfrac{cx_1x_2}{\alpha + x_1} \\ \dot{x}_2 &= -dx_2 + \dfrac{cx_1x_2}{\alpha + x_1} \end{cases} \sim \begin{cases} \dot{x}_1 &= rx_1(\alpha + x_1)(1-x_1) - cx_1x_2 \\ \dot{x}_2 &= -\alpha dx_2 + (c-d)x_1x_2 \end{cases}$$

At $\alpha_0 = \frac{c-d}{c+d}$ the last system has the equilibrium $\left(x_1^{(0)}, x_2^{(0)}\right) = \left(\frac{d}{c+d}, \frac{rc}{(c+d)^2}\right)$ with eigenvalues $\lambda_{1,2} = \pm i\omega$, where $\omega^2 = \frac{rc^2d(c-d)}{(c+d)^3} > 0$.

Translate the origin of the coordinates to this equilibrium by

$$\begin{cases} x_1 &= x_1^{(0)} + X_1, \\ x_2 &= x_2^{(0)} + X_2. \end{cases}$$

This transforms the system into

$$\begin{cases} \dot{X}_1 &= -\dfrac{cd}{c+d}X_2 - \dfrac{rd}{c+d}X_1^2 - cX_1X_2 - rX_1^3, \\ \dot{X}_2 &= \dfrac{rc(c-d)}{(c+d)^2}X_1 + (c-d)X_1X_2, \end{cases}$$

that can be represented as

$$\dot{X} = AX + \frac{1}{2}B(X,X) + \frac{1}{6}C(X,X,X),$$

where

$$A = \begin{pmatrix} 0 & -\dfrac{cd}{c+d} \\ \dfrac{\omega^2(c+d)}{cd} & 0 \end{pmatrix}, \; B(X,Y) = \begin{pmatrix} -\dfrac{2rd}{c+d}X_1Y_1 - c(X_1Y_2 + X_2Y_1) \\ (c-d)(X_1Y_2 + X_2Y_1) \end{pmatrix}$$

and

$$C(X,Y,Z) = \begin{pmatrix} -6rX_1Y_1Z_1 \\ 0 \end{pmatrix}.$$

The complex vectors

$$q = \begin{pmatrix} cd \\ -i\omega(c+d) \end{pmatrix}, \quad p = \frac{1}{2\omega cd(c+d)} \begin{pmatrix} \omega(c+d) \\ -icd \end{pmatrix}.$$

satisfy $Aq = i\omega q$, $A^{\mathsf{T}} p = -i\omega p$ and $\langle p, q \rangle = 1$.

Then

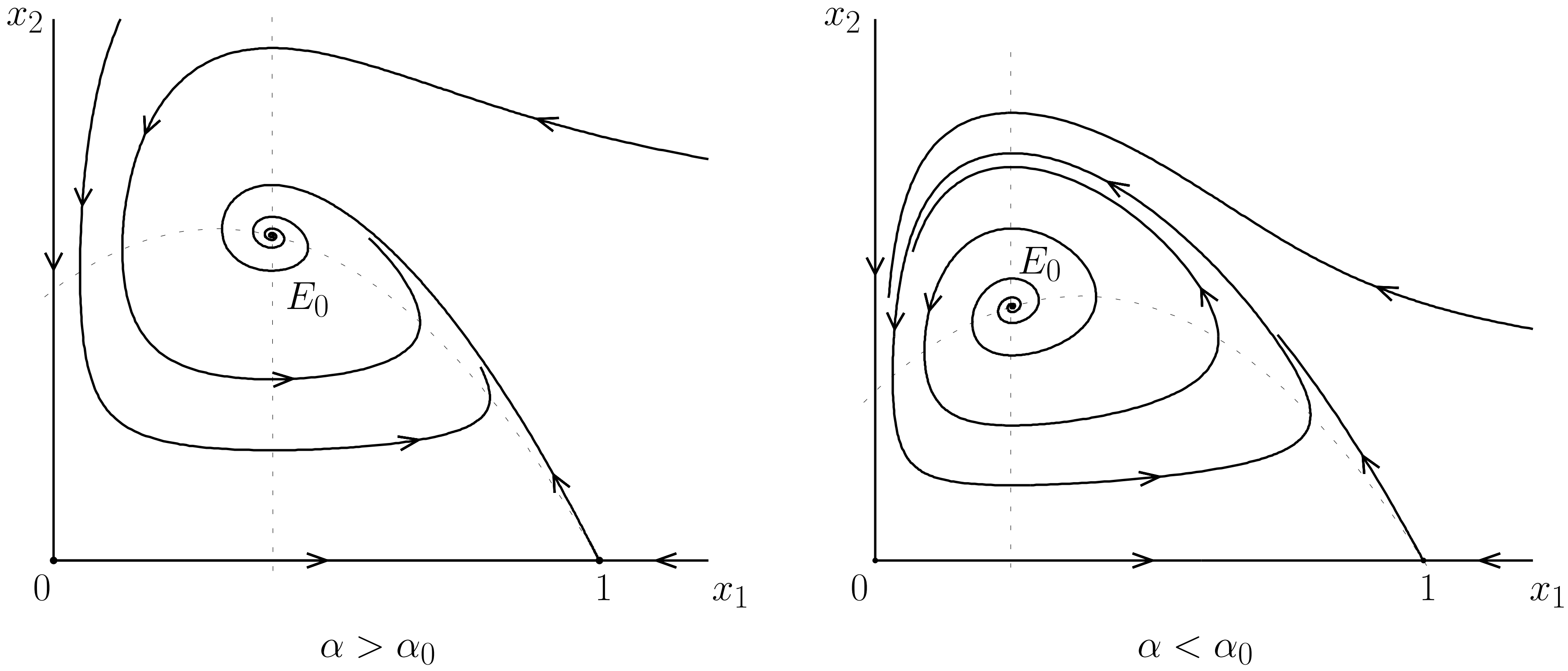
$$g_{20} = \langle p, B(q,q) \rangle = \frac{cd(c^2 - d^2 - rd) + i\omega c(c+d)^2}{(c+d)},$$

$$g_{11} = \langle p, B(q,\bar{q}) \rangle = -\frac{rcd^2}{(c+d)}, \quad g_{21} = \langle p, C(q,q,\bar{q}) \rangle = -3rc^2d^2,$$

and the first Lyapunov coefficient

$$l_1(\alpha_0) = \frac{1}{2\omega^2} \operatorname{Re}(ig_{20}g_{11} + \omega g_{21}) = -\frac{rc^2d^2}{\omega} < 0.$$

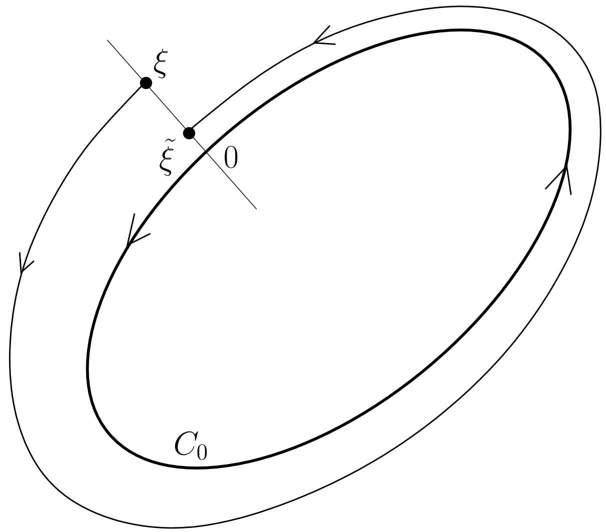
Therefore, a **stable cycle** bifurcates from the equilibrium via the super-critical Hopf bifurcation for $\alpha < \alpha_0$.



One can prove that the cycle is **unique**.

5. CODIM 1 CYCLIC FOLD BIFURCATION

Parameter-dependent Poincaré map:



$$\xi \mapsto \tilde{\xi} = P(\xi, \alpha),$$

where $P(\xi, 0) = \xi + O(2) \quad (\mu = 1)$

Lemma 1 *If*

$$p_2(0) = \frac{1}{2} P_{\xi\xi}(0,0) \neq 0,$$

then there exists a smooth function $\delta = \delta(\alpha)$ such that the substitution $x = \xi + \delta(\alpha)$ reduces the map

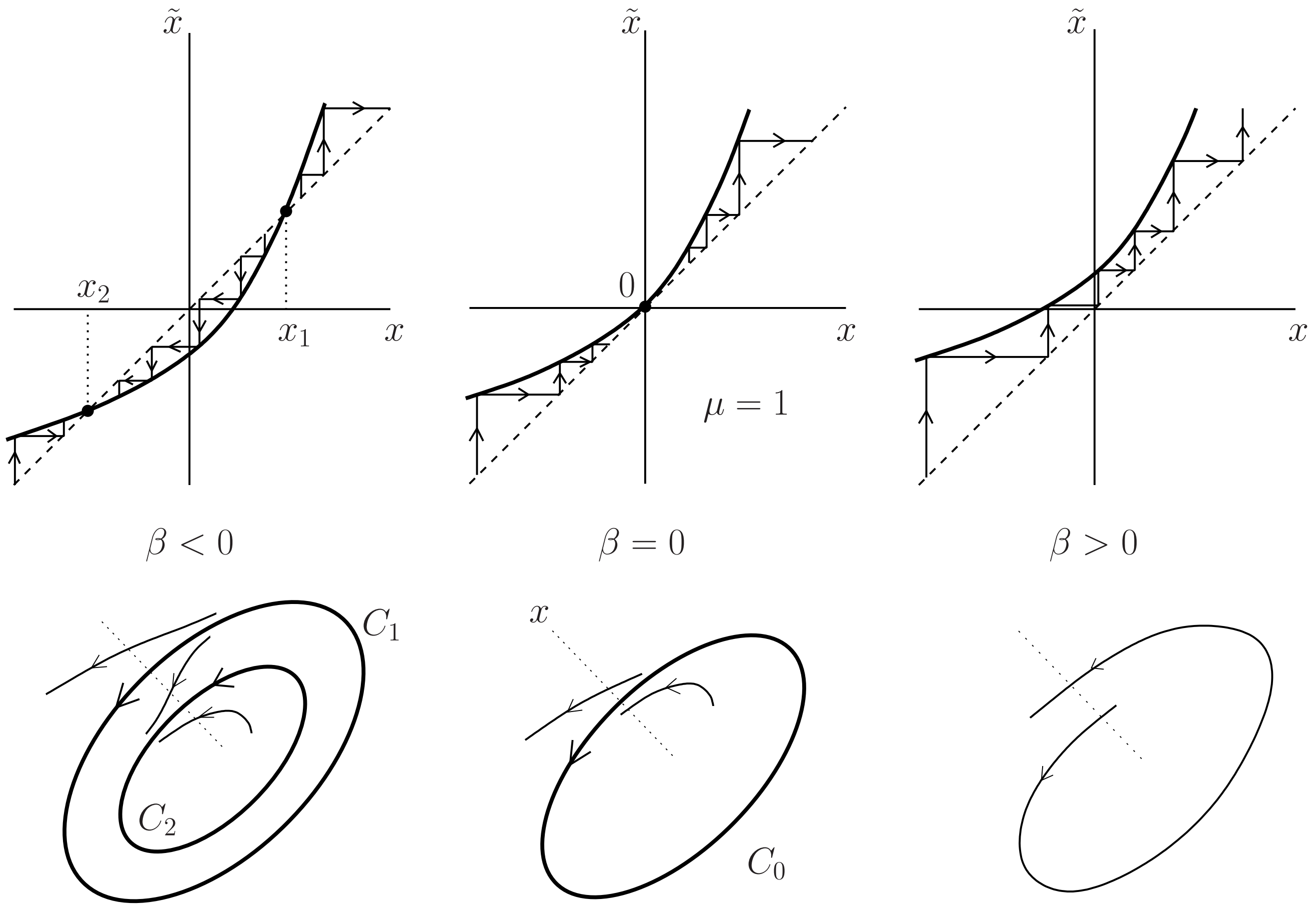
$$\xi \mapsto P(\xi, \alpha) = p_0(\alpha) + [1 + g(\alpha)]\xi + p_2(\alpha)\xi^2 + O(3),$$

where $g(0) = 0, p_0(0) = P(0,0) = 0$, to the form

$$x \mapsto \tilde{x} = \beta(\alpha) + x + b(\alpha)x^2 + O(3)$$

with $\beta(0) = 0$ and $b(0) = p_2(0) \neq 0$.

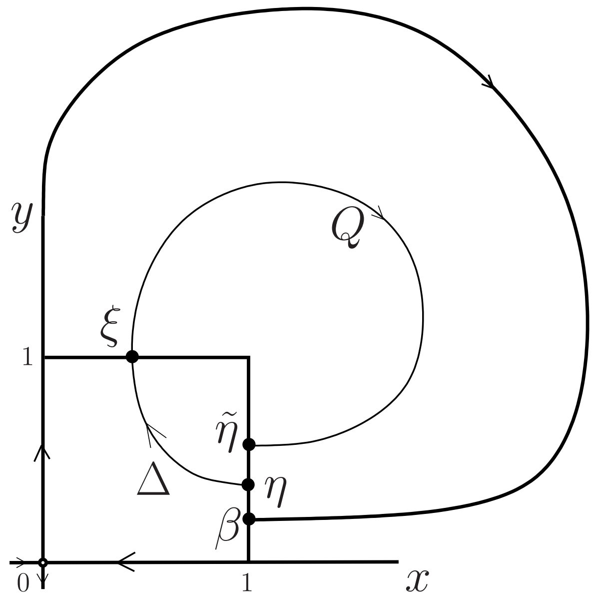
Cyclic fold: $x \mapsto \beta + x + bx^2, \ b > 0$



Two hyperbolic cycles (unstable C_1 and stable C_2) collide forming a non-hyperbolic cycle C_0, and disappear.

6. CODIM 1 BIFURCATIONS OF CONNECTING ORBITS

- **Saddle homoclinic bifurcation**



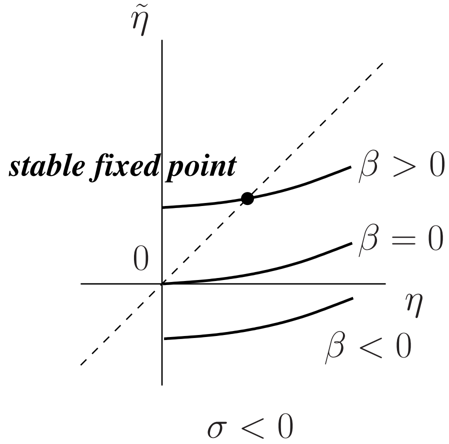
Singular map: $\eta \mapsto \xi = \eta^{-\frac{\lambda_1}{\lambda_2}}.$

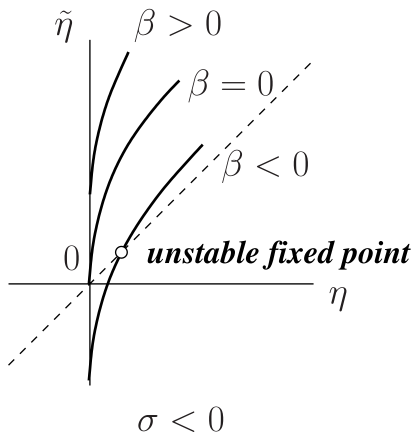
Regular map:

$$\xi \mapsto \tilde{\eta} = \beta(\alpha) + A(\alpha)\xi + O(2), \quad A(0) > 0.$$

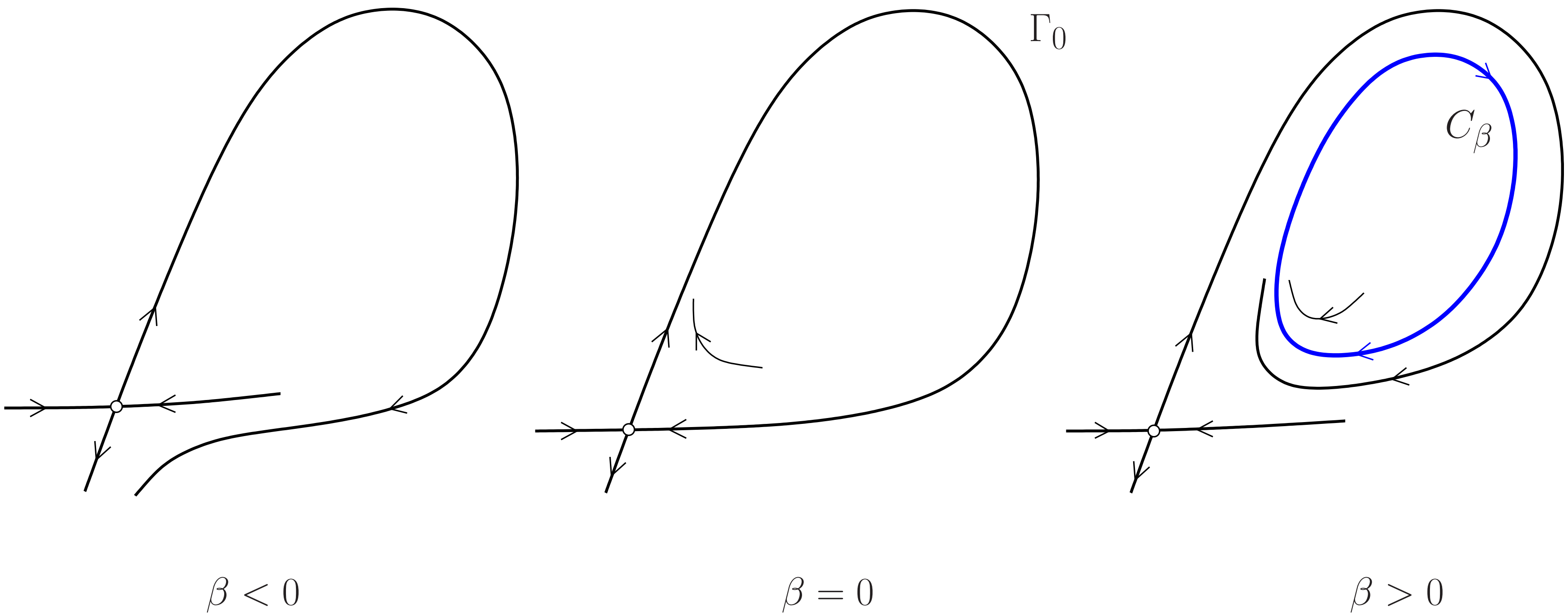
Poincaré map:

$$\eta \mapsto \tilde{\eta} = \beta(\alpha) + A(\alpha)\eta^{-\frac{\lambda_1}{\lambda_2}} + \dots$$



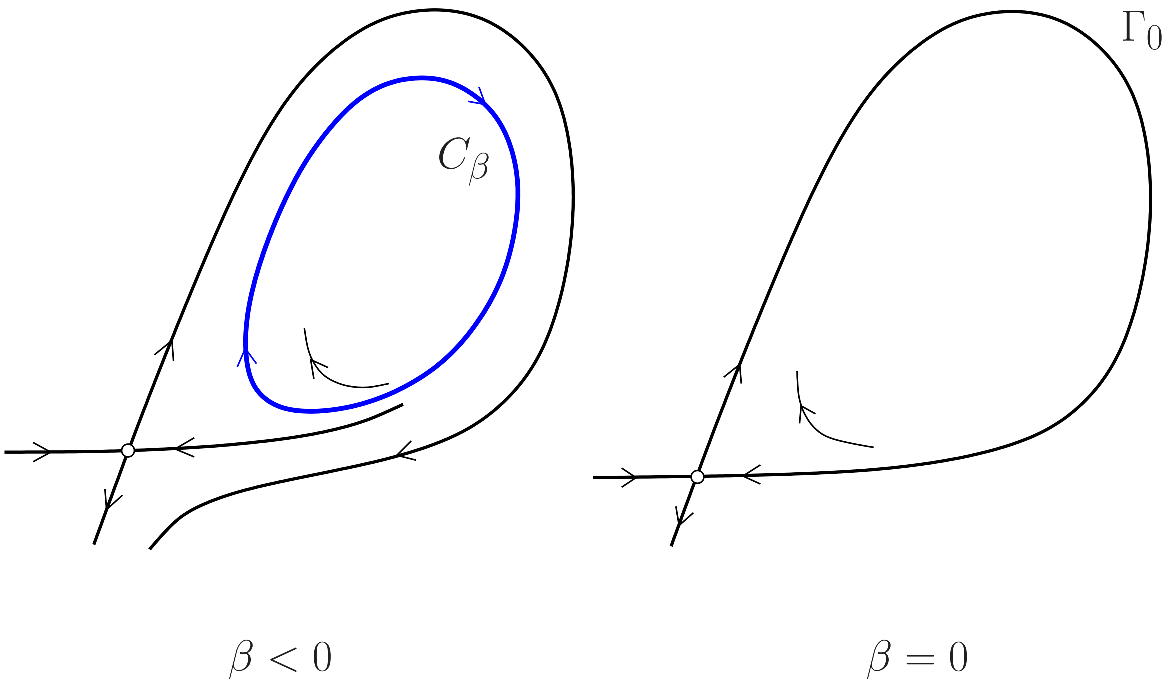


Saddle homoclinic bifurcation: $\sigma < 0$



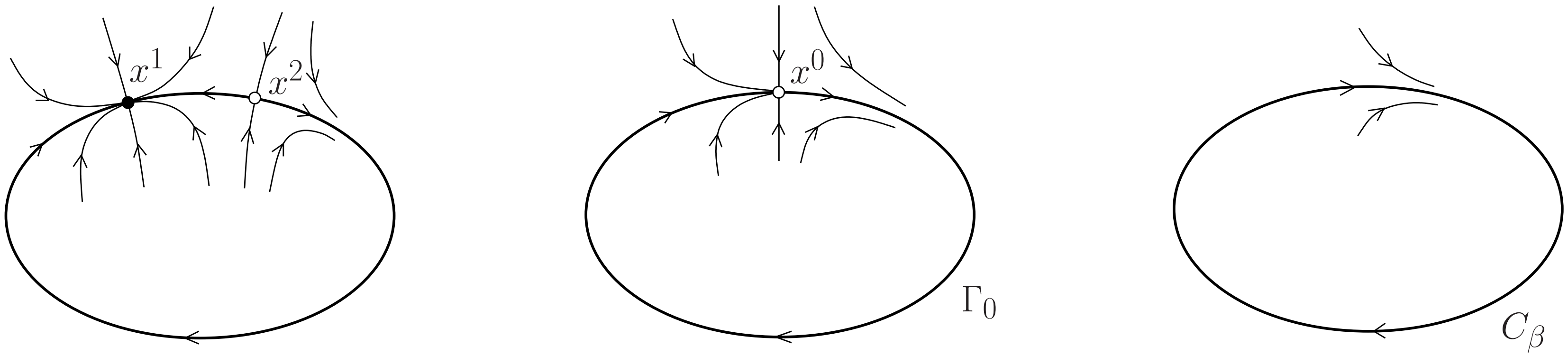
A stable cycle C_β bifurcates from Γ_0 while the separatrices exchange.

Saddle homoclinic bifurcation: $\sigma > 0$



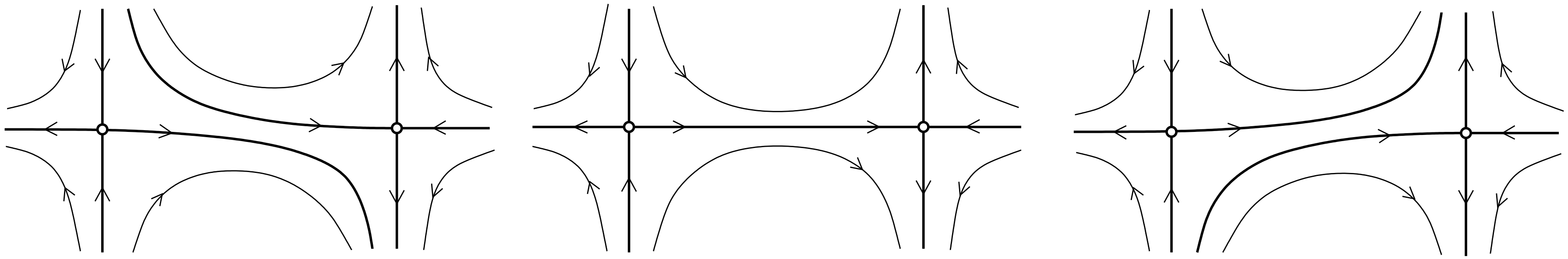
An unstable cycle C_β bifurcates from Γ_0 while the separatrices exchange.

- **Homoclinic saddle-node bifurcation:**



$\beta < 0$ $\beta = 0$ $\beta > 0$

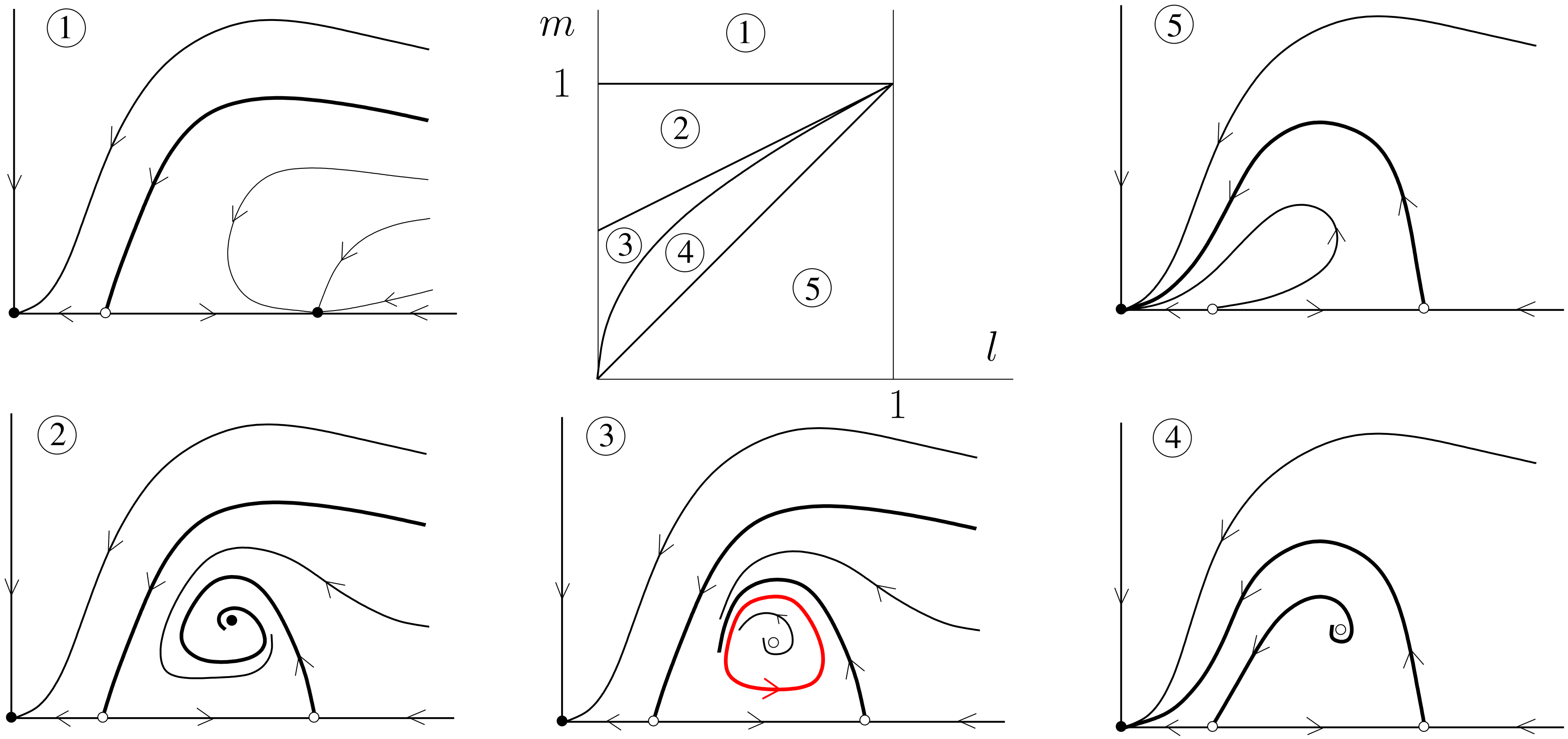
- **Heteroclinic saddle bifurcation:**



$\beta < 0$ $\beta = 0$ $\beta < 0$

Example: Allee effect in a prey-predator system

$$\begin{cases} \dot{x} = x(x-l)(1-x) - xy, \\ \dot{y} = -\gamma y(m-x). \end{cases}$$



7. LOCAL CODIM 2 BIFURCATIONS

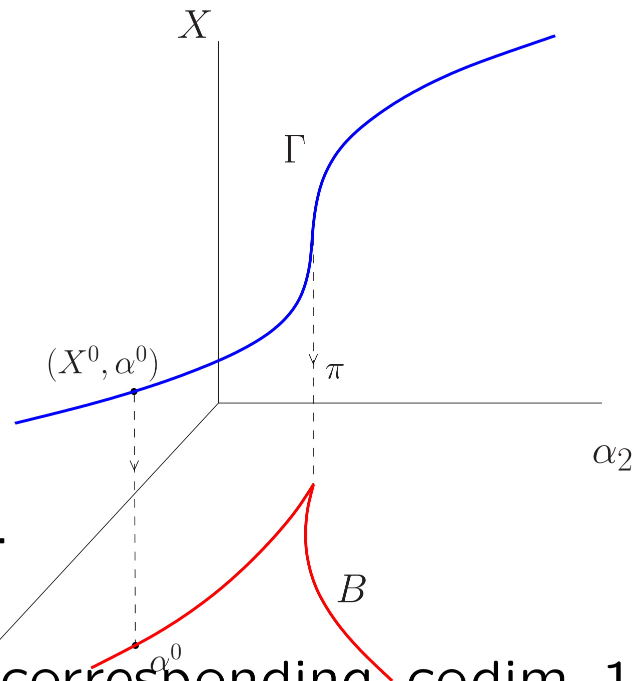
Consider a smooth 2D system depending on two parameters

$$\dot{X} = f(X, \alpha), \quad X \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^2.$$

Curves of codim 1 bifurcations:

$$\text{Fold}: \begin{cases} f(X,\alpha) &= 0, \\ \det\ f_X(X,\alpha) &= 0. \end{cases}$$

$$\text{Hopf}: \begin{cases} f(X,\alpha) &= 0, \\ \text{Tr}\ f_X(X,\alpha) &= 0. \end{cases}$$



In both cases, we have 3=2+1 equations in $\mathbb{R}^4$.

When we cross $B = \pi\Gamma$ in the α-plane, the corresponding codim 1 bifurcation occurs.

One has to check that $\lambda_{1,2} = \pm i\omega$ along the Hopf curve.

Local codim 2 cases in the plane:

$$\text{Fold} : \lambda_1 = 0 \quad \begin{cases} \dot{x} = ax^2 + O(3) \\ \dot{y} = \lambda_2 y + O(2) \end{cases}$$

$$\text{Hopf} : \lambda_{1,2} = \pm i\omega \quad \begin{cases} \dot{\rho} = \omega l_1 \rho^3 + O(4) \\ \dot{\varphi} = \omega + O(1) \end{cases}$$

1 $\lambda_1 = 0,\ a = 0$

2 $\lambda_1 = 0,\ \lambda_2 = 0$

3 $\lambda_{1,2} = \pm i\omega,\ l_1 = 0$

To meet each case, we need to "tune" two parameters while following Γ (or B) $\Rightarrow$ codim 2.

Cusp bifurcation: $\lambda_1 = 0, \ a = 0$

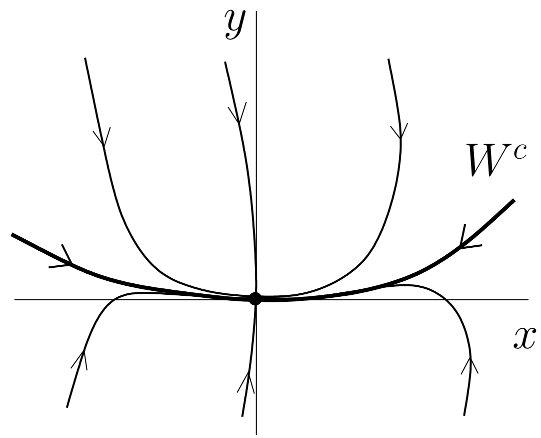
The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to

$$\begin{cases} \dot{x} = p_{11}xy + \frac{1}{2}p_{02}y^2 + \frac{1}{6}p_{30}x^3 + \dots, \\ \dot{y} = \lambda_2 y + \frac{1}{2}q_{20}x^2 + q_{11}xy + \frac{1}{2}q_{02}y^2 + O(3). \end{cases}$$

It has an invariant 1D **center manifold** $W^c = \{(x, y) : y = W(x)\}$:

$$y = W(x) = \frac{1}{2}w_2 x^2 + O(3)$$

where $w_2 = -\dfrac{q_{20}}{\lambda_2}$.



Thus, the restriction of $\dot{X} = f(X, 0)$ to W^c is

$$\dot{x} = cx^3 + O(4), \quad \text{where} \quad c = \frac{1}{6}\left(p_{30} - \frac{3}{\lambda_2}q_{20}p_{11}\right).$$

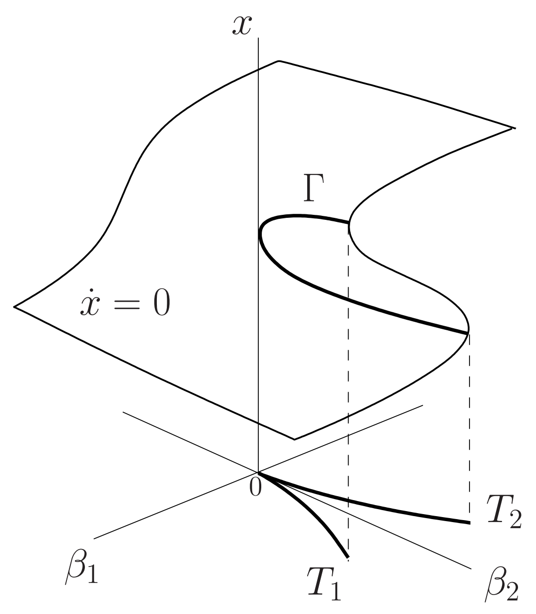
Theorem 7 (Cusp normal form) *If $c \neq 0$, then $\dot{X} = f(X, \alpha)$ is locally topologically equivalent near the cusp bifurcation to*

$$\begin{cases} \dot{x} = \beta_1(\alpha) + \beta_2(\alpha)x + sx^3, \\ \dot{y} = \lambda_2 y, \end{cases}$$

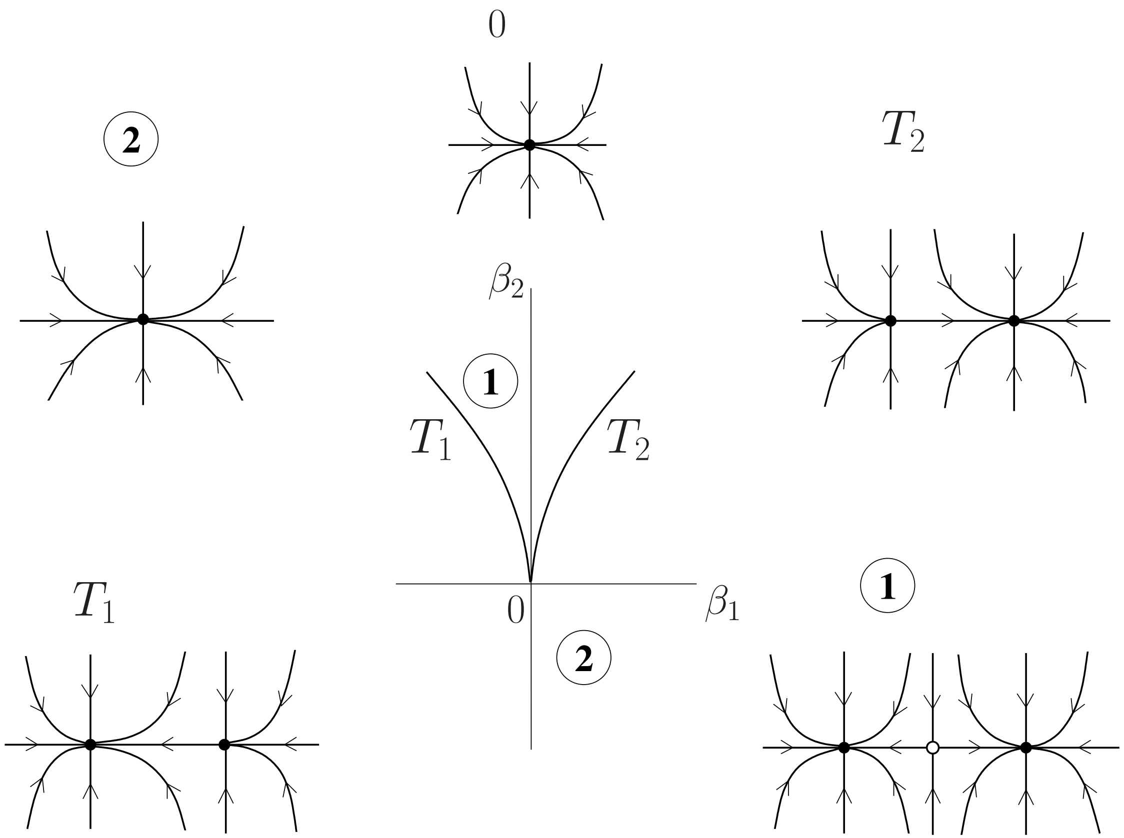
where $\beta_1(0) = \beta_2(0) = 0$ and $s = \text{sign}(c) = \pm 1$.

Fold curve(s) $4\beta_2^3 + 27s\beta_1^2 = 0$

Equilibrium manifold:



Cusp bifurcation diagram $(c < 0,\ \lambda_2 < 0)$



Three equilibria exist inside the wedge, pairwise colliding at its borders $T_{1,2}$ and leaving one equilibrium outside.

Bogdanov-Takens bifurcation: $\lambda_1 = \lambda_2 = 0$

The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to

$$\begin{cases} \dot{x} &= y + \frac{1}{2}p_{20}x^2 + p_{11}xy + \frac{1}{2}p_{02}y^2 + O(3) \equiv P(x,y), \\ \dot{y} &= \frac{1}{2}q_{20}x^2 + q_{11}xy + \frac{1}{2}q_{02}y^2 + \frac{1}{6}q_{03}x^2 + O(3). \end{cases}$$

By a nonlinear local diffeomorphism (change of variables)

$$\begin{cases} \xi &= x, \\ \eta &= P(x,y), \end{cases}$$

this system can be reduced near the origin to

$$\begin{cases} \dot{\xi} &= \eta, \\ \dot{\eta} &= a\xi^2 + b\xi\eta + \dots, \end{cases}$$

where

$$a = \frac{1}{2}q_{20}, \quad b = p_{20} + q_{11}.$$

Theorem 8 (Bogdanov-Takens normal form) *If* $ab \neq 0$*, then*

$$\dot{X} = f(X, \alpha)$$

is locally topologically equivalent near the BT-bifurcation to

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= \beta_1(\alpha) + \beta_2(\alpha)x + x^2 + sxy, \end{cases}$$

where $\beta_1(0) = \beta_2(0) = 0$ *and* $s = \mathrm{sign}(ab) = \pm 1$*.*

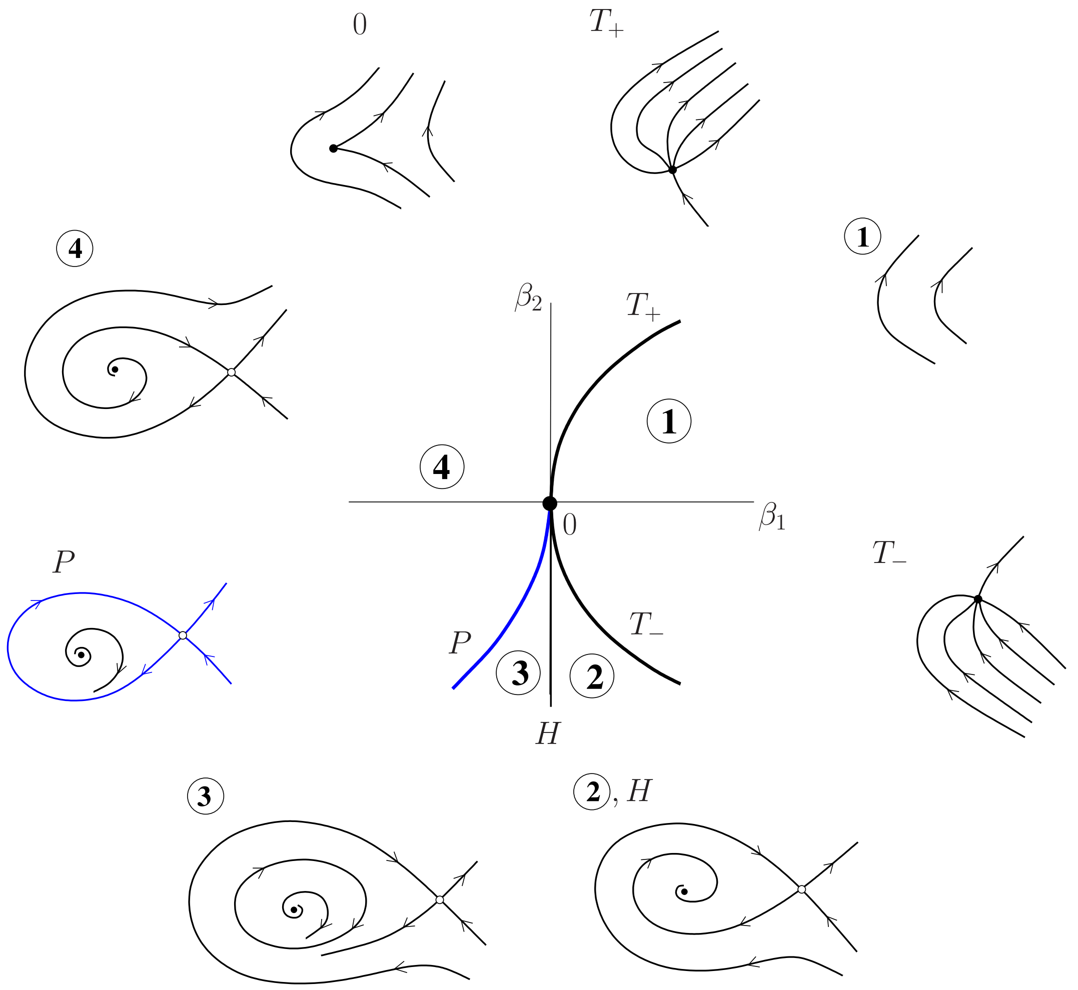
Bifurcation curves $(ab < 0)$:

- **fold** $T : \beta_1 = \frac{1}{4}\beta_2^2$

- **Andronov-Hopf** $H : \beta_1 = 0,\ \beta_2 < 0$

- **saddle homoclinic** $P : \beta_1 = -\frac{6}{25}\beta_2^2 + O(3),\ \beta_2 < 0$ (**global bifurcation**)

BT bifurcation diagram $(ab < 0)$



A unique limit cycle appears at Andronov-Hopf bifurcation curve H and disappears via the saddle homoclinic orbit at the curve P.

Bautin ("generalized Hopf") bifurcation: $\lambda_{1,2} = \pm i\omega,\ l_1 = 0$

The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to the complex form

$$\dot{z} = i\omega z + \sum_{2 \le j+k \le 5} \frac{1}{j!k!} g_{jk} z^k \bar{z}^j + O(6),$$

which is locally smoothly equivalent to the **Poincaré normal form**

$$\dot{w} = i\omega w + c_1 w|w|^2 + c_2 w|w|^4 + O(6),$$

where the **Lyapunov coefficients**

$$l_j = \frac{1}{\omega} \Re(c_j)$$

satisfy

$$2l_1 = \frac{1}{\omega}\left(\Re(g_{21}) - \frac{1}{\omega}\Im(g_{20}g_{11})\right) \ \Rightarrow \ l_1 = \frac{1}{2\omega^2}\Re(ig_{20}g_{11} + \omega g_{21})$$

If $l_1 = 0$ then

$$
\begin{aligned}
12 l_2(0) &= \frac{1}{\omega}\Re(g_{32}) \\
&+ \frac{1}{\omega^2}\Im[g_{20}\bar{g}_{31} - g_{11}(4g_{31} + 3\bar{g}_{22}) - \frac{1}{3}g_{02}(g_{40} + \bar{g}_{13}) - g_{30}g_{12}] \\
&+ \frac{1}{\omega^3}\{\Re[g_{20}(\bar{g}_{11}(3g_{12} - \bar{g}_{30}) + g_{02}\left(\bar{g}_{12} - \frac{1}{3}g_{30}\right) + \frac{1}{3}\bar{g}_{02}g_{03}) \\
&\qquad + g_{11}(\bar{g}_{02}\left(\frac{5}{3}\bar{g}_{30} + 3g_{12}\right) + \frac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30})] \\
&\qquad + 3\Im(g_{20}g_{11})\ \Im(g_{21})\} \\
&+ \frac{1}{\omega^4}\left\{\Im\left[g_{11}\bar{g}_{02}\left(\bar{g}_{20}^2 - 3\bar{g}_{20}g_{11} - 4g_{11}^2\right)\right]\right. \\
&\qquad \left. + \ \Im(g_{20}g_{11})\left[3\Re(g_{20}g_{11}) - 2|g_{02}|^2\right]\right\}
\end{aligned}
$$

Theorem 9 (Normal form for Bautin bifurcation) *If $l_2 \neq 0$ and $\omega \neq 0$, then $\dot{X} = f(X, \alpha)$ is locally topologically equivalent near Bautin bifurcation to the normal form in the polar coordinates:*

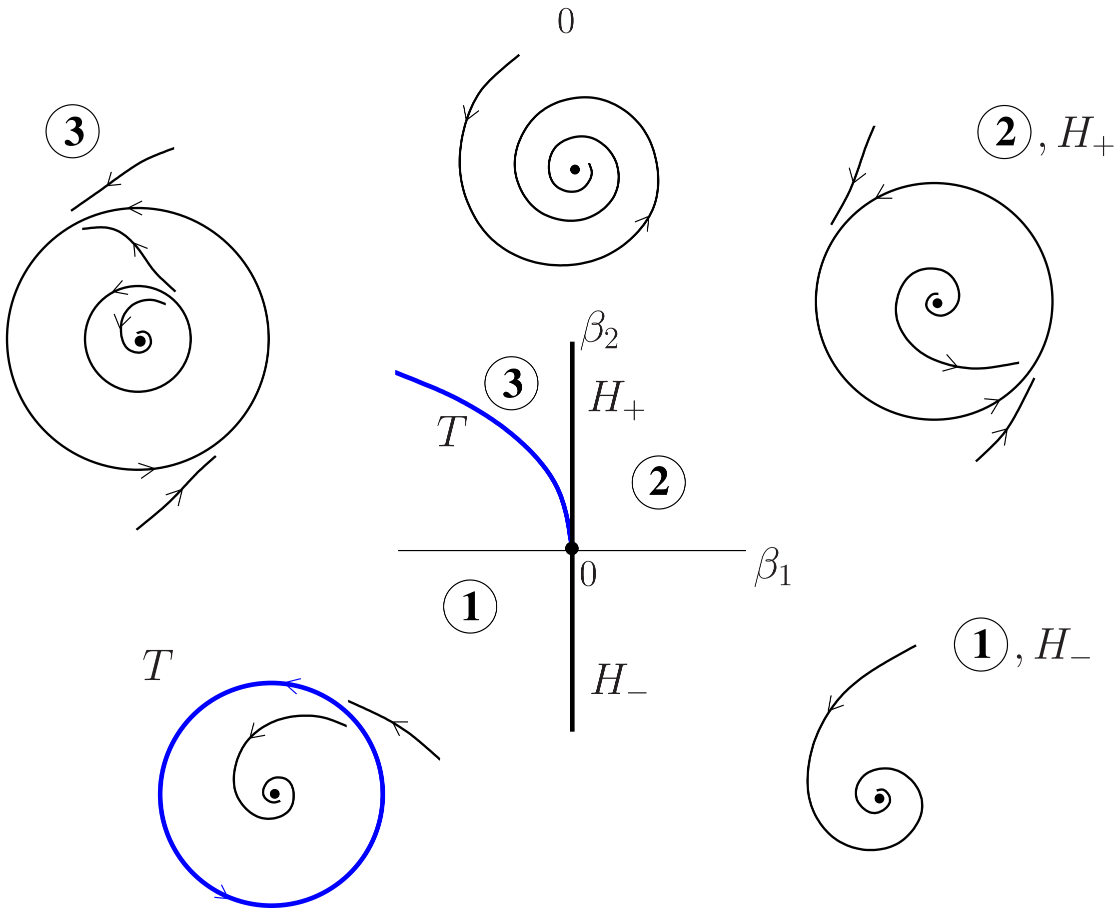
$$\begin{cases} \dot{\rho} = \rho(\beta_1(\alpha) + \beta_2(\alpha)\rho^2 + s\rho^4), \\ \dot{\varphi} = 1, \end{cases}$$

where $\beta_1(0) = \beta_2(0) = 0$ and $s = \text{sign}(l_2) = \pm 1$.

Bifurcation curves $(l_2 < 0)$:

- **superctitical Andronov-Hopf** $H^- : \beta_1 = 0,\ \beta_2 < 0$
- **subctitical Andronov-Hopf** $H^+ : \beta_1 = 0,\ \beta_2 > 0$
- **cyclic fold** $T_c : \beta_1 = \frac{1}{4}\beta_2^2,\ \beta_2 > 0$ (**global bifurcation**)

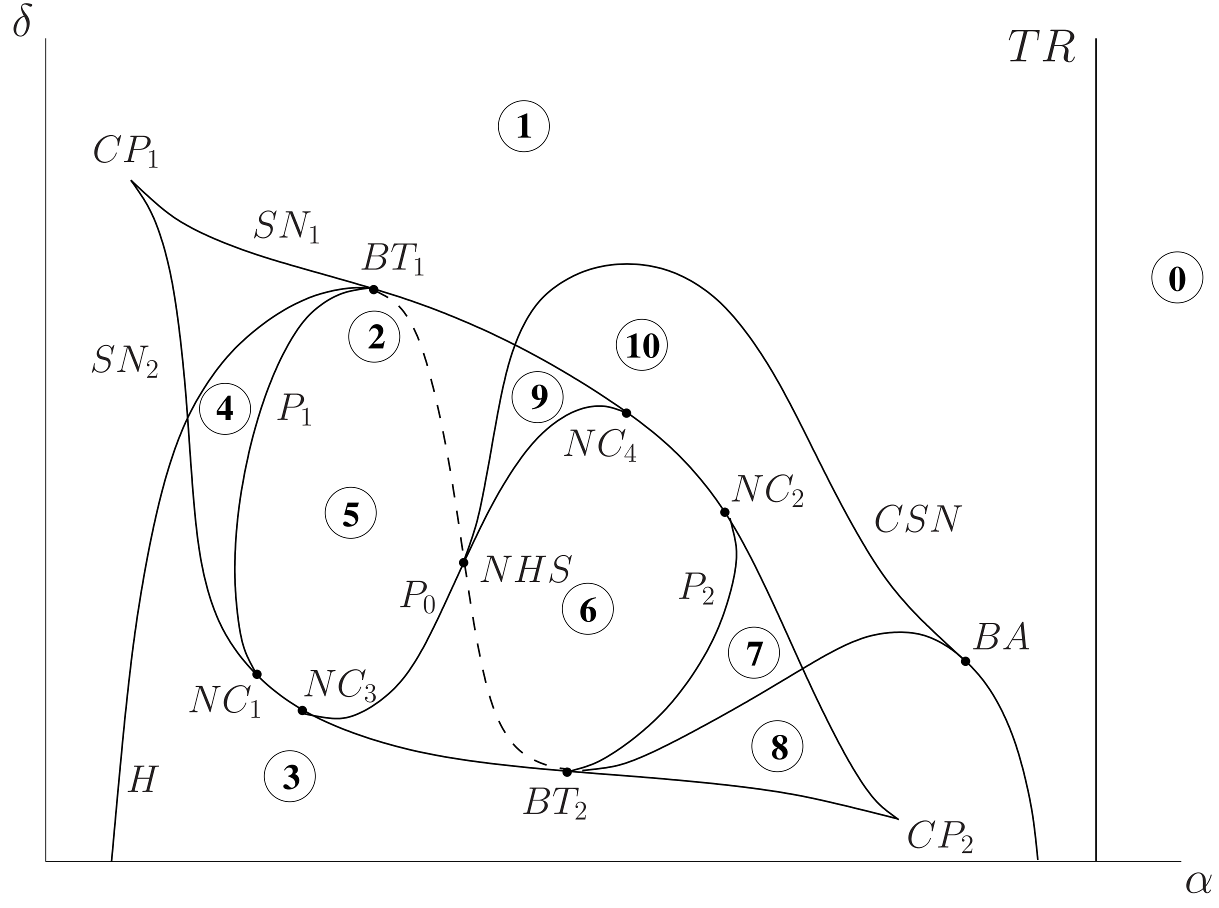
Bautin bifurcation diagram $(l_2 < 0)$



In the wedge between H^+ and T_c there exist two limit cycles born via different Andronov-Hopf bifurcations, which merge and disappear at the cyclic fold curve T_c.

Example: Bazykin's prey-predator model

$$\begin{cases} \dot{x}_1 &=& x_1 - \dfrac{x_1 x_2}{1+\alpha x_1} - \varepsilon x_1^2, \\ \dot{x}_2 &=& -\gamma x_2 + \dfrac{x_1 x_2}{1+\alpha x_1} - \delta x_2^2 \ . \end{cases}$$



Generic phase portraits: