BIFURCATION PHENOMENA

Lecture 1: Qualitative theory of planar ODEs

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Contents

Literature

1. A.A. Andronov, E.A. Leontovich, I.I. Gordon, and A.G. Maier *Qualitative Theory of Second-Order Dynamic Systems*, Willey & Sons, London, 1973

2. L.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, and L.O. Chua *Methods of Qualitative Theory in Nonlinear Dynamics*, Part I, World Scientific, Singapore, 1998

3. F. Dumortier, J. Llibre, and J.C. Artés *Qualitative Theory of Planar Differential Systems*, Universitext, Springer-Verlag, Berlin, 2006

1. SOLUTIONS AND ORBITS

Newton's Second Law: $m\ddot{x} = F(x, \dot{x}) \quad \Rightarrow \quad \begin{cases} \dot{x} &= y, \\ \dot{y} &= \frac{1}{m} F(x, y) \end{cases}$

General planar system:

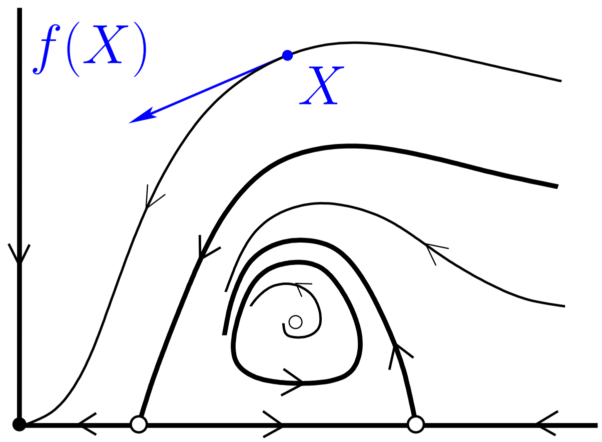
$$\begin{cases} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y) \end{cases} \quad \text{or} \quad \dot{X} = f(X), \quad X \in \mathbb{R}^2,$$

where

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(X) = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}.$$

Theorem 1 *If f is smooth than for any inital point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ there exists a unique locally defined solution $t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ such that $x(0) = x_0$ and $y(0) = y_0$.*

Definition 1 *Let I be the maximal definition interval of a solution $t \mapsto X(t)$, $t \in I$. The oriented by the advance of time image $X(I) \subset \mathbb{R}^2$ is called the* **orbit**.



Vector field: $X \mapsto f(X)$

$f(X) \neq 0$ is tangent to the orbit through X
$\Rightarrow$ orbits do not cross.

Definition 2 **Phase portrait** *of a planar system is the collection of all its orbits in $\mathbb{R}^2$.*

We draw only key orbits, which determine the topology of the phase portrait.

Types of orbits:

1. **Equilibria**: $X(t) = X_0$ so that $f(X_0) = 0$.

2. **Periodic orbits (cycles):** $X(t) \not\equiv X_0,\ X(t+T) = X(t), t \in \mathbb{R}$
   The minimal $T > 0$ is called the **period** of the cycle.

3. **Connecting orbits:** $\lim_{t \to \pm\infty} X(t) = X_\pm$ with $f(X_\pm) = 0$.

   If $X_- = X_+$ the connecting orbit is called **homoclinic**
   If $X_- \neq X_+$ the connecting orbit is called **heteroclinic**.

4. **All other orbits**

Theorem 2 (Poincaré-Bendixson)

A bounded orbit of a smooth system

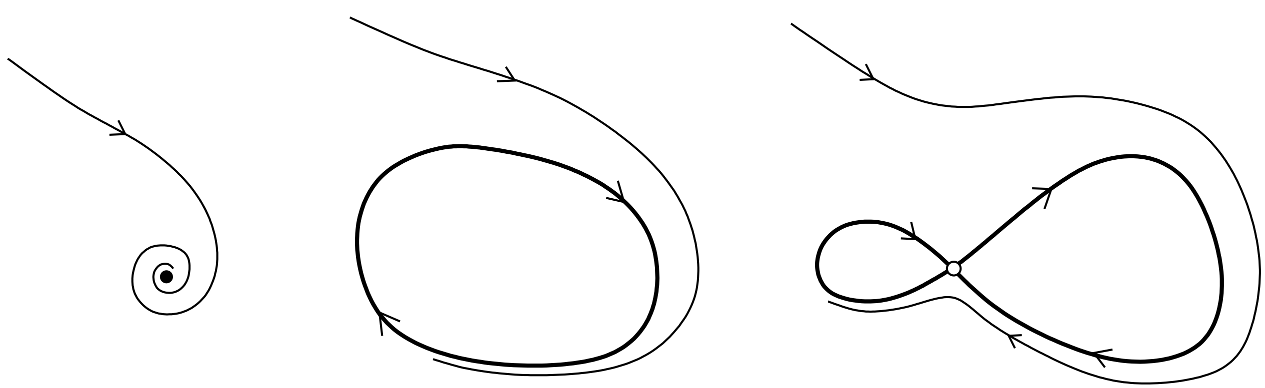
$$\dot{X} = f(X), \quad X \in \mathbb{R}^2,$$

tends to one of the following sets in the phase plane:

(i) *an equilibrium point;*

(ii) *a periodic orbit;*

(iii) *a union of equilibria and their connecting orbits.*

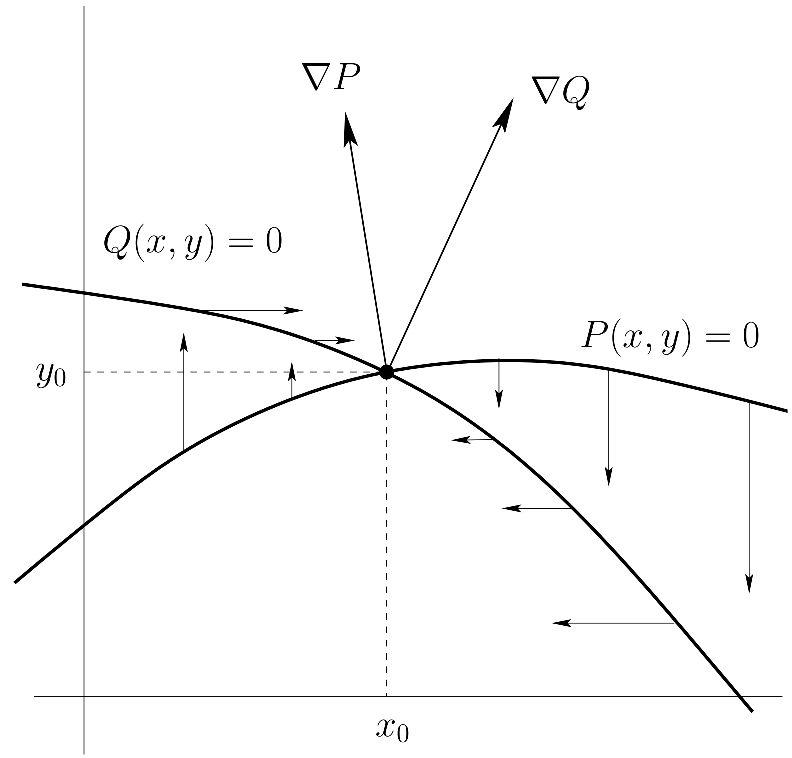


2. EQUILIBRIA: Null-isoclines

$$f(X) = 0 \quad \Leftrightarrow \quad \begin{cases} P(x,y) &= 0, \\ Q(x,y) &= 0. \end{cases}$$

$$\nabla P = \begin{pmatrix} P_x \\ P_y \end{pmatrix} \quad \text{and} \quad \nabla Q = \begin{pmatrix} Q_x \\ Q_y \end{pmatrix}$$

are orthogonal to $P = 0$ and $Q = 0$, resp.



Jacobian matrix of the equilibrium X_0:

$$A = f_X(X_0) = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}\Bigg|_{x=x_0, y=y_0}$$

If $\det A \neq 0 \Rightarrow$ the null-isoclines intersect transversally at X_0.
If $\det A = 0 \Rightarrow$ the null-isoclines are tangent at X_0.

Eigenvalues of the equilibrium X_0 are the eigenvalues of A, i.e. the solutions of

$$\lambda^2 - \sigma\lambda + \Delta = 0,$$

where

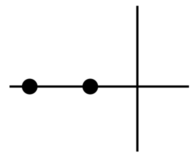
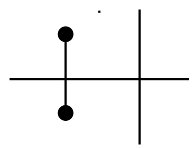
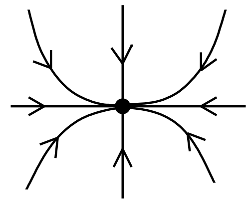
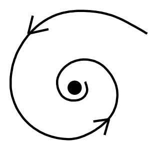
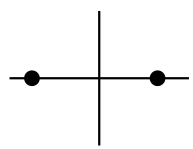
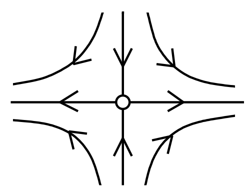
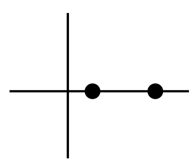
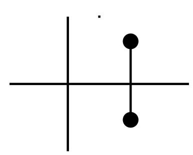
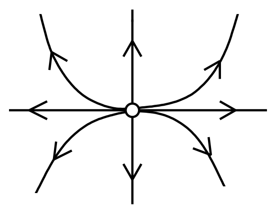
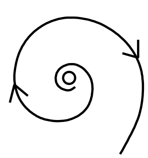
$$\begin{aligned} \sigma &= \lambda_1 + \lambda_2 = \mathrm{Sp}A = P_x(x_0, y_0) + Q_y(x_0, y_0), \\ \Delta &= \lambda_1\lambda_2 = \det A = P_x(x_0, y_0)Q_y(x_0, y_0) - P_y(x_0, y_0)Q_x(x_0, y_0). \end{aligned}$$

$$\lambda_{1,2} = -\frac{\sigma}{2} \pm \sqrt{\frac{\sigma^2}{4} - \Delta}$$

Definition 3 *An equilibrium* X_0 *is* **hyperbolic** *if* $\Re(\lambda) \neq 0$.

Equilibrium X_0 *with* $\lambda_1 = 0$ *(i.e.* $\det A = 0$*) is called* **multiple**.
Equilibrium X_0 *with* $\lambda_1 + \lambda_2 = 0$ *(i.e.* $\mathrm{Sp}\ A = 0$*) is called* **neutral**.

Phase portraits of planar linear systems $\dot{Y} = AY$

(n_u, n_s)	Eigenvalues	Phase portrait	Stability
(0, 2)	 	 node  focus	stable
(1, 1)		 saddle	unstable
(2, 0)	 	 node  focus	unstable

Definition 4 *Two systems are called* **topologically equivalent** *if their phase portraits are homeomorphic, i.e. there is a continuous invertible transformation that maps orbits of one system onto orbits of the other, preserving their orientation.*

Theorem 3 (Grobman-Hartman) *Consider a smooth nonlinear system*

$$\dot{X} = AX + F(X), \quad F = \mathcal{O}(\|X\|^2) \equiv O(2),$$

and its **linearization**

$$\dot{Y} = AY.$$

If $\Re(\lambda) \neq 0$ for all eigenvalues of A, then these systems are locally topologically equivalent near the origin.

Warning: A stable/unstable node is locally topologically equivalent to a stable/unstable focus.

Trivial topological equivalences

1. **Orbital equivalence**:

$$\dot{X} = f(X) \ \sim \ \dot{Y} = g(Y)f(Y)$$

where $g : \mathbb{R}^2 \mapsto \mathbb{R}$ is smooth positive function; $Y = h(X) = X$ preserves orbits.

2. **Smooth equivalence**:

$$\dot{X} = f(X) \ \sim \ \dot{Y} = h_X(h^{-1}(Y))f(h^{-1}(Y)),$$

where $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth diffeomorphism; the substitution $Y = h(X)$ transforms solutions onto solutions:

$$\dot{Y} = h_X(X)\dot{X} = h_X(X)f(X) \quad \text{where} \quad X = h^{-1}(Y).$$

3. **Smooth orbital equivalence**: 1. + 2.

Simplest critical cases

- $\lambda_1 = 0,\ \lambda_2 \neq 0$

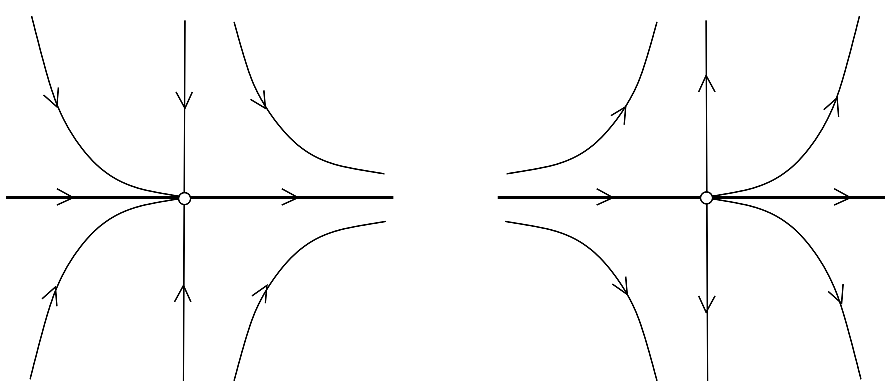
  By a linear diffeomorphism, $\dot{X} = f(X)$ can be transformed into

$$\begin{cases} \dot{x} &=& ax^2 + bxy + cy^2 + O(3), \\ \dot{y} &=& \lambda_2 y + O(2). \end{cases}$$

  If $a \neq 0$ then $\dot{X} = f(X)$ is locally topologically equivalent near the origin to

$$\begin{cases} \dot{x} &=& ax^2, \\ \dot{y} &=& \lambda_2 y. \end{cases}$$

  Saddle-node ($a > 0$):



$\lambda_2 < 0$ $\qquad\qquad$ $\lambda_2 > 0$

- $\lambda_{1,2} = \pm i\omega,\ \omega > 0$

  By a linear diffeomorphism, $\dot{X} = f(X)$ can be transformed into

$$\begin{cases} \dot{x} &= -\omega y + R(x,y), \quad R = O(2), \\ \dot{y} &= \omega x + S(x,y), \quad S = O(2). \end{cases}$$

  Introduce $z = x + iy \in \mathbb{C}$. Then this system becomes

$$\dot{z} = i\omega z + g(z, \bar{z}),$$

  where

$$g(z,\bar{z}) = R\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iS\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right).$$

  Write its Taylor expansion in $z, \bar{z}$:

$$g(z,\bar{z}) = \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \ldots$$

Definition 5 *The* **first Lyapunov coefficient** *is*

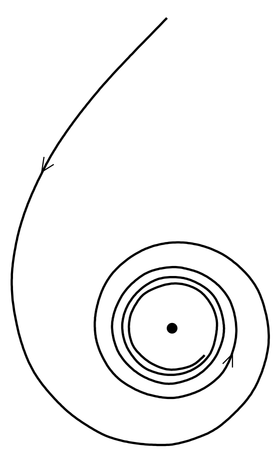
$$l_1 = \frac{1}{2\omega^2}\Re(ig_{20}g_{11} + \omega g_{21}).$$

If $l_1 \neq 0$ then $\dot{X} = f(X)$ is locally topologically equivalent near the origin to

$$\begin{cases} \dot{\rho} = l_1 \rho^3, \\ \dot{\varphi} = 1, \end{cases}$$

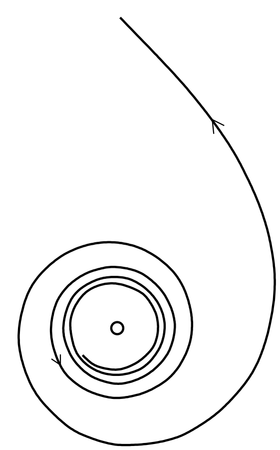
where (ρ, φ) are polar coordinates: $z = \rho e^{i\varphi}$.

Weak focus:



stable

$l_1 < 0$

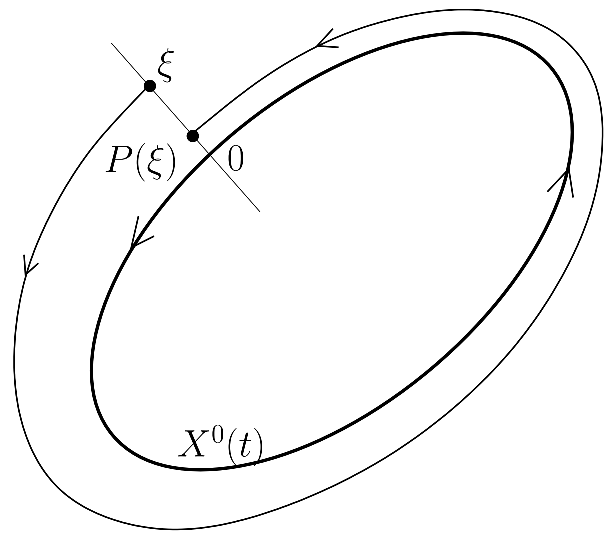


unstable

$l_1 > 0$

3. PERIODIC ORBITS AND LIMIT CYCLES

Poincaré map:



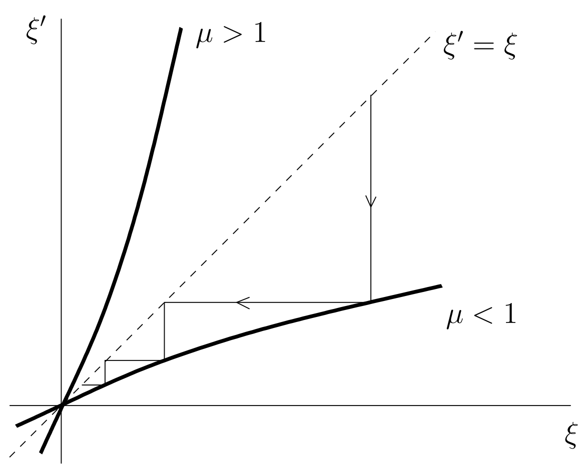
$$\xi \mapsto P(\xi) = \mu\xi + O(2),$$

where the **multiplier**

$$\mu = \exp\left(\int_0^T (\text{div } f)(X^0(t))\, dt\right) > 0$$

Definition 6 *A cycle of the planar system is* **hyperbolic** *if* $\mu \neq 1$.

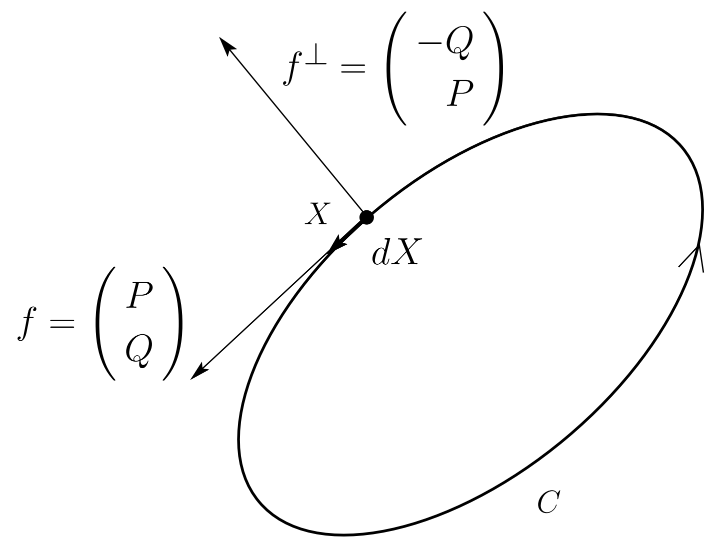
The cycle is stable if $\mu < 1$ and is unstable if $\mu > 1$.



Theorem 4 (Bendixson-Dulac)

If (div f)$(X) > 0$ (< 0) *in a disc* $D \in \mathbb{R}^2$*, then* $\dot{X} = f(X)$ *has no periodic orbits in* D*.*

Proof: Suppose, there is such a cycle C.



$$\oint_C P dy - Q dx = \oint_C \langle f^{\perp}, dX \rangle \equiv 0$$

but

$$\oint_C P dy - Q dx = \iint_D (\text{div } f) dX > 0 \ (< 0),$$

contradiction. $\square$

Implications:

1. If div$(gf) > 0$ (< 0) in a disc $D \subset \mathbb{R}^2$ for a smooth positive function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, then $\dot{X} = f(X)$ has **no** periodic orbits in D.

2. If div$(gf) > 0$ (< 0) is an annulus $A \subset \mathbb{R}^2$ for a smooth positive function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, then $\dot{X} = f(X)$ has **at most one** periodic orbit in A.

3. If $f(X) \neq 0$ and div$(gf) < 0$ in a trapping annulus $A \in \mathbb{R}^2$ for a smooth positive function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, then $\dot{X} = f(X)$ has a **unique** stable periodic orbit in A.

Example:

Consider

$$\begin{cases} \dot{x} &= y \equiv P(x,y), \\ \dot{y} &= ax + by + \alpha x^2 + \beta y^2 \equiv Q(x,y). \end{cases}$$

Define

$$g(x,y) = e^{-2\beta x} > 0$$

in $\mathbb{R}^2$. Then

$$\begin{aligned} \frac{\partial}{\partial x}(gP) + \frac{\partial}{\partial y}(gP) &= \frac{\partial}{\partial x}(e^{-2\beta x} y) + \frac{\partial}{\partial y}(e^{-2\beta x}(ax + by + \alpha x^2 + \beta y^2)) \\ &= -2\beta e^{-2\beta x} y + b e^{-2\beta x} + 2\beta e^{-2\beta x} y \\ &= b e^{-2\beta x} \neq 0 \end{aligned}$$

in $\mathbb{R}^2$ if $b \neq 0$. $\Rightarrow$ **no** periodic orbits.

Reversible systems

Definition 7 *A smooth system $\dot{X} = f(X)$ is called* **reversible** *if*

$$f(JX) = -Jf(X)$$

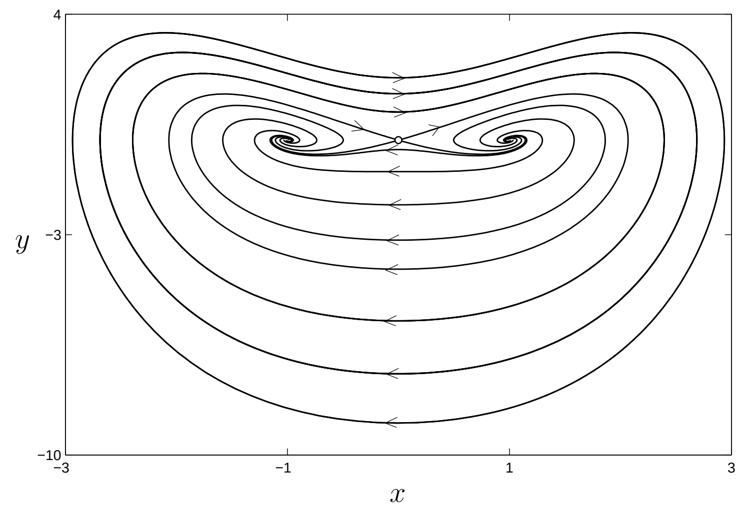
for a matrix J such that $J^2 = E$. The transformation $X \mapsto JX$ is called an **involution**.

If there is an orbit segment without equilibria connecting two points in the fixed subspace $\{Y : JY = Y\}$ of the involution, there is a periodic orbit of $\dot{X} = f(X)$. $\Rightarrow$ Periodic orbits occur in continuos familes.

Example:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= x + xy - x^3, \end{cases}$$

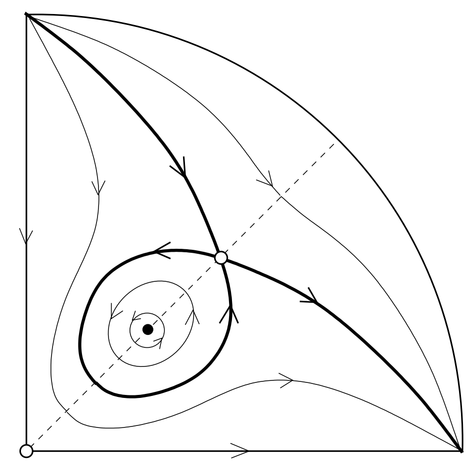
$$J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$



Example: A prey-predator model

$$\begin{cases} \dot{\xi} &=& \xi - \dfrac{\xi\eta}{(1+\alpha\xi)(1+\beta\eta)} &\equiv& f_1(\xi,\eta), \\ \dot{\eta} &=& -\eta + \dfrac{\xi\eta}{(1+\alpha\xi)(1+\beta\eta)} &\equiv& f_2(\xi,\eta), \end{cases}$$

where $\alpha, \beta > 0$ and $x, y \geq 0$.



- There is a family of closed orbits for $\alpha = \beta$ if

$$0 < \alpha = \beta < \frac{1}{4}.$$

  since the system is *reversible* with involution $J : (\xi, \eta) \mapsto (\eta, \xi)$.

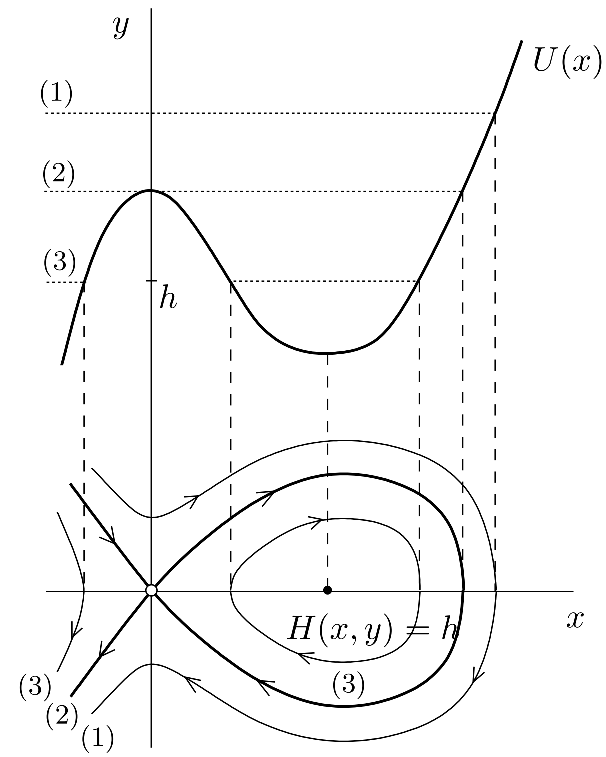
- There are no closed orbits if $\alpha \neq \beta$, since the choice

$$g(\xi,\eta) = \xi^a \eta^b (1+\alpha\xi)(1+\beta\eta),$$

  with appropriate a and b implies $\text{div}(gf) = (\alpha - \beta)\xi g$.

Planar Hamiltonian systems: $H : \mathbb{R}^2 \to \mathbb{R}$ **(smooth)**

$$\begin{cases} \dot{x} &= H_y(x,y), \\ \dot{y} &= -H_x(x,y). \end{cases} \quad \Rightarrow \quad \dot{H} = H_x\dot{x} + H_y\dot{y} \equiv 0 \quad \Rightarrow \quad H(x(t),y(t)) = h$$



Potential system:

$$H(x,y) = \frac{y^2}{2} + U(x)$$

(reversible: $y \mapsto -y, t \mapsto -t$).

$$T = \frac{dS}{dh}\Big|_{h=H_0}$$

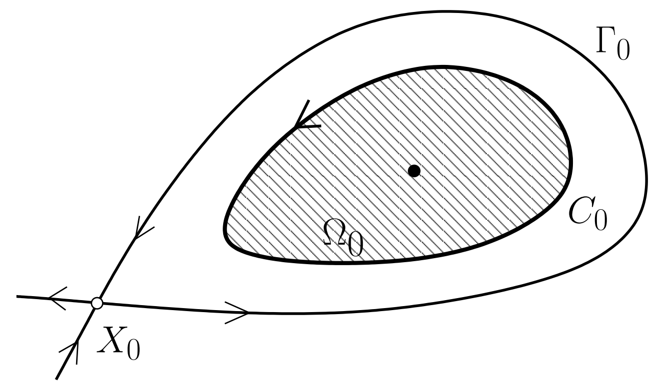
where

$$S(h) = \langle \text{area inside } \frac{y^2}{2} + U(x) = h \rangle$$

The Lotka-Volterra prey-predator model is orbitally equivalent to a Hamiltonian system.

Dissipative perturbations of 2D Hamiltonian systems

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon P(x,y), \\ \dot{y} = -H_x(x,y) + \varepsilon Q(x,y), \end{cases} \qquad F(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}.$$



Let $X^0(t)$ correspond to the T_0-periodic orbit C_0 at $\varepsilon = 0$ and let $\Omega_0 = \langle$domain inside $C_0\rangle$.

Theorem 5 (Pontryagin-Melnikov) *If*

$$\iint_{\Omega_0} \text{div } F(X)\ dX = 0 \quad \text{but} \quad \int_0^{T_0} \text{div } F(X^0(t))\ dt \neq 0$$

then there exists an annulus contaning C_0 in which the system has a unique periodic orbit C_ε for all sufficiently small ε, such that $C_\varepsilon \to C_0$ as $\varepsilon \to 0$.

Example: Van der Pol equation $\ddot{x} + x = \varepsilon\dot{x}(1 - x^2)$

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon y(1 - x^2), \end{cases}$$

For $\varepsilon = 0$, $H(x, y) = \frac{1}{2}(x^2 + y^2)$ and

$$X^0(t) = \begin{pmatrix} r \sin t \\ r \cos t \end{pmatrix}, \quad C_0 = \{(x, y) : x^2 + y^2 = r^2,\ r > 0\}$$

with $T_0 = 2\pi$.

$$F(x, y) = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix} = \begin{pmatrix} 0 \\ y(1 - x^2) \end{pmatrix}.$$

Then

$$\iint_{C_0} \text{div } F\, dxdy = -\oint_{C_0} Pdy - Qdx = \int_0^{2\pi} r^2 \cos^2 t(1 - r^2 \sin^2 t)dt = \frac{\pi}{4} r^2 (4 - r^2)$$

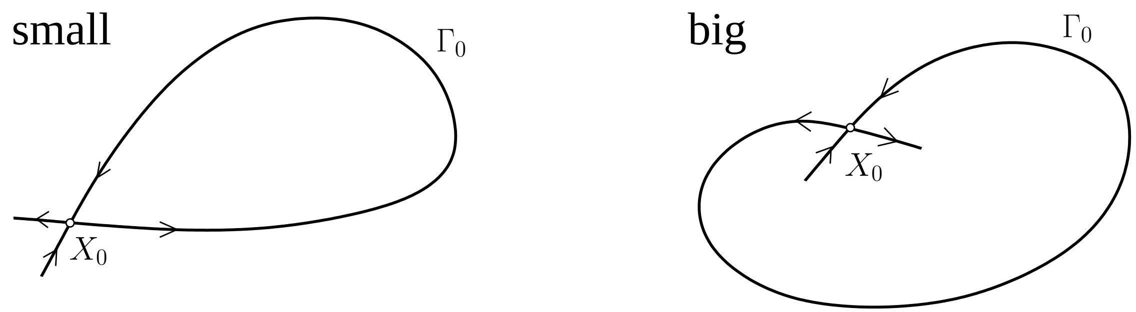
and

$$\int_0^{T_0} \text{div } F(X^0(t))\ dt = \int_0^{2\pi} (1 - 4\sin^2 t)dt = -2\pi$$

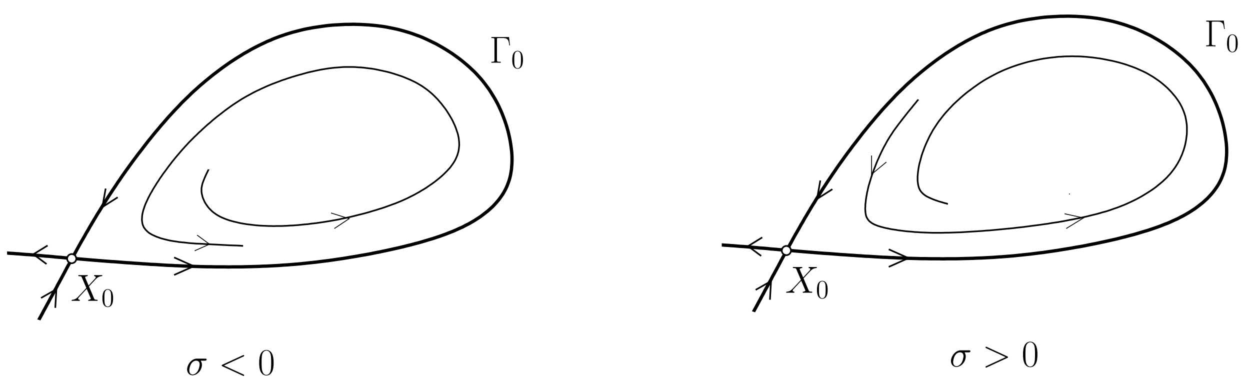
$\Rightarrow$ A **cycle** close to $r = 2$ exists for small $\varepsilon \neq 0$.

4. HOMOCLINIC ORBITS

Homoclinic orbits to saddles:



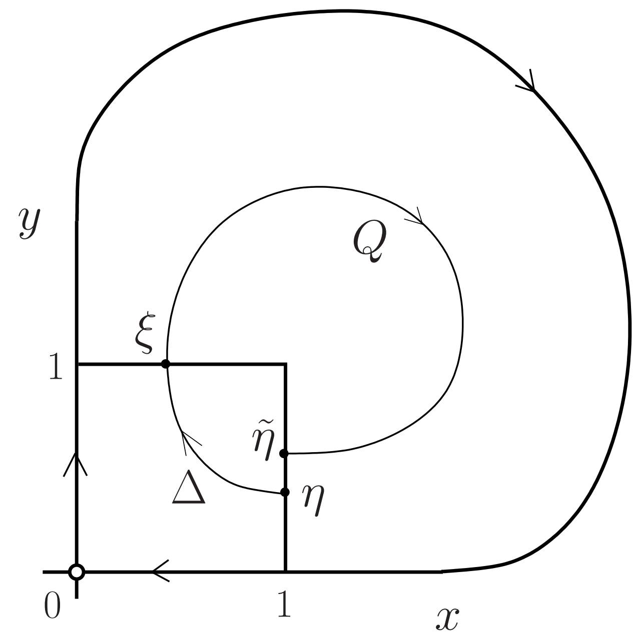
Definition 8 *The real number* $\sigma = \lambda_1 + \lambda_2 = (\text{div } f)(X_0)$ *is called the* **saddle quantity** *of* X_0.



Singular map:

$$\begin{cases} \dot{x} &=& \lambda_1 x \\ \dot{y} &=& \lambda_2 y \end{cases}$$

$$\xi = \Delta(\eta) = \eta^{-\frac{\lambda_1}{\lambda_2}}$$



Regular map:

$$\tilde{\eta} = Q(\xi) = A\xi + O(2), \quad A > 0.$$

Poincaré map:

$$\eta \mapsto \tilde{\eta} = Q(\Delta(\eta)) = A\eta^{-\frac{\lambda_1}{\lambda_2}} + \dots$$

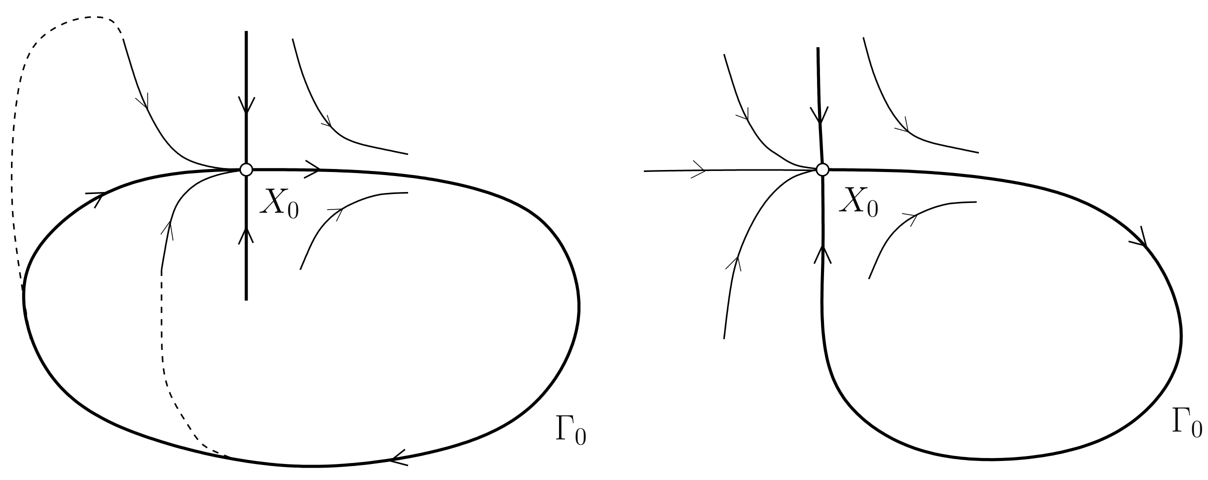
The homoclinic orbit is stable if $\sigma < 0$ and is unstable if $\sigma > 0$.

If $\sigma = \lambda_1 + \lambda_2 = 0$, then

if $\int_{-\infty}^{\infty} (\text{div } f)(X^0(t))\ dt < 0$ the homoclinic orbit is stable;

if $\int_{-\infty}^{\infty} (\text{div } f)(X^0(t))\ dt > 0$ the homoclinic orbit is unstable.

Homoclinic orbits to saddle-nodes:



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