NBA Lecture 1

Simplest bifurcations in n-dimensional ODEs

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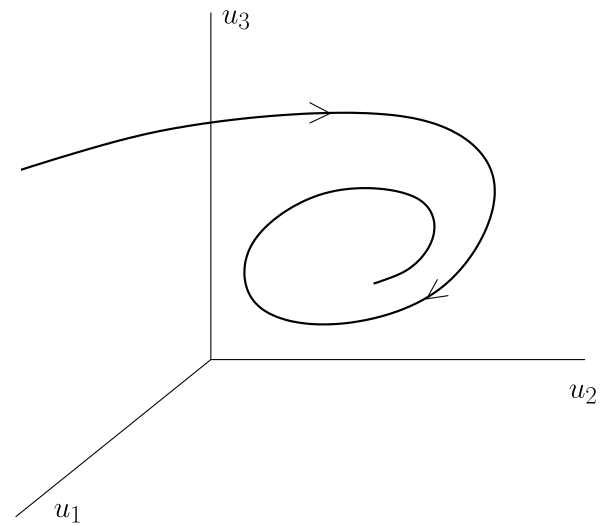
Contents

1. Solutions and orbits



Consider a smooth system

$$\dot{u} = f(u), \quad u \in \mathbb{R}^n.$$

Th. 1 *If f is smooth than for any inital point u_0 there exists a unique locally defined solution $t \mapsto u(t)$ such that $u(0) = u_0$.*

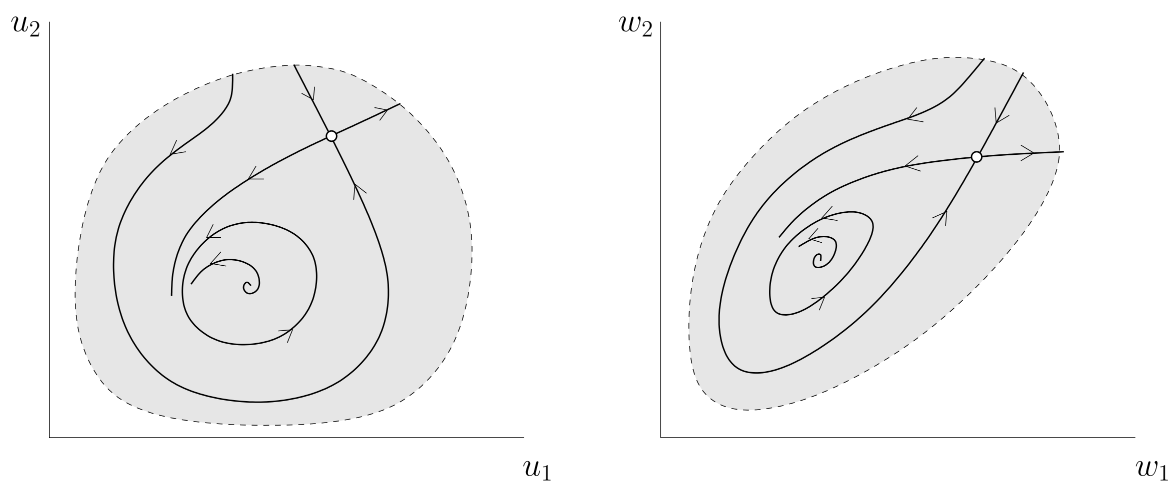
Def. 1 *Let I be the maximal definition interval of a solution $t \mapsto u(t)$, $t \in I$. The oriented by the advance of time image $u(I) \subset \mathbb{R}^n$ is called the* **orbit**.

Def. 2 Phase portrait *of an ODE system is the collection of all its orbits in $\mathbb{R}^n$.*

Def. 3 *Two systems are called* **topologically equivalent** *if their phase portraits are homeomorphic, i.e. there is a continuous invertible transformation*

$$h : \mathbb{R}^n \to \mathbb{R}^n, \quad u \mapsto w = h(u),$$

that maps orbits of one system onto orbits of the other, preserving their orientation.



Equilibria of ODEs

- An equilibrium u_0 satisfies

$$f(u_0) = 0$$

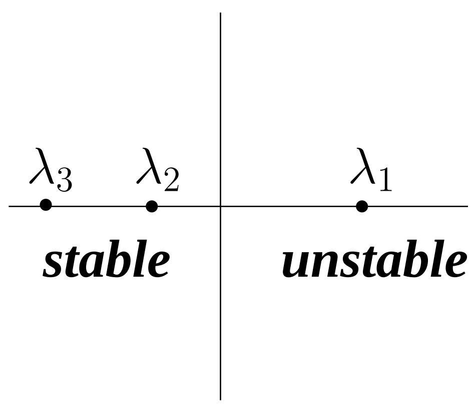
  and its Jacobian matrix $A = f_u(u_0)$ has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

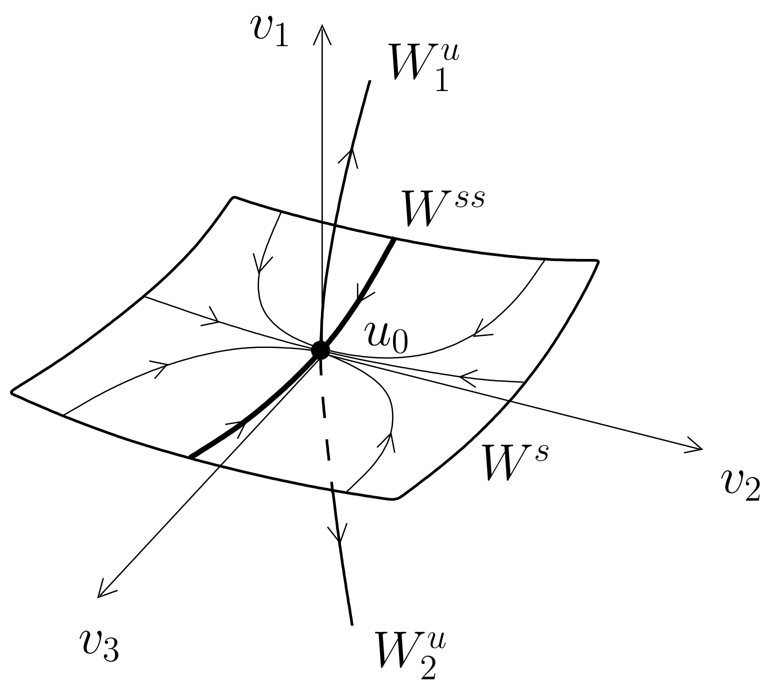
- Linearized stability of u_0:
  - If $\Re(\lambda_j) < 0$ for $j = 1, 2, \ldots, n$, the equilibrium is stable;
  - If $\Re(\lambda_k) > 0$ for some $k \in \{1, 2, \ldots, n\}$, the equilibrium is unstable.

Def. 4 *An equilibrium* u_0 *is* **hyperbolic** *if* $\Re(\lambda_j) \neq 0$ *for* $j = 1, 2, \ldots, n$.

Stable and unstable invariant manifolds of equilibria:

If a hyperbolic equilibrium u_0 has n_s eigenvalues with $\Re(\lambda) < 0$ and n_u eigenvalues with $\Re(\lambda) > 0$, it has the n_s-dimensional smooth invariant manifold W^s composed of all orbits approaching u_0 as $t \to \infty$, and the n_u-dimensional smooth invariant manifold W^u composed of all orbits approaching u_0 as $t \to -\infty$





$$Av_j = \lambda_j v_j$$

Periodic orbits (cycles)

- A **limit cycle** C_0 corresponds to a periodic solution $u_0(t+T_0) = u_0(t)$ of

$$\dot{u} = f(u), \quad u \in \mathbb{R}^n.$$

Floquet multipliers $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 1$ are the eigenvalues of $M(T_0)$:

$$\dot{M}(t) - f_u(u_0(t))M(t) = 0, \qquad M(0) = I_n.$$

- Linearized stability of C_0:

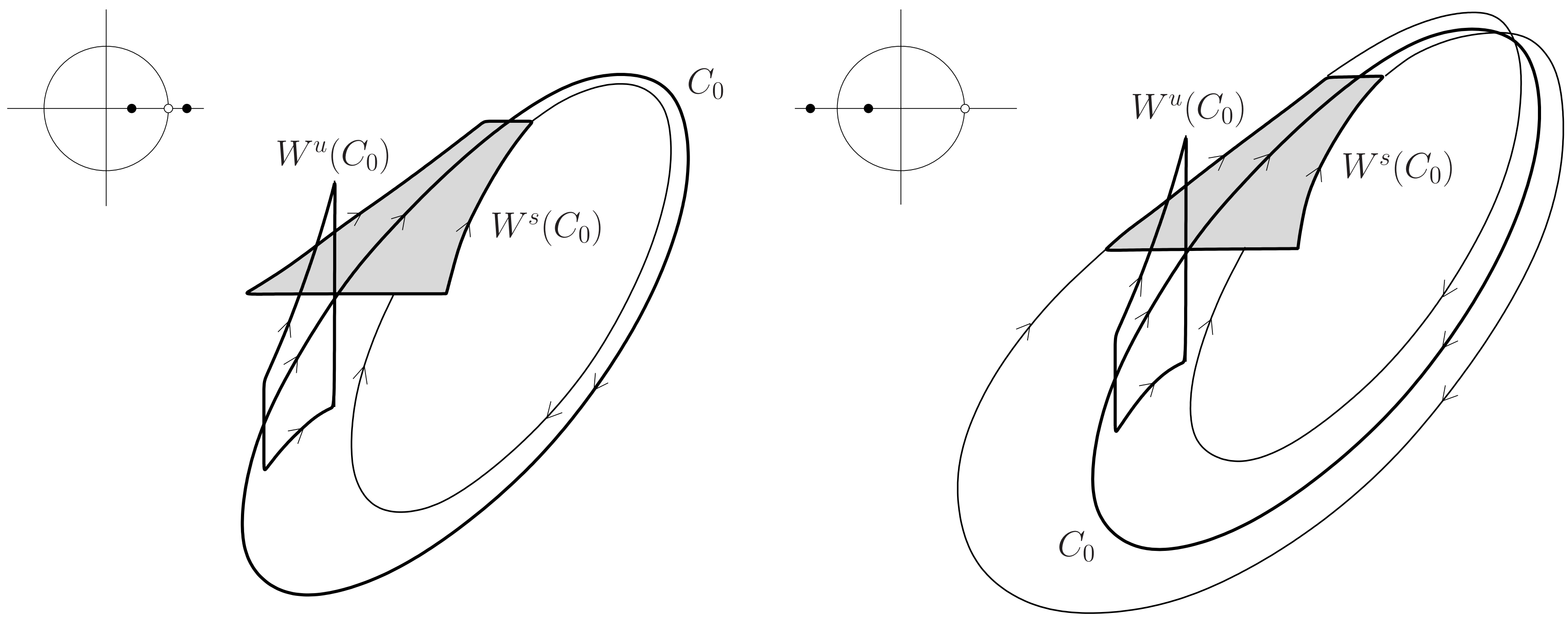
  – If $|\mu_j| < 1$ for $j = 1, 2, \ldots, n-1$, the cycle is stable;

  – If $|\mu_k| > 1$ for some $k \in \{1, 2, \ldots, n-1\}$, the cycle is unstable.

Def. 5 *A cycle* C_0 *is* **hyperbolic** *if* $|\mu_j| \neq 1$ *for* $j = 1, 2, \ldots, n-1$.

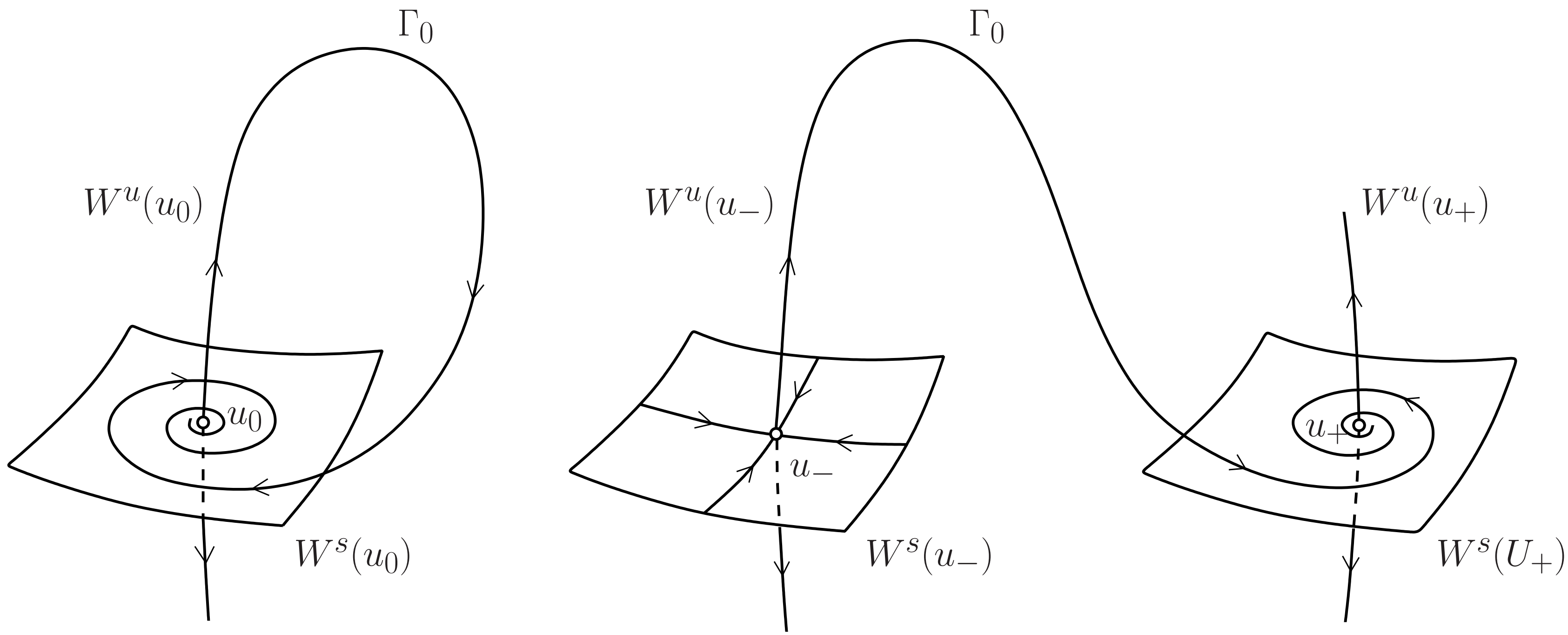
Stable and unstable invariant manifolds of cycles:

If a hyperbolic cycle C_0 has m_s multipliers with $|\mu| < 1$ and m_u multipliers with $|\mu| > 1$, it has the $(m_s + 1)$-dimensional smooth invariant manifold W^s composed of all orbits approaching C_0 as $t \to \infty$, and the $(m_u + 1)$-dimensional smooth invariant manifold W^u composed of all orbits approaching C_0 as $t \to -\infty$

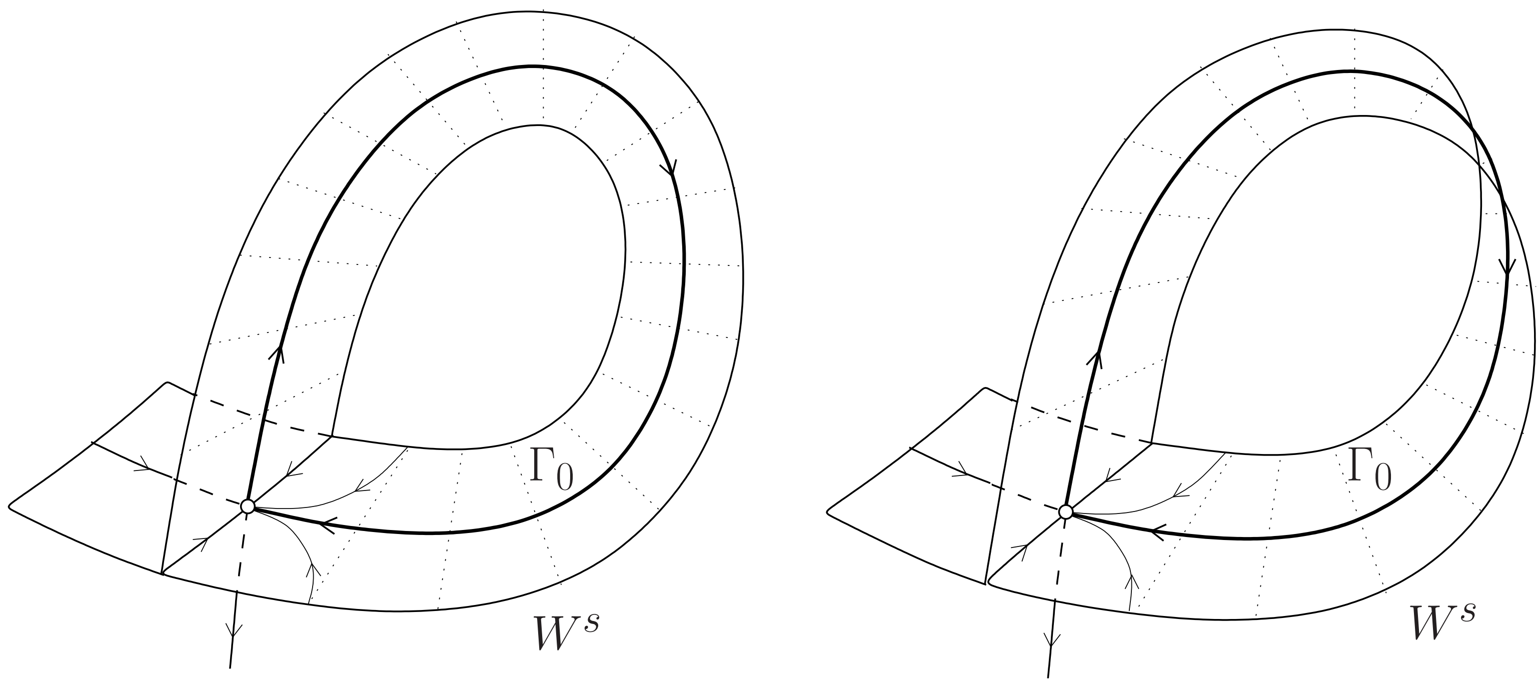


Connecting orbits

Homoclinic orbits are intersections of W^u and W^s of an equilibrium/cycle. **Heteroclinic orbits** are intersections of W^u and W^s of two different equilibria/cycles.



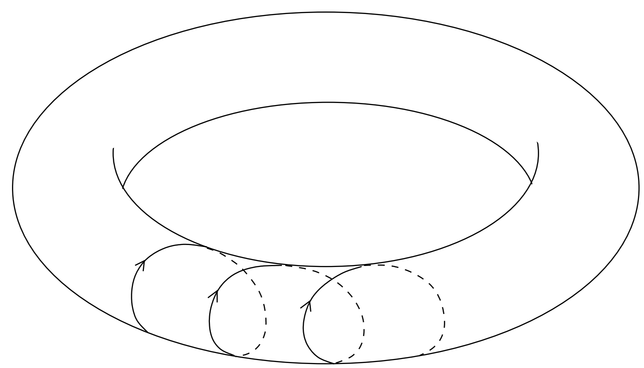
Generically, the closure of the 2D invariant manifold near a homoclinic orbit Γ_0 to an equilibriun with real eigenvalues (**saddle**) in $\mathbb{R}^3$ is either **simple (orientable)** or **twisted (non-orientable)**:



Compact invariant manifolds

1. tori

   Example: 2D-torus $\mathbb{T}^2$ with periodic or quasi-periodic orbits

2. spheres

3. Klein bottles

Strange (chaotic) invariant sets

- have **fractal** structure (not a manifold);

- contain **infinite** number of hyperbolic cycles;

- demonstrate **sensitive dependence** of solutions on initial conditions;

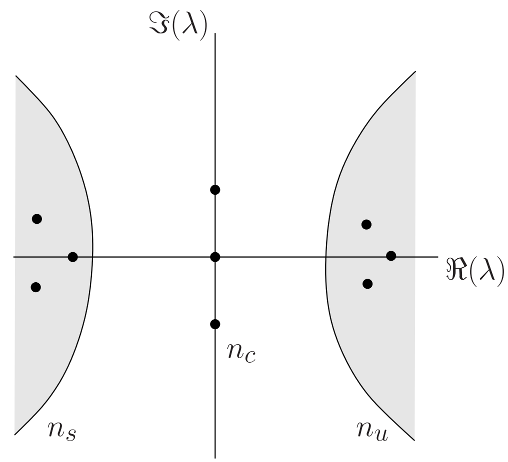
- can be attracting (**strange attractors**);

- orbits can be coded by sequences of symbols (**symbolic dynamics**).

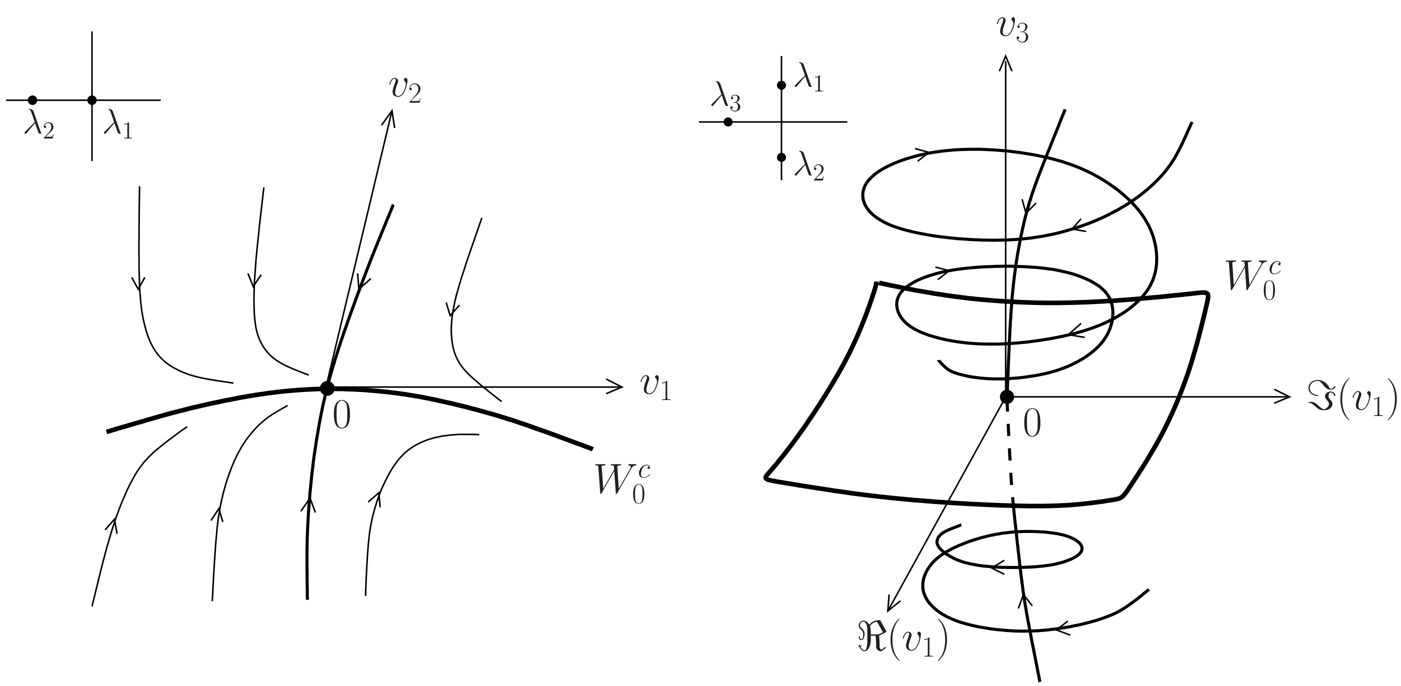
2. Bifurcations of n-dimensional ODEs $\dot{u} = f(u, \alpha)$

- **Local (equilibrium) bifurcations**

  Center manifold reduction: Let $u_0 = 0$ at $\alpha = 0$ be non-hyperbolic with stable, usntable, and critical eigenvalues:



Th. 2 *For all sufficiently small $\|\alpha\|$, there exists a local invariant* **center manifold** W^c_α *of dimension n_c that is locally attracting if $n_u = 0$, repelling if $n_s = 0$, and of saddle type if $n_s n_u > 0$. Moreover W^c_0 is tangent to the critical eigenspace of $A = f_u(0,0)$.*



Remark: W_0^c is **not unique**; however, all W_0^c have the same Taylor expansion.

Th. 3 *If $\dot{\xi} = g(\xi, \alpha)$ is the restriction of $\dot{u} = f(u, \alpha)$ to W_α^c, then this system is locally topologically equivalent to*

$$\begin{cases} \dot{\xi} &= g(\xi, \alpha), \quad \xi \in \mathbb{R}^{n_c}, \alpha \in \mathbb{R}^m, \\ \dot{x} &= -x, \quad x \in \mathbb{R}^{n_s}, \\ \dot{y} &= +y, \quad y \in \mathbb{R}^{n_u}. \end{cases}$$

Codimension 1 bifurcations of equilibria

- Consider a smooth ODE system

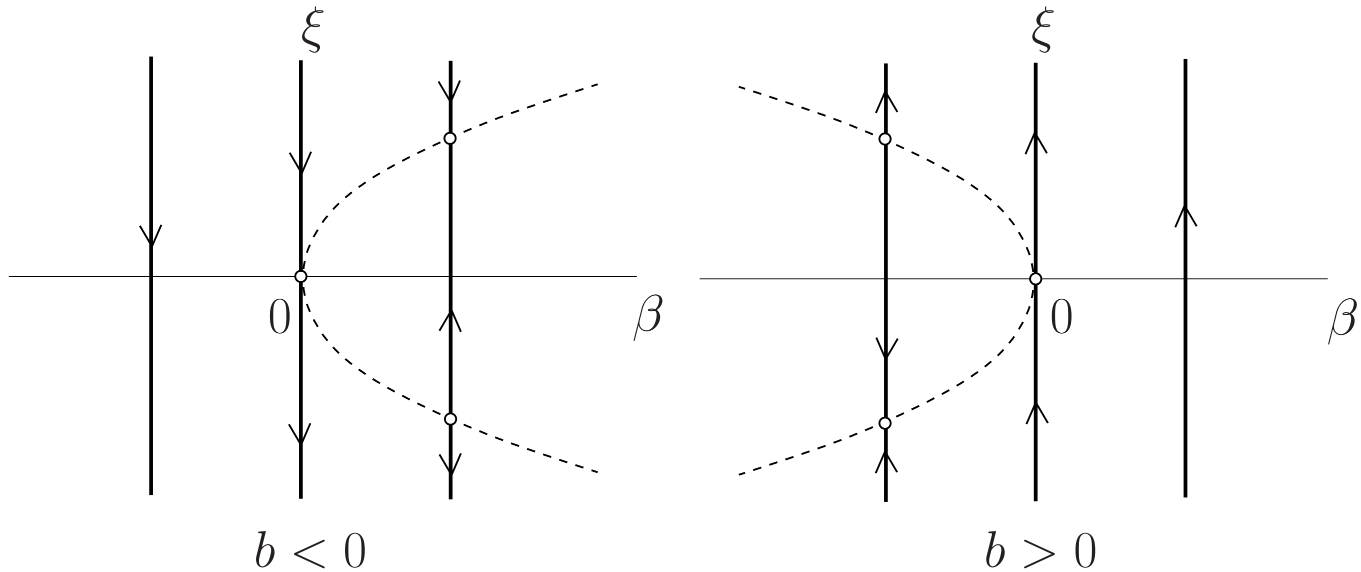
$$\dot{u} = f(u,\alpha), \quad u \in \mathbb{R}^n, \alpha \in \mathbb{R}.$$

- Critical cases:

  - **Fold (limit point, LP)**: $\lambda_1 = 0$;
  - **Andronov-Hopf (H)**: $\lambda_{1,2} = \pm i\omega_0, \quad \omega_0 > 0$.

LP normal form on $W^c_{\beta(\alpha)}$

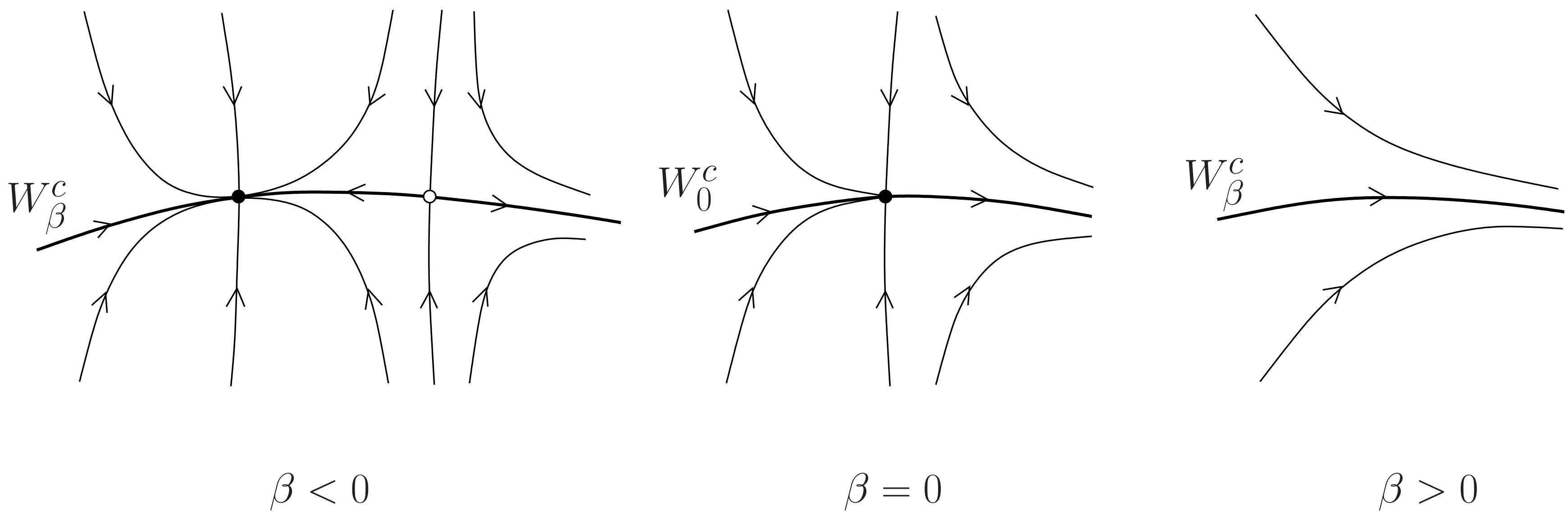
- $\dot{\xi} = \beta(\alpha) + b(\alpha)\xi^2 + O(|\xi|^3), \quad b(0) \neq 0.$



- Approximation of equilibria:

$$\beta + b\xi^2 = 0 \Rightarrow \xi_{1,2} = \pm\sqrt{-\frac{\beta}{b}}$$

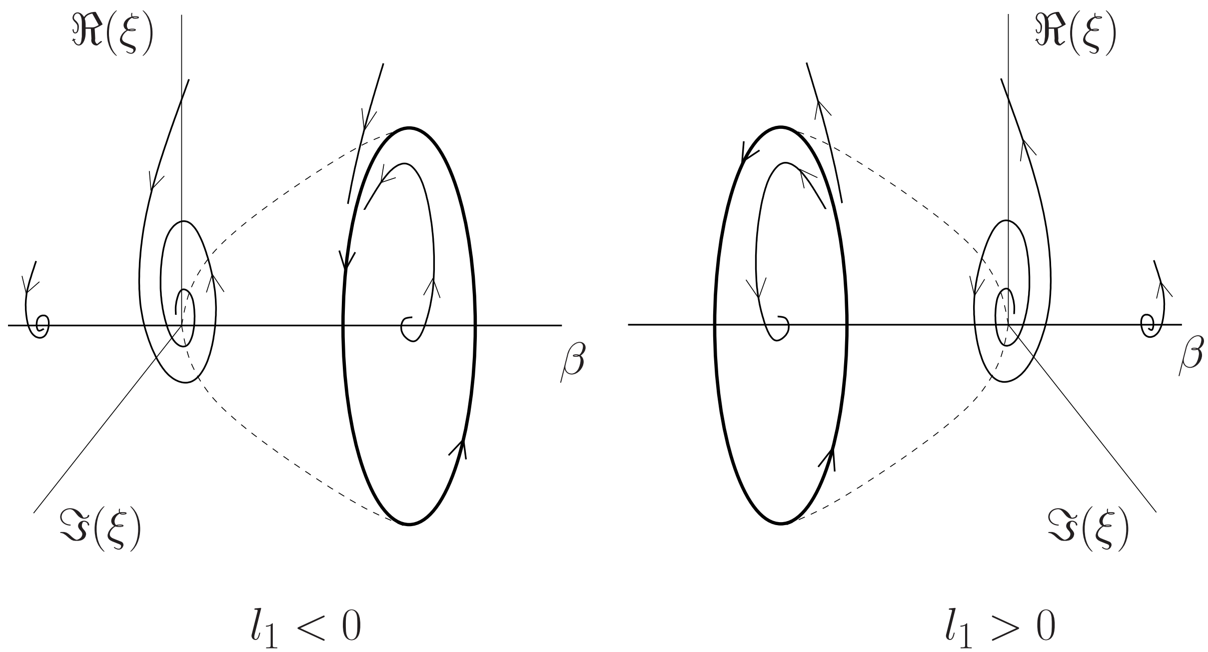
Generic LP bifurcation: $\lambda_1 = 0 \quad (b > 0)$



Collision of two equilibria.

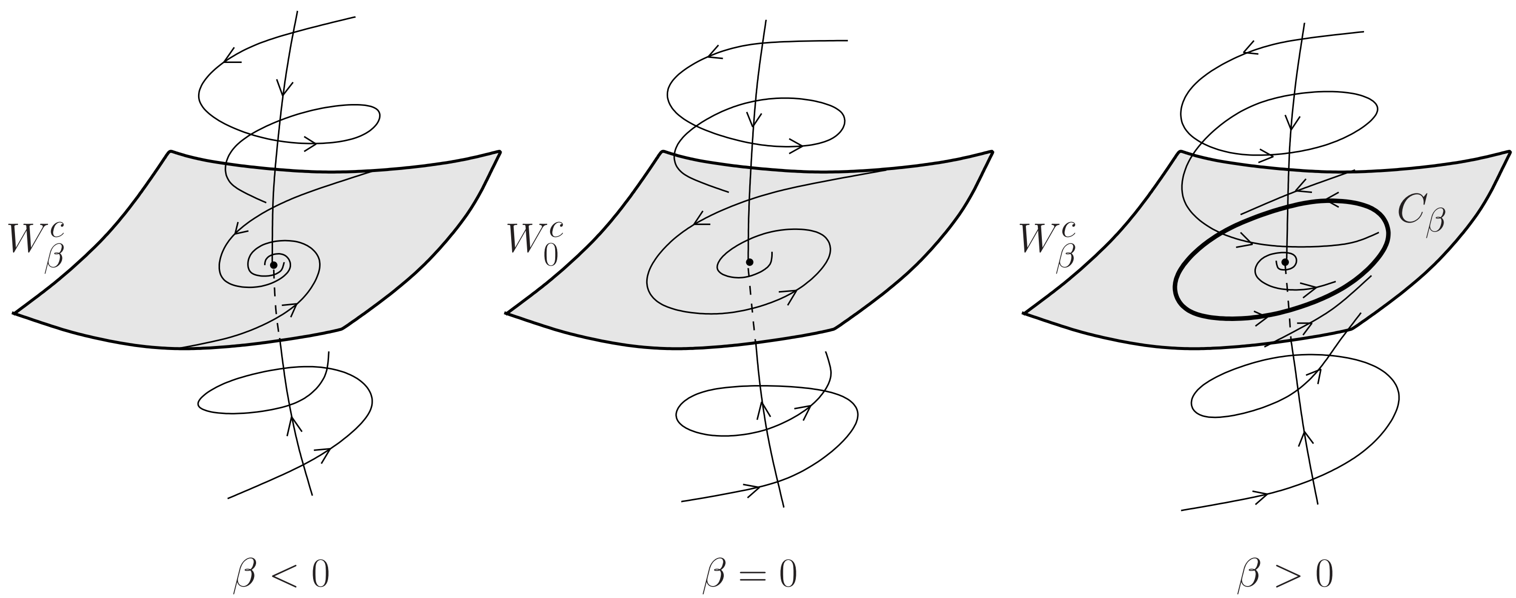
Hopf normal form on $W^c_{\beta(\alpha)}$

- $\dot{\xi} = (\beta(\alpha) + i\omega(\alpha))\xi + c(\alpha)\xi|\xi|^2 + O(|\xi|^4), \quad \omega(0) = \omega_0, l_1 \neq 0$
- **First Lyapunov coeffeicient**: $l_1 = \dfrac{1}{\omega_0}\Re(c(0))$



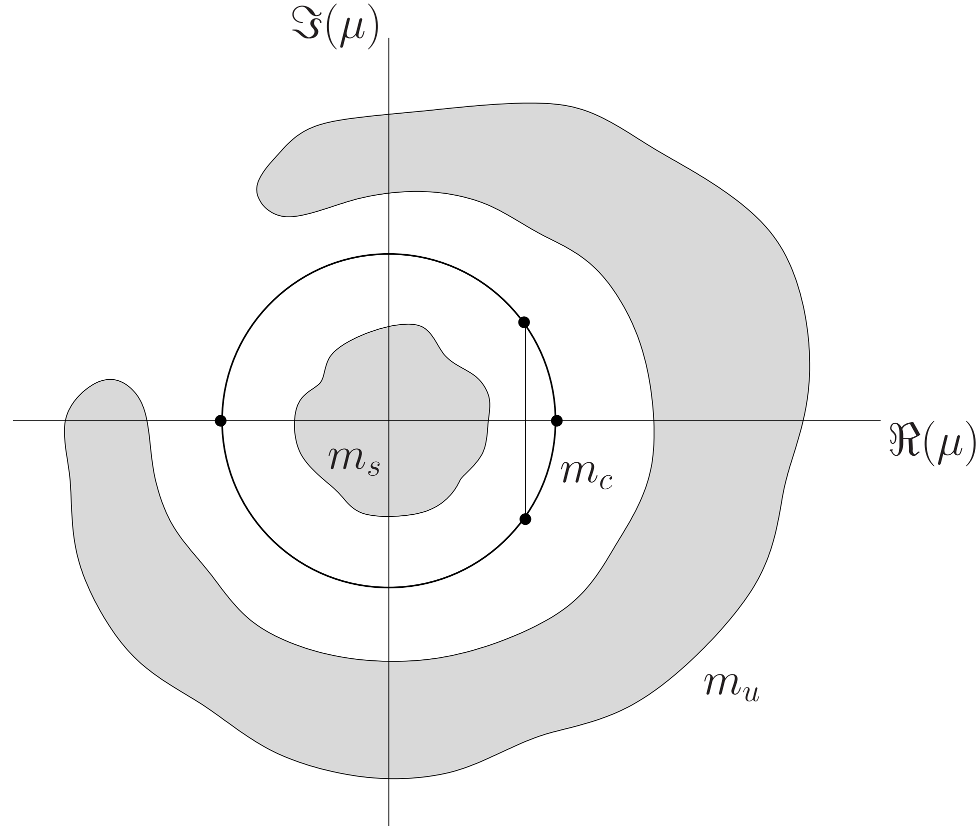
- Approximate cycle: $\left\{ \begin{array}{lcl} \dot{\rho} & = & \rho(\beta + \Re(c)\rho^2), \\ \dot{\varphi} & = & \omega + \Im(c)\rho^2, \end{array} \right. \Rightarrow \rho_0 = \sqrt{-\dfrac{\beta}{\Re(c)}}$

Generic Hopf bifurcation: $\lambda_{1,2} = \pm i\omega_0$



Birth of a limit cycle.

Codimension 1 local bifurcations of cycles



$$\dim W^c_\alpha = m_c$$

Critical cases:

- **cyclic fold (LPC)**: $\mu_1 = 1$
- **period-doubling (PD)**: $\mu_1 = -1$
- **Neimark-Sacker (NS):** $\mu_{1,2} = e^{\pm i\theta_0},\ 0 < \theta_0 < \pi,\ \theta_0 \neq \frac{\pi}{2}, \frac{2\pi}{3}$

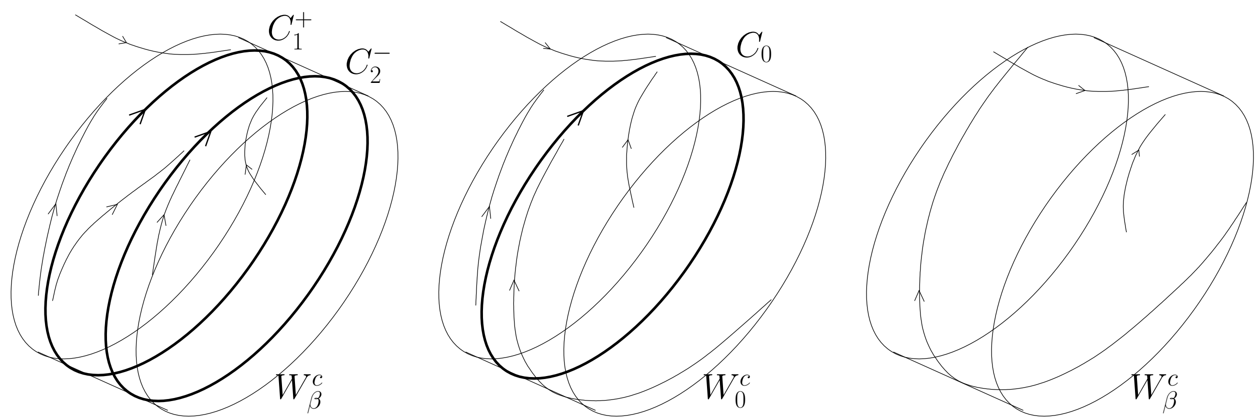
Generic LPC bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\beta) - \xi + a(\beta)\xi^2 + \mathcal{O}(\xi^3), \\ \dfrac{d\xi}{dt} = \beta + b(\beta)\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

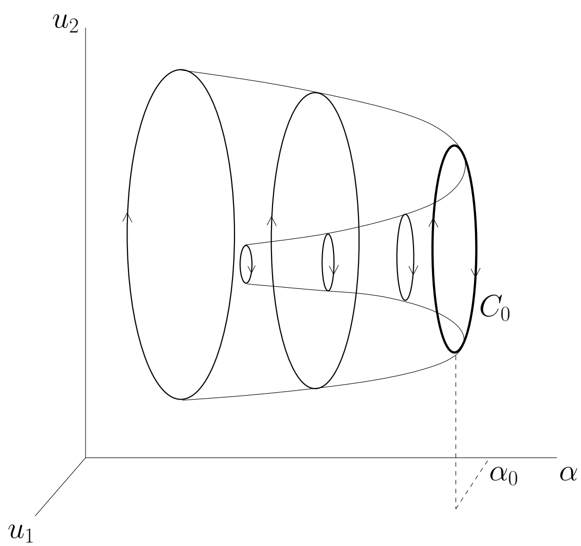
where $a, b \in \mathbb{R}$ and the $\mathcal{O}(\xi^3)$-terms are T_0-periodic in τ.

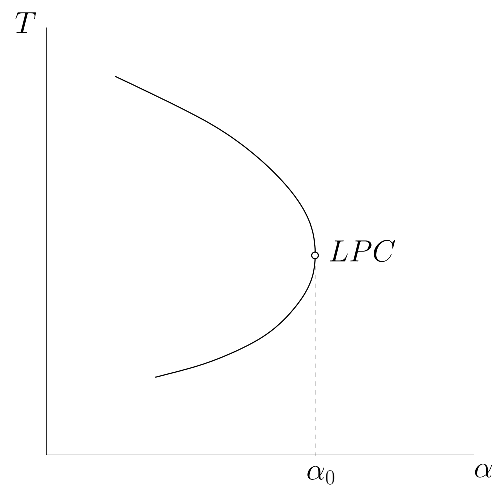
- Collision and disappearance of two limit cycles ($b(0) > 0$):



$\beta < 0$ $\beta = 0$ $\beta > 0$

Cycle manifold near LPC





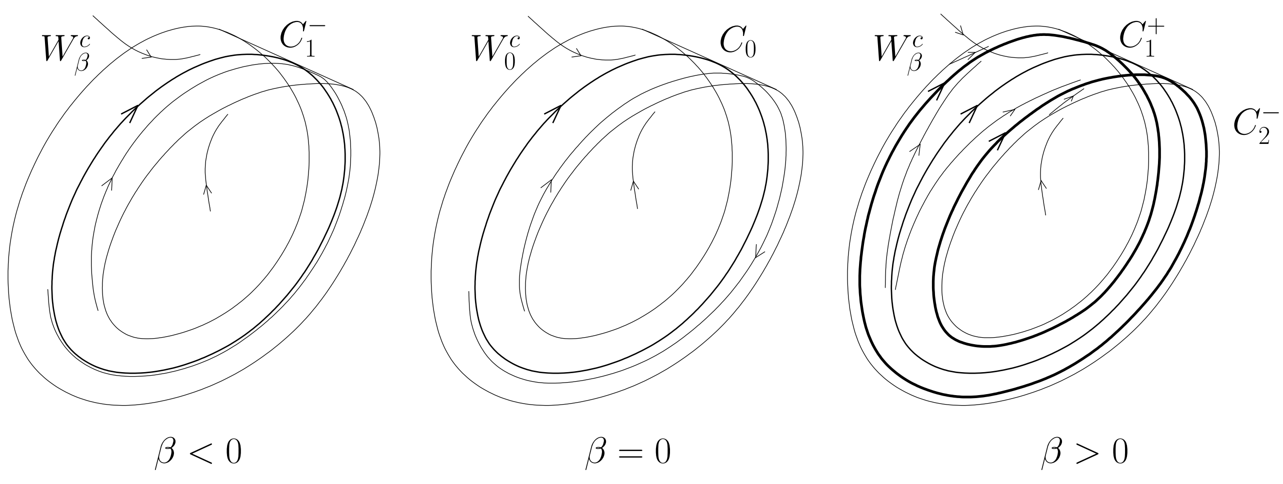
Generic PD bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} = 1 + \nu(\beta) + a(\beta)\xi^2 + \mathcal{O}(\xi^4), \\ \dfrac{d\xi}{dt} = \beta\xi + c(\beta)\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$ and the $\mathcal{O}(\xi^3)$-terms are $2T_0$-periodic in τ.

- Period-doubling ($c(0) < 0$):



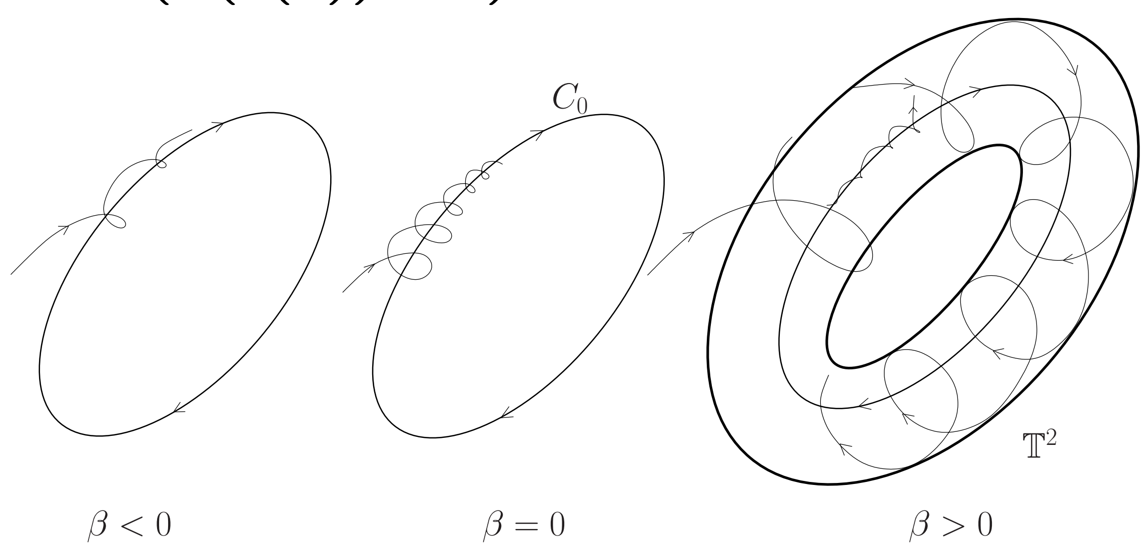
Generic NS bifurcation

- Periodic parameter-dependent normal form on W^c_β:

$$\begin{cases} \dfrac{d\tau}{dt} &= 1 + \nu(\beta) + a(\beta)|\xi|^2 + \mathcal{O}(|\xi|^4), \\ \dfrac{d\xi}{dt} &= \left(\beta + \dfrac{i\theta(\beta)}{T(\beta)}\right)\xi + d(\beta)\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{cases}$$

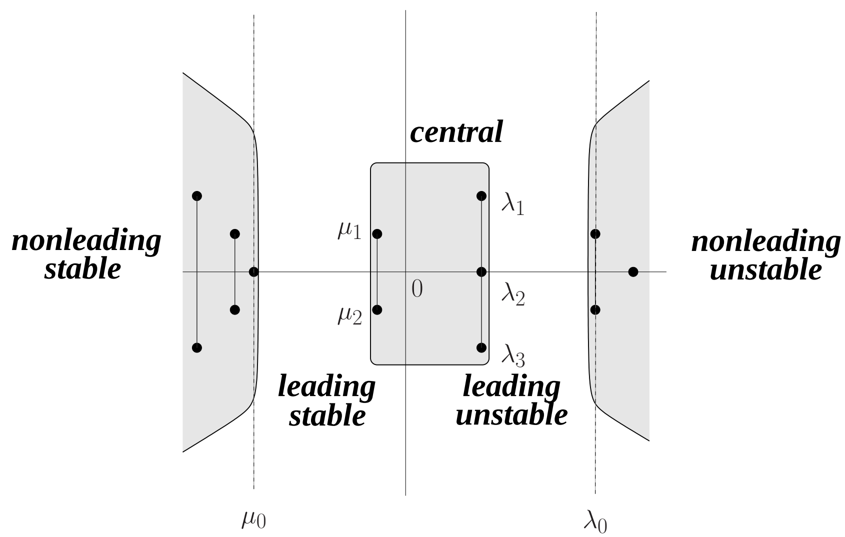
where $a \in \mathbb{R}, d \in \mathbb{C}$ and the $\mathcal{O}(\|\xi\|^4)$-terms are T_0-periodic in τ

- Torus generation ($\Re(d(0)) < 0$):



Codim 1 bifurcations of homoclinic orbits to equilibria

- **Homoclinic orbit to a hyperbolic equilibrium**:

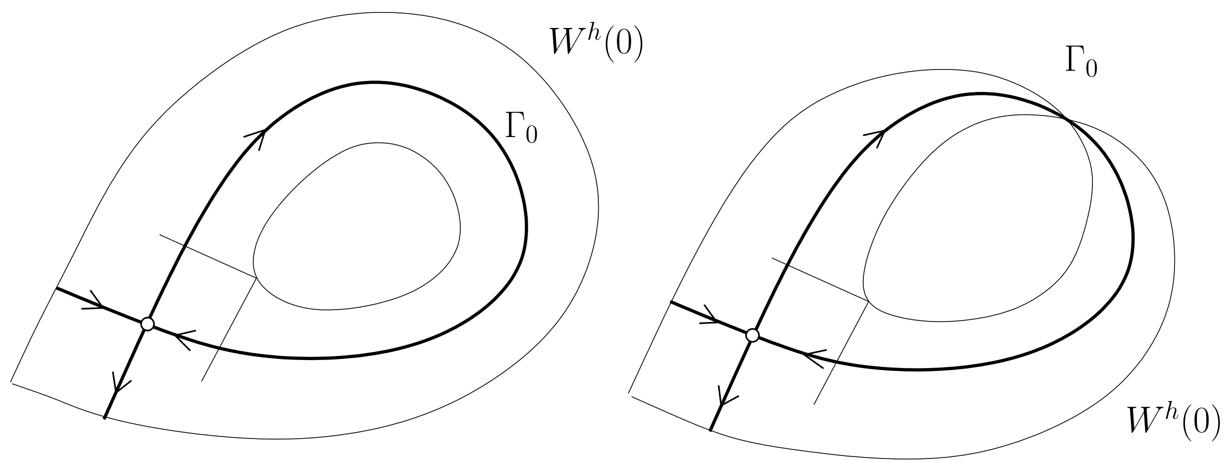


Def. 6 Saddle quantity $\sigma = \Re(\mu_1) + \Re(\lambda_1)$.

Th. 4 (Homoclinic Center Manifold) *Generically, there exists an invariant finitely-smooth manifold $W^h(\alpha)$ that is tangent to the central eigenspace at the homoclinic bifurcation.*

Saddle homoclinic orbit: $\sigma = \mu_1 + \lambda_1$

Assume that Γ_0 approaches u_0 along the leading eigenvectors.



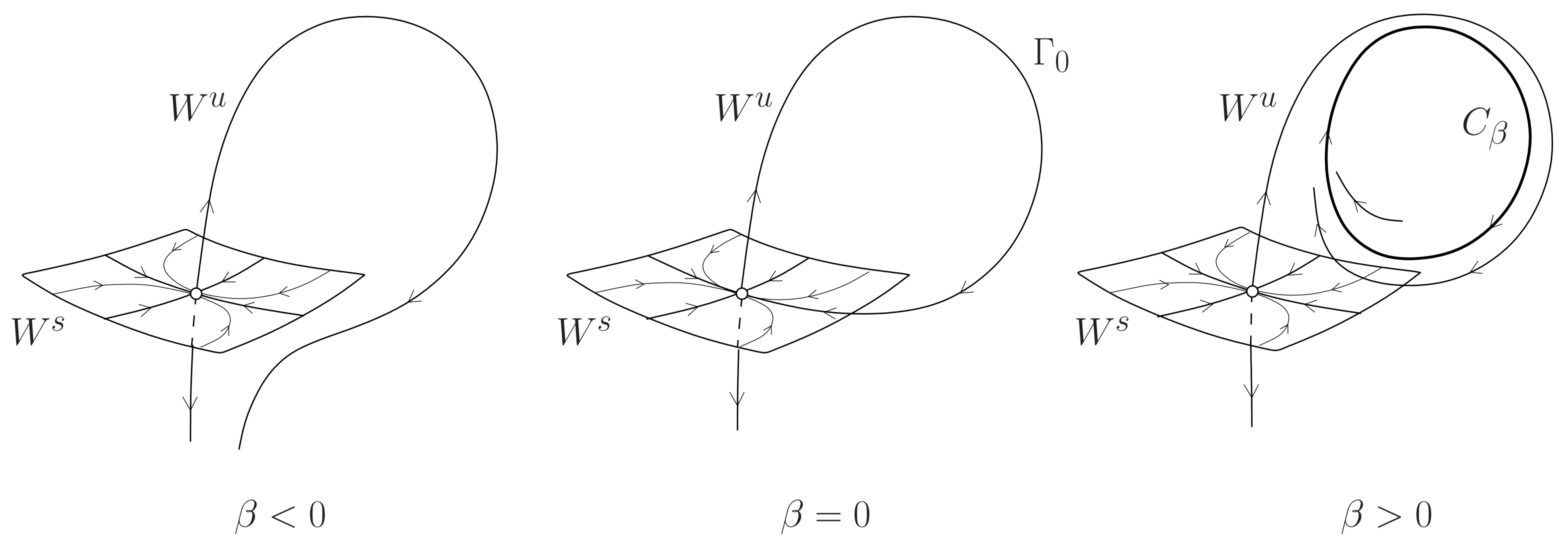
The Poincaré map near Γ_0:

$$\xi \mapsto \tilde{\xi} = \beta + A\xi^{-\frac{\mu_1}{\lambda_1}} + \dots$$

where generically $A \neq 0$, so that a unique hyperbolic cycle bifurcates from Γ_0 (stable in W^h if $\sigma < 0$ and unstable in W^h if $\sigma > 0$).

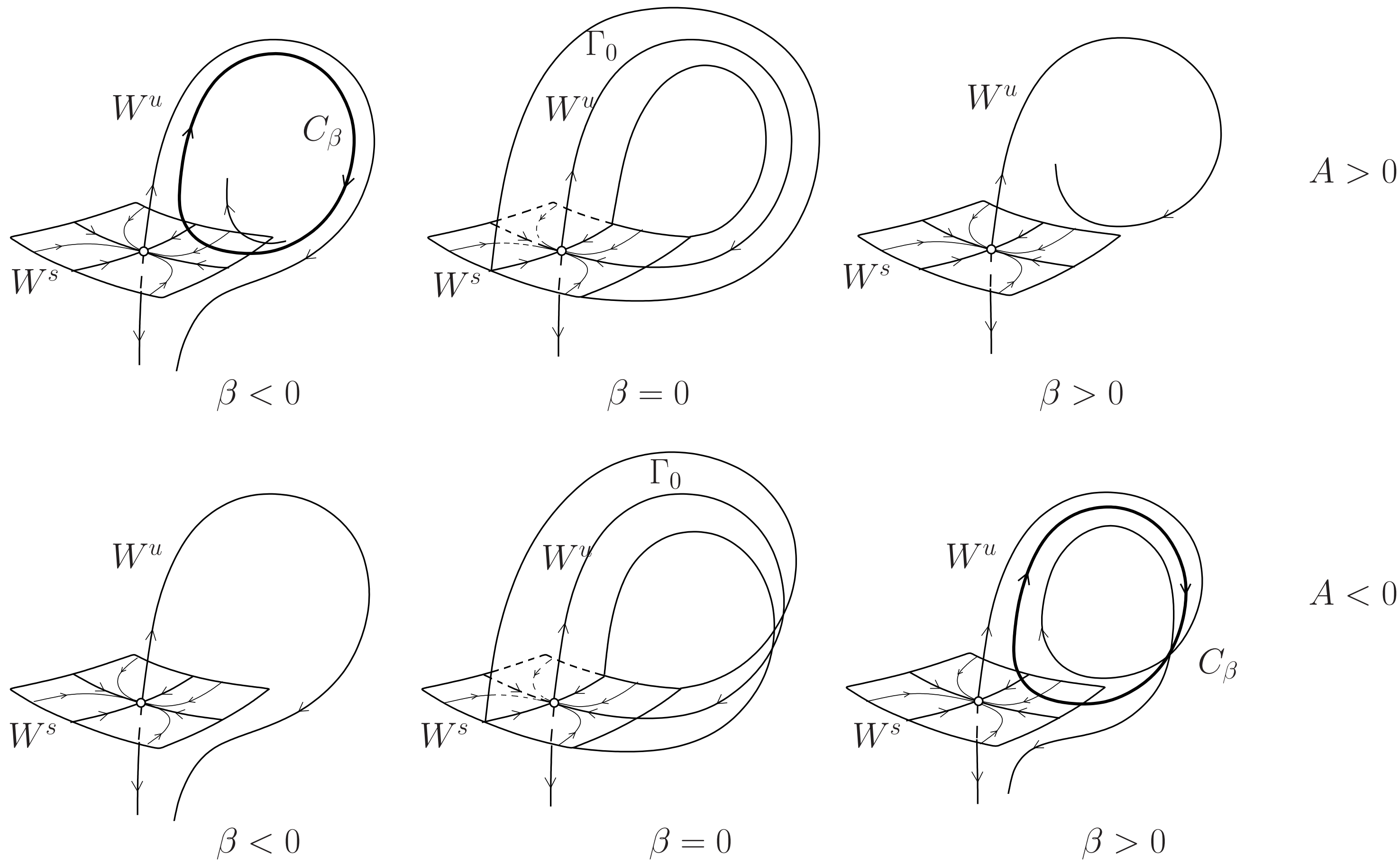
3D saddle homoclinic bifurcation with $\sigma < 0$:

Assume that $\mu_2 < \mu_1 < 0 < \lambda_1$ (otherwise reverse time: $t \mapsto -t$).



3D saddle homoclinic bifurcation with $\sigma > 0$:

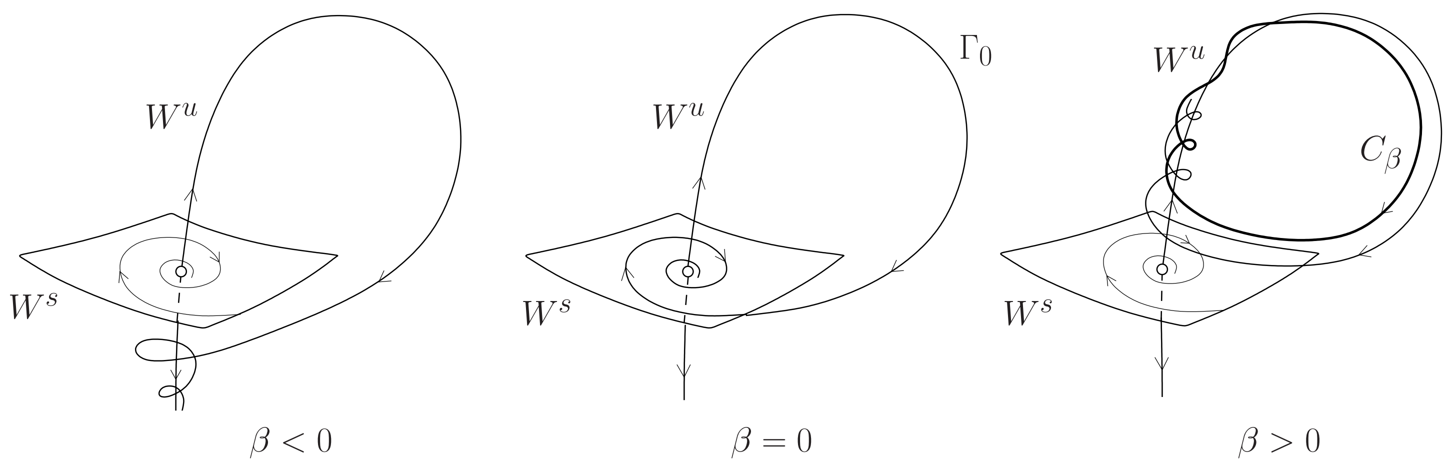
Assume that $\mu_2 < \mu_1 < 0 < \lambda_1$ (otherwise reverse time: $t \mapsto -t$).



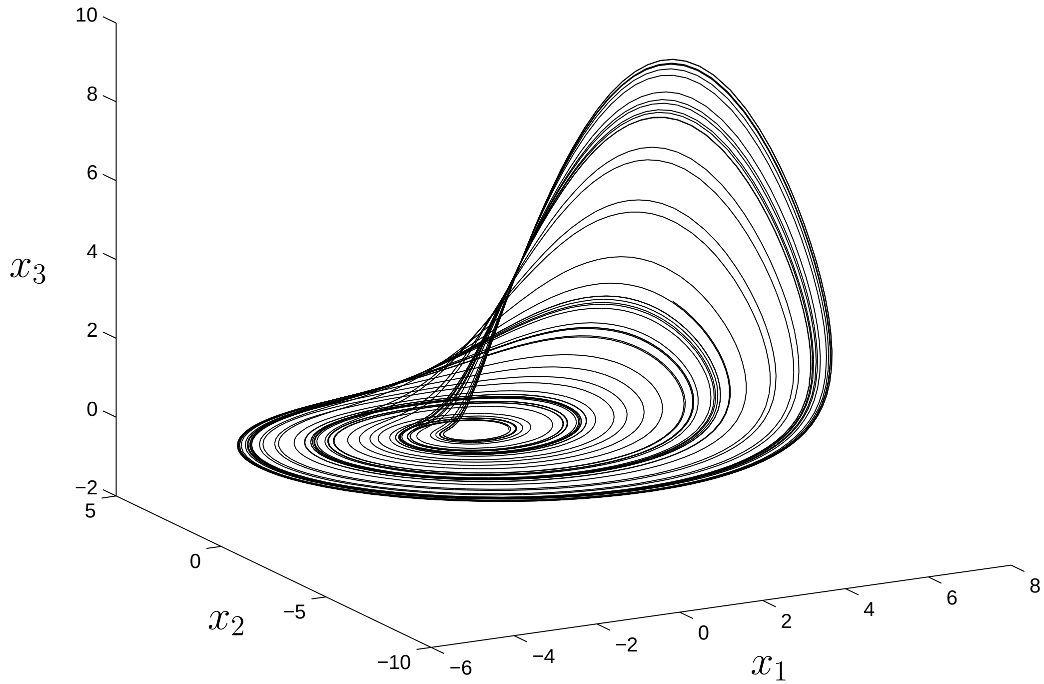
Saddle-focus homoclinic orbit: $\sigma = \Re(\mu_1) + \lambda_1$

3D saddle-focus homoclinic bifurcation with $\sigma < 0$:

Assume that $\Re(\mu_2) = \Re(\mu_1) < 0 < \lambda_1$ (otherwise reverse time: $t \mapsto -t$).



3D saddle-focus homoclinic bifurcation with $\sigma > 0$:

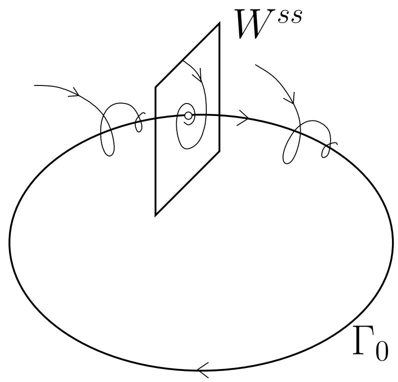


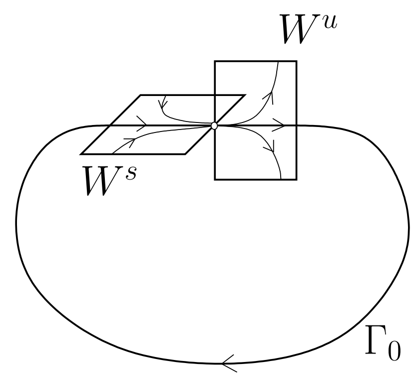
CHAOTIC INVARIANT SET

Focus-focus homoclinic orbit: $\sigma = \Re(\mu_1) + \Re(\lambda_1)$

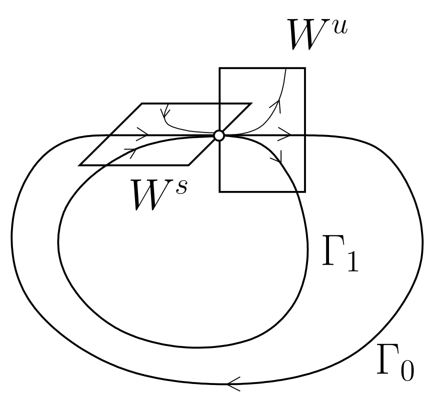
CHAOTIC INVARIANT SET

Homoclinic orbit(s) to a non-hyperbolic equilibrium





One homoclinic orbit: $\Rightarrow$ a unique hyperbolic **cycle**



Several homoclinic orbits: $\Rightarrow$ **CHAOTIC INVARIANT SET**