

NBA Lecture 4

Bifurcations of limit cycles of ODEs and their numerical analysis using BVPs

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1. Codim 1 bifurcations of limit cycles

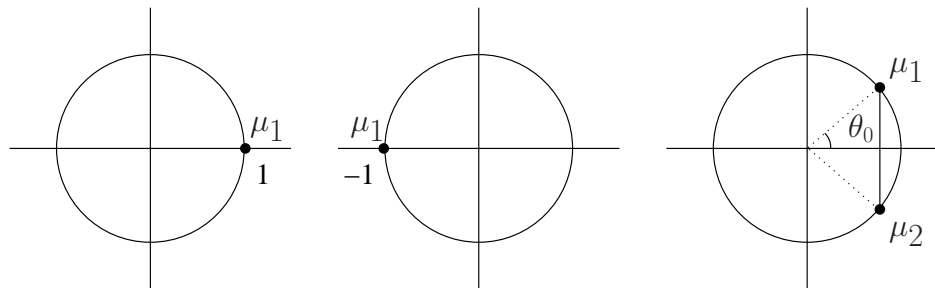
- Consider

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

A **limit cycle** C_0 corresponds to a periodic solution $x_0(t+T_0) = x_0(t)$ and has **Floquet multipliers** $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = 1$, the eigenvalues of $M(T_0)$:

$$\dot{M}(t) - f_x(x_0(t), \alpha_0)M(t) = 0, \quad M(0) = I_n.$$

- Critical cases:



LPC ($\mu_1 = 1$), **PD** ($\mu_1 = -1$), **NS** ($\mu_{1,2} = e^{\pm i\theta_0}, 0 < \theta_0 < \pi$)

Continuation of cycles in one parameter

- Defining system (BVCP):

$$\begin{cases} \dot{u}(\tau) - Tf(u(\tau), \alpha) = 0, & \tau \in [0, 1], \\ u(0) - u(1) = 0, \\ \int_0^1 \langle \dot{v}(\tau), u(\tau) \rangle d\tau = 0, \end{cases}$$

where v is a reference periodic solution.

- Linearization of BVCP: $L \begin{pmatrix} u_1 \\ T_1 \\ \alpha_1 \end{pmatrix} = 0$ where

$$L = \begin{bmatrix} D - Tf_x(u, \alpha) & -f(u, \alpha) & -Tf_\alpha(u, \alpha) \\ \delta_0 - \delta_1 & 0 & 0 \\ \text{Int}_v & 0 & 0 \end{bmatrix}$$

is an operator $L : \mathcal{C}^1([0, 1], \mathbb{R}^n) \times \mathbb{R}^2 \rightarrow \mathcal{C}^0([0, 1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$ with $\dim \mathcal{N}(L) = 1$ at a generic solution (u, T, α) to BVCP.

2. Detection of codim 1 bifurcations

- Test functions:

$$\psi_1 = V_{mn}N + n + 2,$$

$$\psi_2 = \det(M(T_0) + I_n)$$

$$= \frac{1}{\det(P_1)} \det(-P_0 + P_1),$$

$$\psi_3 = \det(M(T_0) \odot M(T_0) - I_{\frac{1}{2}n(n-1)})$$

$$= \frac{1}{\det(P_1 \odot P_1)} \det(P_0 \odot P_0 - P_1 \odot P_1),$$

where V is the tangent vector to the curve, \odot is the bialternate product,

$$M(T_0) = -P_1^{-1}P_0$$

is the monodromy matrix.

- Bifurcations:

LPC : $\psi_1 = 0$

PD : $\psi_2 = 0$

NS : $\psi_3 = 0, \psi_1 \neq 0$ ($\mu_1\mu_2 = 1$)

$$M^2(T_0) - 2 \cos(\theta_0)M(T_0) + I_n$$

has rank defect 2.

3. Continuation of codim 1 bifurcations

- PD and LPC:

$$(u, T, \alpha) \in \mathcal{C}^1([0, 1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2$$
$$\begin{cases} \dot{u}(\tau) - T f(u(\tau), \alpha) = 0, & \tau \in [0, 1], \\ u(0) - u(1) = 0, \\ \int_0^1 \langle \dot{v}(\tau), u(\tau) \rangle d\tau = 0, \\ G[u, T, \alpha] = 0. \end{cases}$$

- NS: $(u, T, \alpha, \kappa) \in \mathcal{C}^1([0, 1], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$

$$\begin{cases} \dot{u}(\tau) - T f(u(\tau), \alpha) = 0, & \tau \in [0, 1], \\ u(0) - u(1) = 0, \\ \int_0^1 \langle \dot{v}(\tau), u(\tau) \rangle d\tau = 0, \\ G_{11}[u, T, \alpha, \kappa] = 0, \\ G_{22}[u, T, \alpha, \kappa] = 0. \end{cases}$$

PD-continuation

- There exist $v_{01}, w_{01} \in \mathcal{C}^0([0, 1], \mathbb{R}^n)$, and $w_{02} \in \mathbb{R}^n$, such that $N_1 : \mathcal{C}^1([0, 1], \mathbb{R}^n) \times \mathbb{R} \rightarrow \mathcal{C}^0([0, 1], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$,

$$N_1 = \begin{bmatrix} D - Tf_x(u, \alpha) & w_{01} \\ \delta_0 + \delta_1 & w_{02} \\ \text{Int}_{v_{01}} & 0 \end{bmatrix},$$

is one-to-one and onto near a generic PD bifurcation point.

- Define G by solving

$$N_1 \begin{pmatrix} v \\ G \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Leftrightarrow \begin{cases} \dot{v}(\tau) - Tf_x(u(\tau), \alpha)v(\tau) + Gw_{01}(\tau) = 0, \\ v(0) + v(1) + Gw_{02} = 0, \\ \int_0^1 \langle v_{01}(\tau), v(\tau) \rangle d\tau = 1. \end{cases}$$

NS-continuation

- There exist $v_{01}, v_{02}, w_{11}, w_{12} \in \mathcal{C}^0([0, 2], \mathbb{R}^n)$, and $w_{21}, w_{22} \in \mathbb{R}^n$, such that $N_3 : \mathcal{C}^1([0, 2], \mathbb{R}^n) \times \mathbb{R}^2 \rightarrow \mathcal{C}^0([0, 2], \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^2$,

$$N_3 = \begin{bmatrix} D - Tf_x(u, \alpha) & w_{11} & w_{12} \\ \delta_0 - 2\kappa\delta_1 + \delta_2 & w_{21} & w_{22} \\ \text{Int}_{v_{01}} & 0 & 0 \\ \text{Int}_{v_{02}} & 0 & 0 \end{bmatrix},$$

is one-to-one and onto near a generic NS bifurcation point.

- Define G_{jk} by solving

$$N_3 \begin{pmatrix} r & s \\ G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- At the NS-cycle: $\kappa = \cos \theta$.

Remarks on continuation of bifurcations

- After discretization via orthogonal collocation, all linear BVPs for G 's have sparsity structure that is identical to that of the linearization of the BVP for limit cycles.
- For each defining system holds: *Simplicity of the bifurcation + Transversality \Rightarrow Regularity of the defining BVP.*
- Jacobian matrix of each (discretized) defining BVP can be efficiently computed using adjoint linear BVP.
- Border adaptation using solutions of the adjoint linear BVPs.
- Actually implemented in MATCONT.

5. Periodic normal forms

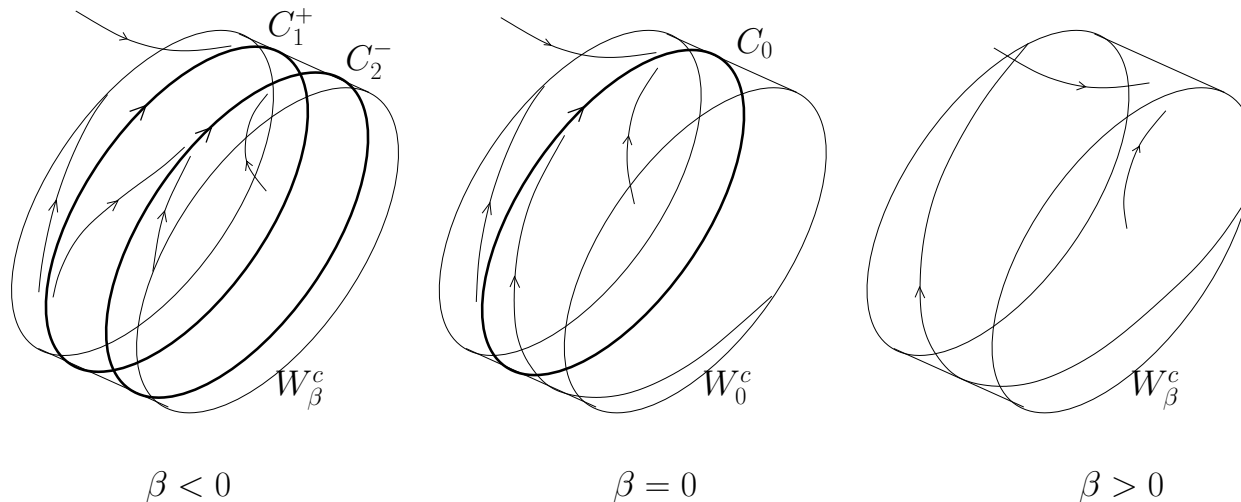
Generic LPC bifurcation

- Periodic parameter-dependent normal form on W_β^c :

$$\begin{cases} \frac{d\tau}{dt} = 1 + \nu(\beta) - \xi + a(\beta)\xi^2 + \mathcal{O}(\xi^3), \\ \frac{d\xi}{dt} = \beta + b(\beta)\xi^2 + \mathcal{O}(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$ and the $\mathcal{O}(\xi^3)$ -terms are T_0 -periodic in τ .

- Phase portraits ($b(0) > 0$):



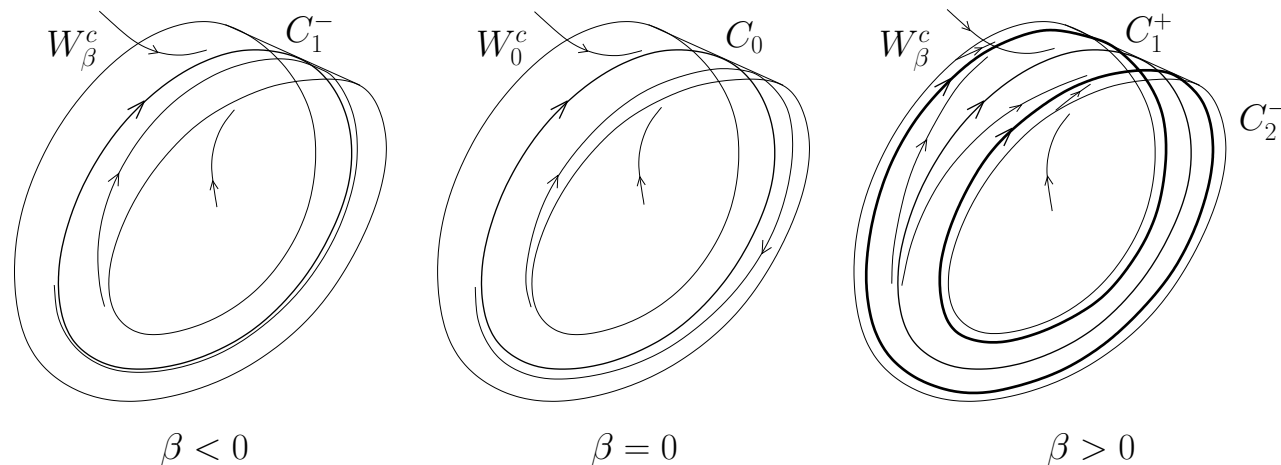
Generic PD bifurcation

- Periodic parameter-dependent normal form on W_β^c :

$$\begin{cases} \frac{d\tau}{dt} = 1 + \nu(\beta) + a(\beta)\xi^2 + \mathcal{O}(\xi^4), \\ \frac{d\xi}{dt} = \beta\xi + c(\beta)\xi^3 + \mathcal{O}(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$ and the $\mathcal{O}(\xi^3)$ -terms are $2T_0$ -periodic in τ .

- Phase portraits ($c(0) < 0$):



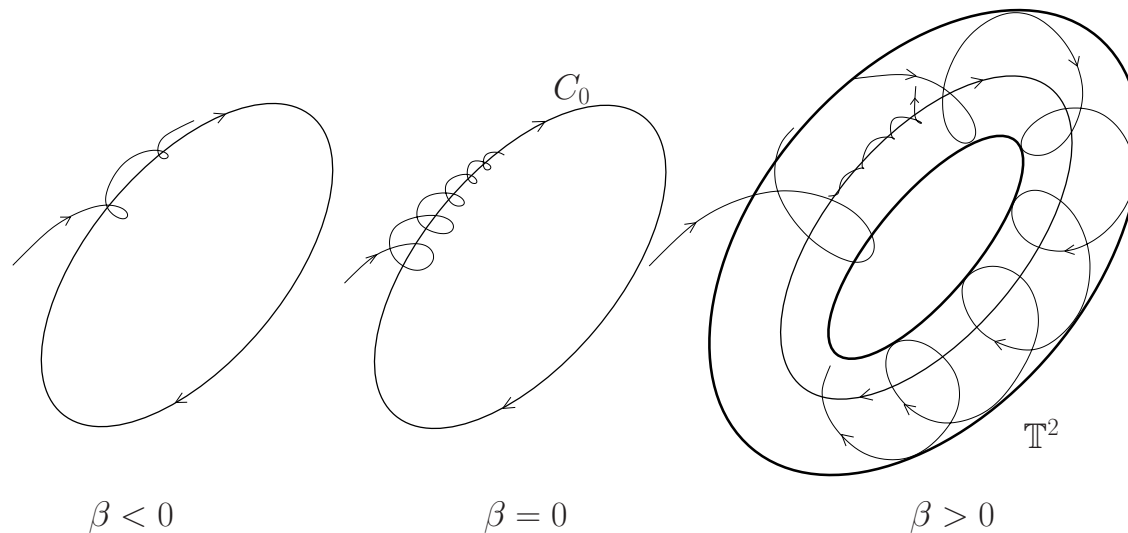
Generic NS bifurcation (no strong resonances)

- Periodic parameter-dependent normal form on W_β^c :

$$\begin{cases} \frac{d\tau}{dt} = 1 + \nu(\beta) + a(\beta)|\xi|^2 + \mathcal{O}(|\xi|^4), \\ \frac{d\xi}{dt} = \left(\beta + \frac{i\theta(\beta)}{T(\beta)} \right) \xi + d(\beta)\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{cases}$$

where $a \in \mathbb{R}, d \in \mathbb{C}$ and the $\mathcal{O}(\|\xi\|^4)$ -terms are T_0 -periodic in τ

- Phase portraits ($\Re(d(0)) < 0$):



5. Critical normal form coefficients

- Assume $\varphi, \varphi^* \in \mathcal{C}^1([0, T_0], \mathbb{R}^n)$ satisfy

$$\begin{cases} \dot{\varphi}(\tau) - A(\tau)\varphi(\tau) = 0, & \tau \in [0, T_0], \\ \varphi(0) - \varphi(T_0) = 0, \\ \int_0^{T_0} \langle \varphi(\tau), \varphi(\tau) \rangle d\tau - 1 = 0, \end{cases}$$

and

$$\begin{cases} \dot{\varphi}^*(\tau) + A^\top(\tau)\varphi^*(\tau) = 0, & \tau \in [0, T_0], \\ \varphi^*(0) - \varphi^*(T_0) = 0. \end{cases}$$

- If $h \in \mathcal{C}^1([0, T_0], \mathbb{R}^n)$ is a solution to

$$\begin{cases} \dot{h}(\tau) - A(\tau)h(\tau) = g(\tau), & \tau \in [0, T_0], \\ h(0) - h(T_0) = 0, \end{cases}$$

with $g \in \mathcal{C}^0([0, T_0], \mathbb{R}^n)$, then

$$\int_0^{T_0} \langle \varphi^*(\tau), g(\tau) \rangle d\tau = 0$$

(Fredholm solvability condition) and there is a unique solution h satisfying $\int_0^{T_0} \langle \varphi^*(\tau), h(\tau) \rangle d\tau = 0$.

Multilinear forms

At a codimension-one point write

$$\begin{aligned} f(x_0(t) + v, \alpha_0) &= f(x_0(t), \alpha_0) \\ &+ A(t)v + \frac{1}{2}B(t; v, v) \\ &+ \frac{1}{6}C(t; v, v, v) + \mathcal{O}(\|v\|^4), \end{aligned}$$

where $A(t) = f_x(x_0(t), \alpha_0)$ and the components of the multilinear functions B and C are given by

$$B_i(t; u, v) = \sum_{j,k=1}^n \frac{\partial^2 f_i(x, \alpha_0)}{\partial x_j \partial x_k} \Big|_{x=x_0(t)} u_j v_k$$

and

$$C_i(t; u, v, w) = \sum_{j,k,l=1}^n \frac{\partial^3 f_i(x, \alpha_0)}{\partial x_j \partial x_k \partial x_l} \Big|_{x=x_0(t)} u_j v_k w_l,$$

for $i = 1, 2, \dots, n$. These are T_0 -periodic in t .

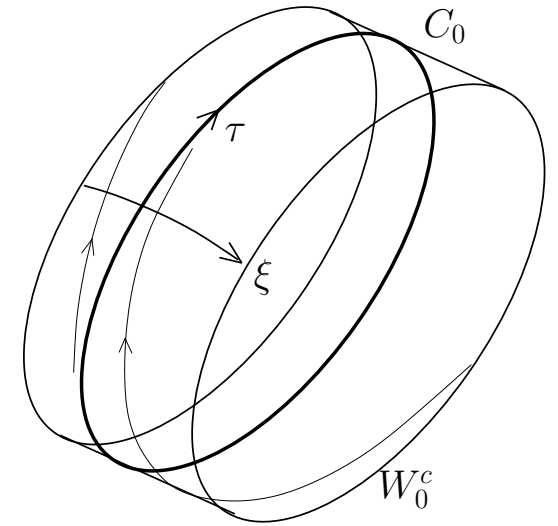
Fold (LPC): $\mu_1 = 1$

- Critical center manifold $W_0^c : \tau \in [0, T_0], \xi \in \mathbb{R}$

$$x = x_0(\tau) + \xi v(\tau) + H(\tau, \xi),$$

where $H(T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2}h_2(\tau)\xi^2 + O(\xi^3)$$



- Critical periodic normal form on W_0^c :

$$\begin{cases} \frac{d\tau}{dt} = 1 - \xi + a\xi^2 + O(\xi^3), \\ \frac{d\xi}{dt} = b\xi^2 + O(\xi^3), \end{cases}$$

where $a, b \in \mathbb{R}$, while the $O(\xi^3)$ -terms are T_0 -periodic in τ .

LPC: Generalized and adjoint eigenfunctions

$$\begin{cases} \dot{v}(\tau) - A(\tau)v(\tau) - f(x_0(\tau), \alpha_0) = 0, & \tau \in [0, T_0], \\ v(0) - v(T_0) = 0, \\ \int_0^{T_0} \langle v(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau = 0, \end{cases}$$

implying

$$\int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau = 0,$$

where φ^* satisfies

$$\begin{cases} \dot{\varphi}^*(\tau) + A^\top(\tau)\varphi^*(\tau) = 0, & \tau \in [0, T_0], \\ \varphi^*(0) - \varphi^*(T_0) = 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), v(\tau) \rangle d\tau - 1 = 0. \end{cases}$$

LPC: Computation of b

- Substitute into

$$\frac{dx}{dt} = \frac{\partial x}{\partial \xi} \frac{d\xi}{dt} + \frac{\partial x}{\partial \tau} \frac{d\tau}{dt}$$

- Collect

$$\xi^0 : \dot{x}_0 = f(x_0, \alpha_0),$$

$$\xi^1 : \dot{v} - A(\tau)v = \dot{x}_0,$$

$$\xi^2 : \dot{h}_2 - A(\tau)h_2 = B(\tau; v, v) - 2af(x_0, \alpha_0) + 2\dot{v} - 2bv.$$

- Fredholm solvability condition

$$b = \frac{1}{2} \int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), v(\tau)) + 2A(\tau)v(\tau) \rangle d\tau.$$

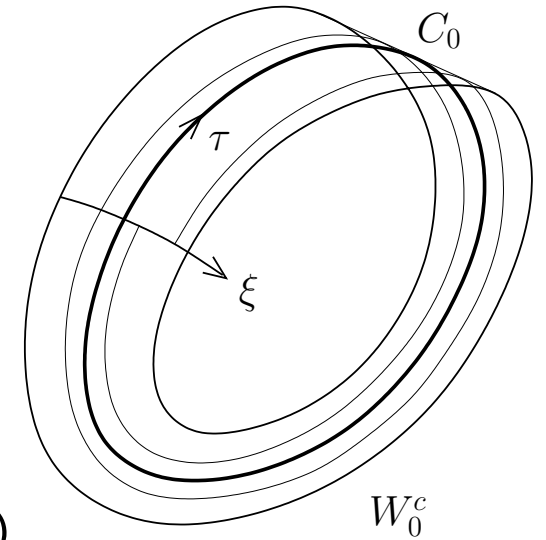
Flip (PD): $\mu_1 = -1$

- Critical center manifold $W_0^c : \tau \in [0, 2T_0], \xi \in \mathbb{R}$

$$x = x_0(\tau) + \xi w(\tau) + H(\tau, \xi),$$

where $H(2T_0, \xi) = H(0, \xi)$,

$$H(\tau, \xi) = \frac{1}{2}h_2(\tau)\xi^2 + \frac{1}{6}h_3(\tau)\xi^3 + O(\xi^4)$$



- Critical periodic normal form on W_0^c :

$$\begin{cases} \frac{d\tau}{dt} = 1 + a\xi^2 + O(\xi^4), \\ \frac{d\xi}{dt} = c\xi^3 + O(\xi^4), \end{cases}$$

where $a, c \in \mathbb{R}$, while the $O(\xi^4)$ -terms are $2T_0$ -periodic in τ .

PD: Eigenfunctions

$$w(\tau) = \begin{cases} v(\tau), & \tau \in [0, T_0], \\ -v(\tau - T_0), & \tau \in [T_0, 2T_0], \end{cases}$$

$$w^*(\tau) = \begin{cases} v^*(\tau), & \tau \in [0, T_0], \\ -v^*(\tau - T_0), & \tau \in [T_0, 2T_0], \end{cases}$$

with

$$\begin{cases} \dot{v}(\tau) - A(\tau)v(\tau) = 0, & \tau \in [0, T_0], \\ v(0) + v(T_0) = 0, \\ \int_0^{T_0} \langle v(\tau), v(\tau) \rangle d\tau - 1 = 0, \end{cases}$$

$$\begin{cases} \dot{v}^*(\tau) + A^T(\tau)v^*(\tau) = 0, & \tau \in [0, T_0], \\ v^*(0) + v^*(T_0) = 0, \\ \int_0^{T_0} \langle v^*(\tau), v(\tau) \rangle d\tau - 1/2 = 0. \end{cases}$$

PD: Quadratic terms

$$\xi^2 : \dot{h}_2 - A(\tau)h_2 = B(\tau; w, w) - 2a\dot{x}_0, \quad \tau \in [0, 2T_0].$$

Since $\text{Ker} \left(\frac{d}{d\tau} - A(\tau) \right) = \text{span}\{w, \psi = \dot{x}_0\}$, we must have

$$\begin{cases} \int_0^{2T_0} \langle w^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau) \rangle d\tau = 0, \\ \int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)) - 2a\dot{x}_0(\tau) \rangle d\tau = 0, \end{cases}$$

where ψ^* satisfies

$$\begin{cases} \dot{\psi}^*(\tau) + A^\top(\tau)\psi^*(\tau) = 0, \quad \tau \in [0, T_0], \\ \psi^*(0) - \psi^*(T_0) = 0, \\ \int_0^{T_0} \langle \psi^*(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau - 1/2 = 0, \end{cases}$$

and is extended to $[T_0, 2T_0]$ by periodicity.

PD: Computation of a and h_2

- The first Fredholm condition holds identically for all a , while the second gives

$$a = \frac{1}{2} \int_0^{2T_0} \langle \psi^*(\tau), B(\tau; w(\tau), w(\tau)) \rangle d\tau = \int_0^{T_0} \langle \psi^*(\tau), B(\tau; v(\tau), v(\tau)) \rangle d\tau.$$

- Define h_2 on $[0, T_0]$ as the unique solution to

$$\begin{cases} \dot{h}_2(\tau) - A(\tau)h_2(\tau) - B(\tau; v(\tau), v(\tau)) + 2af(x_0(\tau), \alpha_0) = 0, \\ h_2(0) - h_2(T_0) = 0, \\ \int_0^{T_0} \langle \psi^*(\tau), h_2(\tau) \rangle d\tau = 0, \end{cases}$$

and extend it by periodicity to $[T_0, 2T_0]$.

PD: Computation of c

Cubic terms:

$$\xi^3 : \dot{h}_3 - A(\tau)h_3 = C(\tau; w, w, w) + 3B(\tau; w, h_2) - 6a\dot{w} - 6cw.$$

The Fredholm solvability condition implies

$$\begin{aligned} 6c &= \int_0^{2T_0} \langle w^*(\tau), C(\tau; w(\tau), w(\tau), w(\tau)) + 3B(\tau; w(\tau), h_2(\tau)) \rangle d\tau \\ &\quad - \int_0^{2T_0} \langle w^*(\tau), 6aA(\tau)w(\tau) \rangle d\tau \end{aligned}$$

or

$$c = \frac{1}{3} \int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), v(\tau)) + 3B(\tau; v(\tau), h_2(\tau)) - 6aA(\tau)v(\tau) \rangle d\tau$$

Torus (NS): $\mu_{1,2} = e^{\pm i\theta_0}$ with $e^{i\nu\theta_0} \neq 1, \nu = 1, 2, 3, 4$

- Critical center manifold $W_0^c : \tau \in [0, T_0], \xi \in \mathbb{C}$

$$x = x_0(\tau) + \xi v(\tau) + \bar{\xi} \bar{v}(\tau) + H(\tau, \xi, \bar{\xi}), \quad H(T_0, \xi, \bar{\xi}) = H(0, \xi, \bar{\xi}),$$

$$H(\tau, \xi, \bar{\xi}) = \frac{1}{2}h_{20}(\tau)\xi^2 + h_{11}(\tau)\xi\bar{\xi} + \frac{1}{2}h_{02}(\tau)\bar{\xi}^2 \\ + \frac{1}{6}h_{30}(\tau)\xi^3 + \frac{1}{2}h_{21}(\tau)\xi^2\bar{\xi} + \frac{1}{2}h_{12}(\tau)\xi\bar{\xi}^2 + \frac{1}{6}h_{03}(\tau)\bar{\xi}^3 + O(|\xi|^4).$$

- Critical periodic normal form on W_0^c :

$$\begin{cases} \frac{d\tau}{dt} = 1 + a|\xi|^2 + \mathcal{O}(|\xi|^4), \\ \frac{d\xi}{dt} = \frac{i\theta_0}{T_0}\xi + d\xi|\xi|^2 + \mathcal{O}(|\xi|^4), \end{cases}$$

where $a \in \mathbb{R}, d \in \mathbb{C}$, and the $\mathcal{O}(|\xi|^4)$ -terms are T_0 -periodic in τ .

NS: Complex eigenfunctions

$$\begin{cases} \dot{v}(\tau) - A(\tau)v(\tau) + \frac{i\theta_0}{T_0}v(\tau) = 0, & \tau \in [0, T_0], \\ v(0) - v(T_0) = 0, \\ \int_0^{T_0} \langle v(\tau), v(\tau) \rangle d\tau - 1 = 0. \end{cases}$$

and

$$\begin{cases} \dot{v}^*(\tau) + A^\top(\tau)v^*(\tau) + \frac{i\theta_0}{T_0}v^*(\tau) = 0, & \tau \in [0, T_0], \\ v^*(0) - v^*(T_0) = 0, \\ \int_0^{T_0} \langle v^*(\tau), v(\tau) \rangle d\tau - 1 = 0. \end{cases}$$

NS: Quadratic terms

- $\xi^2 \bar{\xi}^0$: $\dot{h}_{20} - A(\tau)h_{20} + \frac{2i\theta_0}{T_0}h_{20} = B(\tau; v, v)$

Since $e^{2i\theta_0}$ is not a multiplier of the critical cycle, the BVP

$$\begin{cases} \dot{h}_{20}(\tau) - A(\tau)h_{20}(\tau) + \frac{2i\theta_0}{T_0}h_{20}(\tau) - B(\tau; v(\tau), v(\tau)) = 0, \\ h_{20}(0) - h_{20}(T_0) = 0. \end{cases}$$

has a unique solution on $[0, T_0]$.

- $|\xi|^2$: $\dot{h}_{11} - A(\tau)h_{11} = B(\tau; v, \bar{v}) - a\dot{x}_0$

Here

$$\text{Ker} \left(\frac{d}{d\tau} - A(\tau) \right) = \text{span}(\varphi = \dot{x}_0).$$

NS: Computation of a and h_{11}

- Define φ^* as the unique solution of

$$\begin{cases} \dot{\varphi}^*(\tau) + A^\top(\tau)\varphi^*(\tau) = 0, & \tau \in [0, T_0], \\ \varphi^*(0) - \varphi^*(T_0) = 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), f(x_0(\tau), \alpha_0) \rangle d\tau - 1 = 0. \end{cases}$$

- Fredholm solvability: $a = \int_0^{T_0} \langle \varphi^*(\tau), B(\tau; v(\tau), \bar{v}(\tau)) \rangle d\tau$.

- Then find h_{11} on $[0, T_0]$ from the BVP

$$\begin{cases} \dot{h}_{11}(\tau) - A(\tau)h_{11}(\tau) - B(\tau; v(\tau), \bar{v}(\tau)) + af(x_0(\tau), \alpha_0) = 0, \\ h_{11}(0) - h_{11}(T_0) = 0, \\ \int_0^{T_0} \langle \varphi^*(\tau), h_{11}(\tau) \rangle d\tau = 0. \end{cases}$$

NS: Computation of d

- Cubic terms:

$$\xi^2 \bar{\xi} : \quad \dot{h}_{21} - Ah_{21} + \frac{i\theta_0}{T_0} h_{21} = 2B(\tau; h_{11}, v) + B(\tau; h_{20}, \bar{v}) \\ + C(\tau; v, v, \bar{v}) - 2av - 2dv.$$

- Fredholm solvability condition:

$$d = \frac{1}{2} \int_0^{T_0} \langle v^*(\tau), C(\tau; v(\tau), v(\tau), \bar{v}(\tau)) \rangle d\tau \\ + \frac{1}{2} \int_0^{T_0} \langle v^*(\tau), 2B(\tau; h_{11}(\tau), v(\tau)) + B(\tau; h_{20}(\tau), \bar{v}(\tau)) \rangle d\tau \\ - a \int_0^{T_0} \langle v^*(\tau), A(\tau)v(\tau) \rangle d\tau + \frac{ia\theta_0}{T_0}.$$

Remarks on numerical periodic normalization

- Only the derivatives of $f(x, \alpha_0)$ are used, not those of the Poincaré map.
- Detection of codim 2 points is easy.
- After discretization via orthogonal collocation, all linear BVPs involved have the standard sparsity structure.
- One can re-use solutions to linear BVPs appearing in the continuation to compute the normal form coefficients.
- Actually implemented in MATCONT.