BIFURCATION PHENOMENA

Lecture 3: Two-parameter bifurcations of planar ODEs

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Contents

Literature

1. Yu.A. Kuznetsov *Elements of Applied Bifurcation Theory*, 3rd ed. Applied Mathematical Sciences 112, Springer-Verlag, New York, 2004

2. L.P. Shilnikov, A.L. Shilnikov, D.V. Turaev, and L.O. Chua *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II, World Scientific, Singapore, 2001

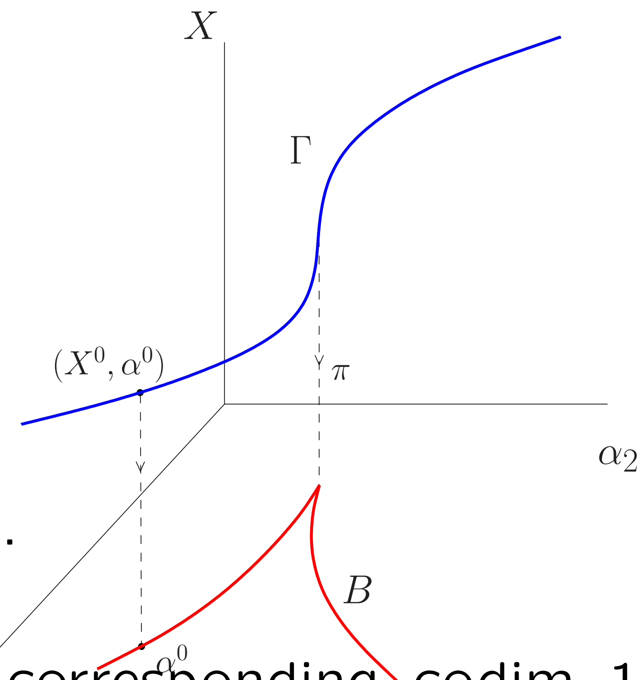
3. V.I. Arnol'd, V.S. Afraimovich, Yu.S. Il'yashenko, and L.P. Shil'nikov *Bifurcation theory*, In: V.I. Arnol'd (ed), Dynamical Systems V. Encyclopaedia of Mathematical Sciences, Springer-Verlag, New York, 1994

4. A.D. Bazykin *Nonlinear Dynamics of Interacting Populations*, World Scientific Publishing, River Edge, NJ, 1998

1. LOCAL CODIM 2 BIFURCATIONS

Consider a smooth 2D system depending on two parameters

$$\dot{X} = f(X, \alpha), \quad X \in \mathbb{R}^2, \ \alpha \in \mathbb{R}^2.$$



Curves of codim 1 bifurcations:

$$\text{Fold}: \begin{cases} f(X,\alpha) &= 0, \\ \det\ f_X(X,\alpha) &= 0. \end{cases}$$

$$\text{Hopf}: \begin{cases} f(X,\alpha) &= 0, \\ \text{Sp}\ f_X(X,\alpha) &= 0. \end{cases}$$

In both cases, we have 3=2+1 equations in $\mathbb{R}^4$.

When we cross $B = \pi\Gamma$ in the α-plane, the corresponding codim 1 bifurcation occurs.

One has to check that $\lambda_{1,2} = \pm i\omega$ along the Hopf curve.

Local codim 2 cases in the plane:

$$\text{Fold} : \lambda_1 = 0 \quad \begin{cases} \dot{x} &= ax^2 + O(3) \\ \dot{y} &= \lambda_2 y + O(2) \end{cases}$$

$$\text{Hopf} : \lambda_{1,2} = \pm i\omega \quad \begin{cases} \dot{\rho} &= l_1 \rho^3 + O(4) \\ \dot{\varphi} &= \omega + O(1) \end{cases}$$

(1) $\lambda_1 = 0,\ a = 0$

(2) $\lambda_1 = 0,\ \lambda_2 = 0$

(3) $\lambda_{1,2} = \pm i\omega,\ l_1 = 0$

To meet each case, we need to "tune" two parameters while following Γ (or B) $\Rightarrow$ codim 2.

Bogdanov–Takens bifurcation: $\lambda_1 = \lambda_2 = 0$

The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to

$$\begin{cases} \dot{x} &= y + \frac{1}{2}p_{20}x^2 + p_{11}xy + \frac{1}{2}p_{02}y^2 + O(3) \equiv P(x,y), \\ \dot{y} &= \frac{1}{2}q_{20} + \frac{1}{2}q_{02}y^2 + \frac{1}{6}q_{03}x^2 + O(3). \end{cases}$$

By a nonlinear local diffeomorphism (change of variables)

$$\begin{cases} \xi &= x, \\ \eta &= P(x,y), \end{cases}$$

this system can be reduced near the origin to

$$\begin{cases} \dot{\xi} &= \eta, \\ \dot{\eta} &= a\xi^2 + b\xi\eta + \dots, \end{cases}$$

where

$$a = \frac{1}{2}q_{20}, \quad b = p_{20} + q_{11}.$$

Bogdanov-Takens normal form

Theorem 1 *If* $ab \neq 0$, *then* $\dot{X} = f(X, \alpha)$ *is locally topologically equivalent near the BT-bifurcation to*

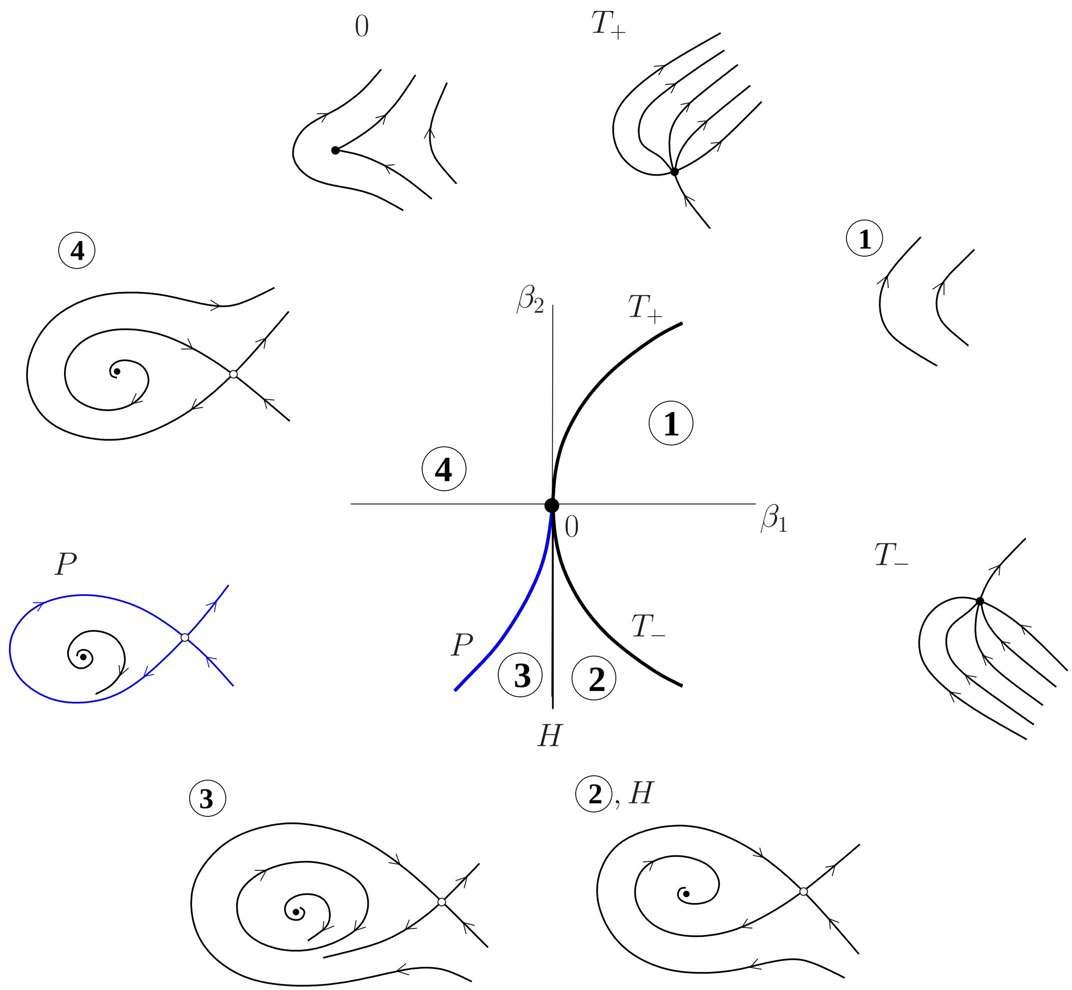
$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= \beta_1(\alpha) + \beta_2(\alpha)x + x^2 + sxy, \end{cases}$$

where $\beta_1(0) = \beta_2(0) = 0$ *and* $s = \mathrm{sign}(ab) = \pm 1$.

Bifurcation curves $(ab < 0)$:

- **fold** $T : \beta_1 = \frac{1}{4}\beta_2^2$
- **Andronov-Hopf** $H : \beta_1 = 0,\ \beta_2 < 0$
- **saddle homoclinic** $P : \beta_1 = -\frac{6}{25}\beta_2^2 + O(3),\ \beta_2 < 0$ (**global bifurcation**)

BT bifurcation diagram $(ab < 0)$



A unique limit cycle appears at Andronov-Hopf bifurcation curve H and disappears via the saddle homoclinic orbit at the curve P.

Bautin ("generalized Hopf") bifurcation: $\lambda_{1,2} = \pm i\omega,\ l_1 = 0$

The critical system $\dot{X} = f(X, 0)$ can be transformed by a linear diffeomorphism to the complex form

$$\dot{z} = i\omega z + \sum_{2 \le j+k \le 5} \frac{1}{j!k!} g_{jk} z^k \bar{z}^j + O(6),$$

which is locally smoothly equivalent to the **Poincaré normal form**

$$\dot{w} = i\omega w + c_1 w|w|^2 + c_2 w|w|^4 + O(6),$$

where the **Lyapunov coefficients**

$$l_j = \frac{1}{\omega} \Re(c_j)$$

satisfy

$$2l_1 = \frac{1}{\omega}\left(\Re(g_{21}) - \frac{1}{\omega}\Im(g_2 0 g_1 1)\right) \ \Rightarrow\ l_1 = \frac{1}{2\omega^2}\Re(i g_{20} g_{11} + \omega g_{21})$$

If $l_1 = 0$ then

$$
\begin{aligned}
12 l_2(0) &= \frac{1}{\omega}\text{Re } g_{32} \\
&+ \frac{1}{\omega^2}\text{Im } \left[g_{20}\bar{g}_{31} - g_{11}(4g_{31} + 3\bar{g}_{22}) - \frac{1}{3}g_{02}(g_{40} + \bar{g}_{13}) - g_{30}g_{12}\right] \\
&+ \frac{1}{\omega^3}\Big\{\text{Re } \Big[g_{20}(\bar{g}_{11}(3g_{12} - \bar{g}_{30}) + g_{02}\left(\bar{g}_{12} - \frac{1}{3}g_{30}\right) + \frac{1}{3}\bar{g}_{02}g_{03}) \\
&\qquad\qquad + g_{11}(\bar{g}_{02}\left(\frac{5}{3}\bar{g}_{30} + 3g_{12}\right) + \frac{1}{3}g_{02}\bar{g}_{03} - 4g_{11}g_{30})\Big] \\
&\qquad\qquad + 3\ \text{Im}(g_{20}g_{11})\ \text{Im } g_{21}\Big\} \\
&+ \frac{1}{\omega^4}\Big\{\text{Im}\left[g_{11}\bar{g}_{02}\left(\bar{g}_{20}^2 - 3\bar{g}_{20}g_{11} - 4g_{11}^2\right)\right] \\
&\qquad + \text{Im}(g_{20}g_{11})\left[3\ \text{Re}(g_{20}g_{11}) - 2|g_{02}|^2\right]\Big\}
\end{aligned}
$$

Bautin normal form

Theorem 2 *If* $l_2 \neq 0$ *and* $\omega \neq 0$*, then* $\dot{X} = f(X, \alpha)$ *is locally topologically equivalent near Bautin bifurcation to the normal form in the polar coordinates:*

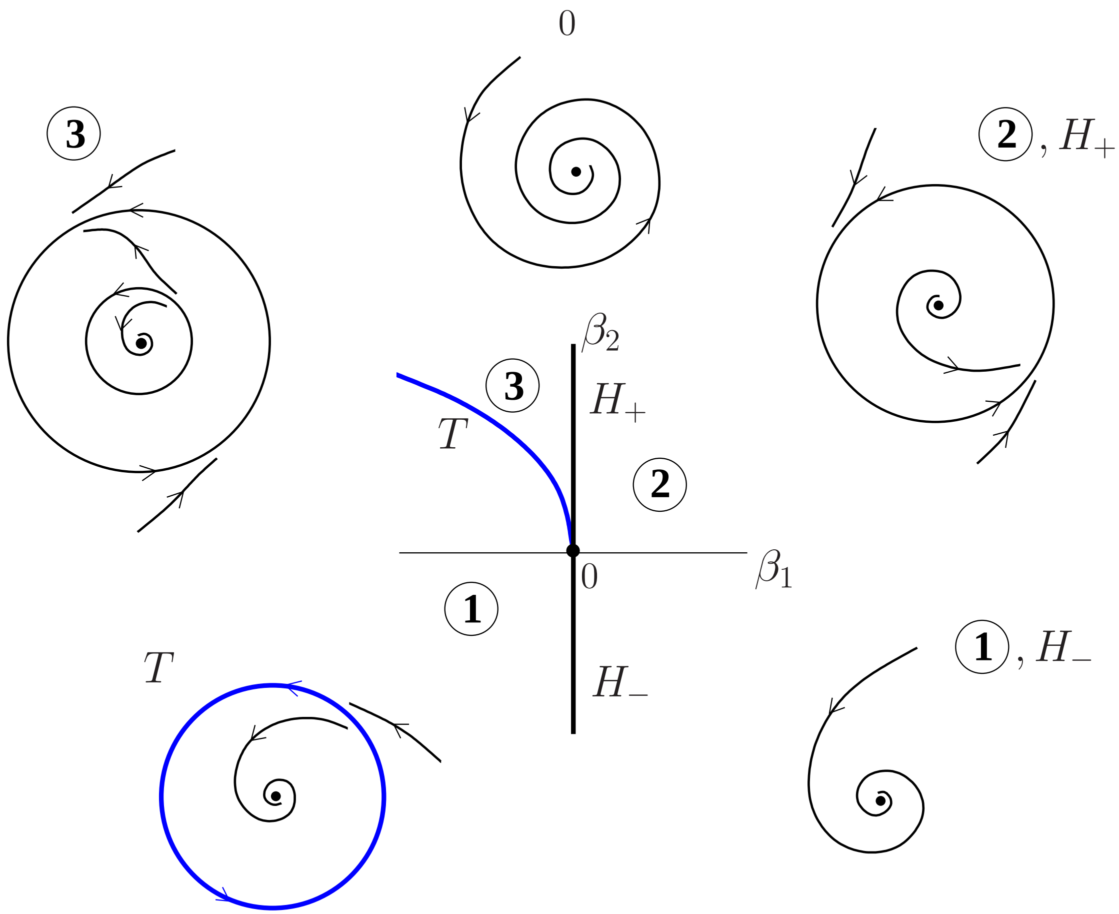
$$\begin{cases} \dot{\rho} & = & \rho(\beta_1(\alpha) + \beta_2(\alpha)\rho^2 + s\rho^4), \\ \dot{\varphi} & = & 1, \end{cases}$$

where $\beta_1(0) = \beta_2(0) = 0$ *and* $s = \text{sign}(l_2) = \pm 1$*.*

Bifurcation curves $(l_2 < 0)$:

- **superctitical Andronov-Hopf** $H^- : \beta_1 = 0,\ \beta_2 < 0$
- **subctitical Andronov-Hopf** $H^+ : \beta_1 = 0,\ \beta_2 > 0$
- **cyclic fold** $T_c : \beta_1 = \frac{1}{4}\beta_2^2,\ \beta_2 > 0$ (**global bifurcation**)

Bautin bifurcation diagram ($l_2 < 0$)



In the wedge between H^+ and T_c there exist two limit cycles born via different Andronov-Hopf bifurcations, which merge and disappear at the cyclic fold curve T_c.

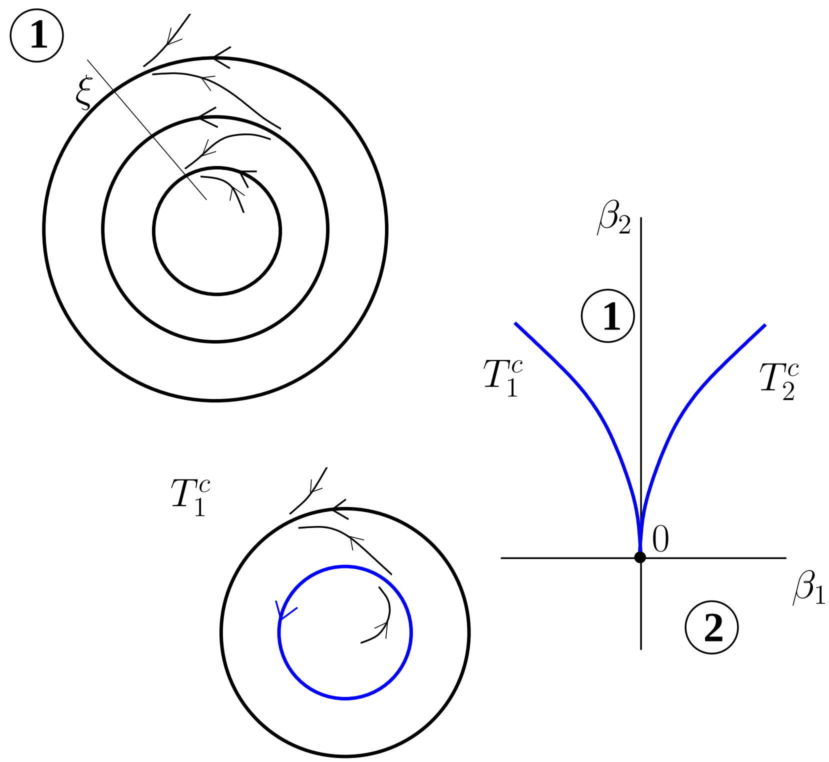
2. SOME GLOBAL BIFURCATIONS

- **Cyclic cusp** ($b = 0$): Critical Poincaré map $\xi \mapsto \xi + c\xi^3 + \ldots$.

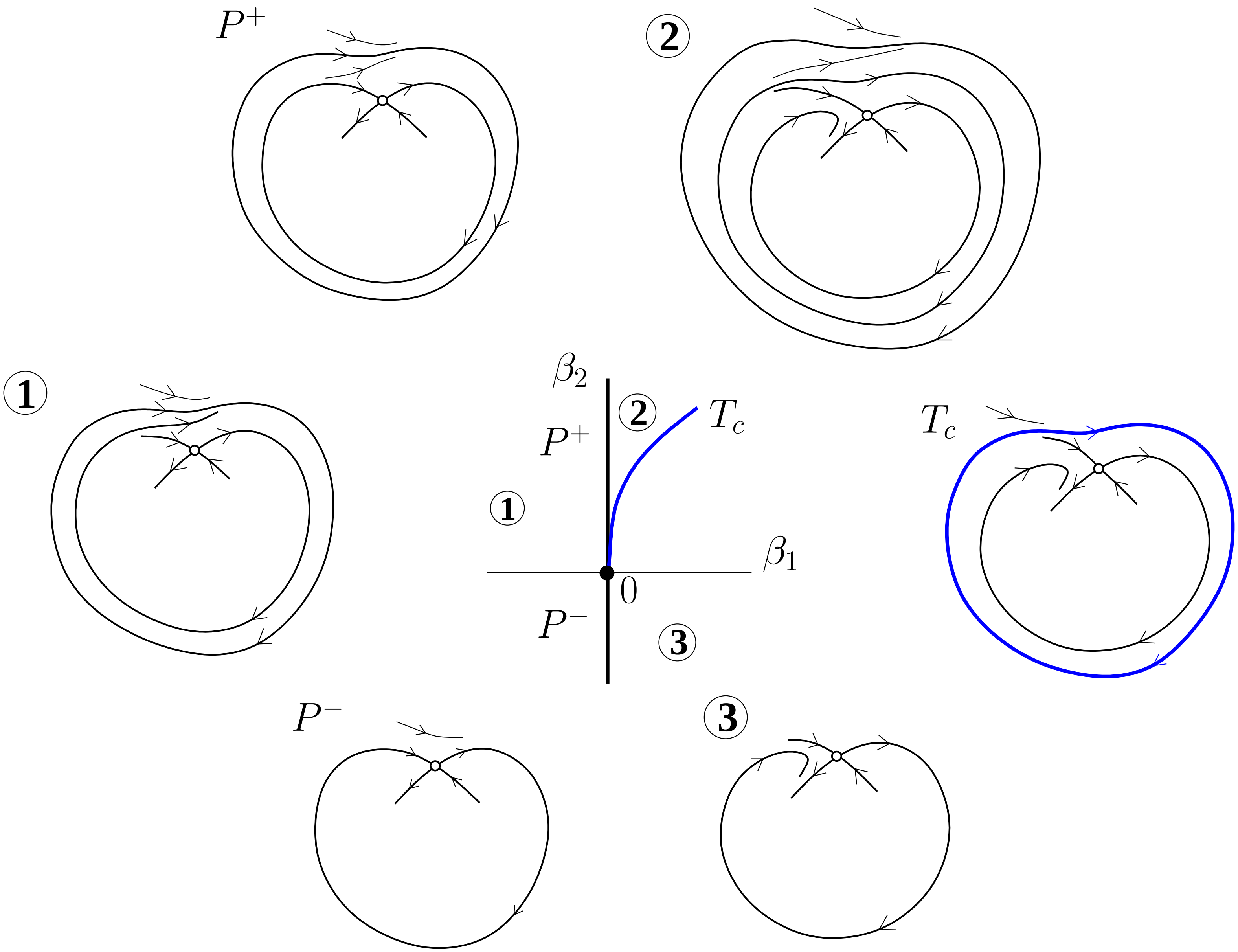
If $c \neq 0$ then the Poincaré map is locally topologically equivalent to

$$\xi \mapsto \beta_1(\alpha) + \beta_2(\alpha)\xi + \xi + s\xi^3,$$

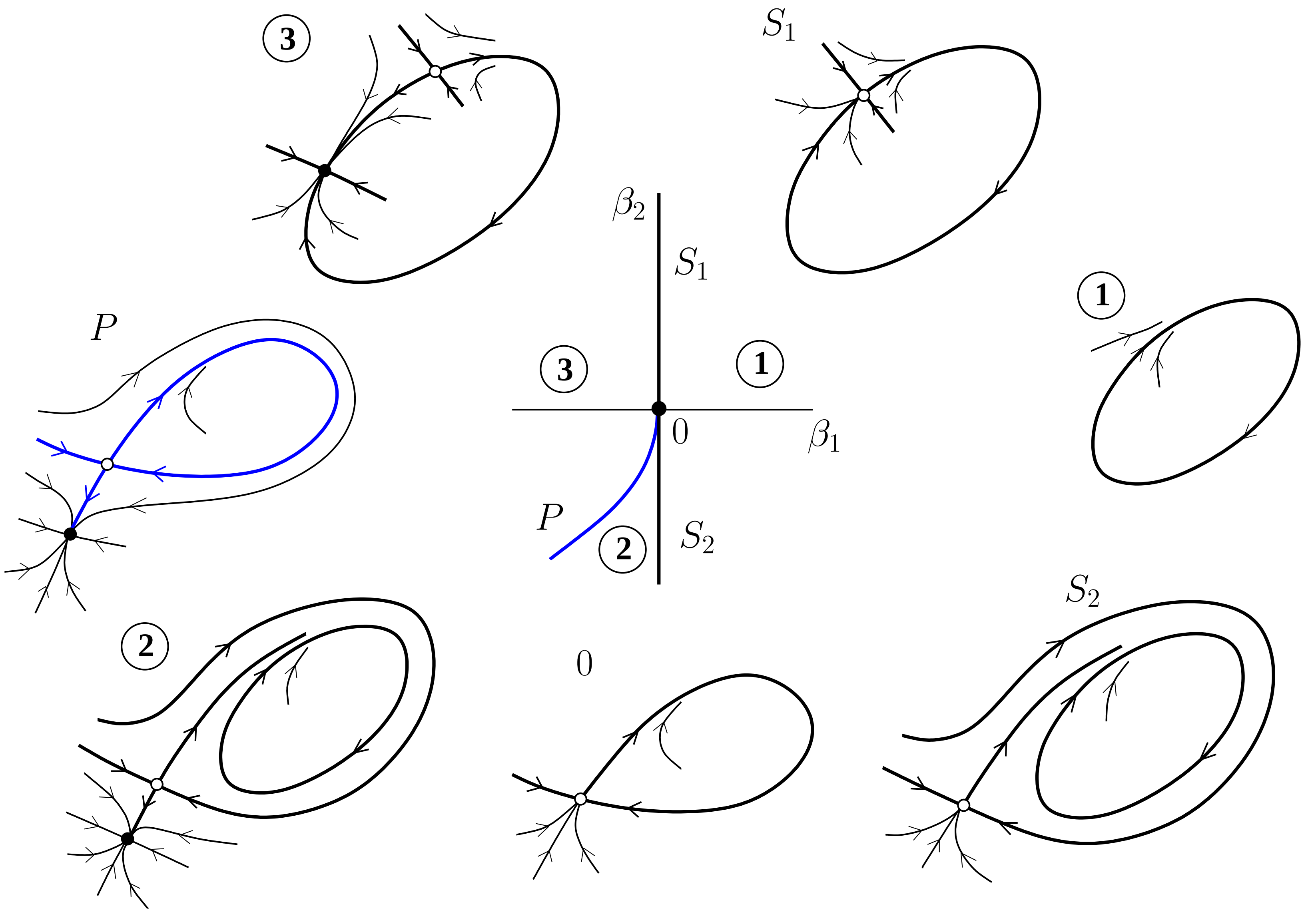
where $\beta_1(0) = \beta_2(0) = 0$ and $s = \text{sign}(c) = \pm 1$.



- **Neutral saddle homoclinic orbit:** $\int_{-\infty}^{\infty} (\text{div } f)(X^0(t))dt < 0$

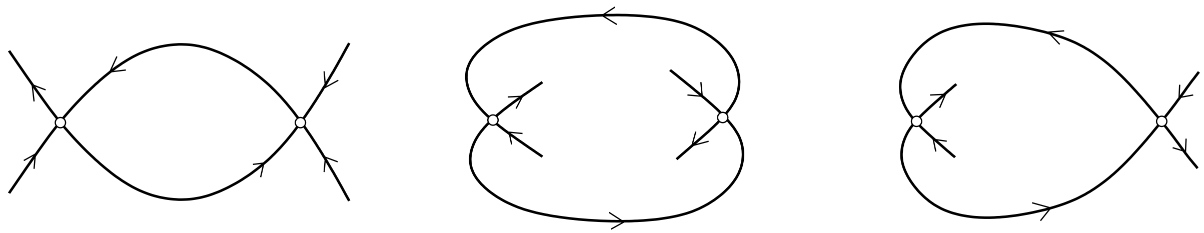


- **Non-central saddle-node homoclinic orbit**

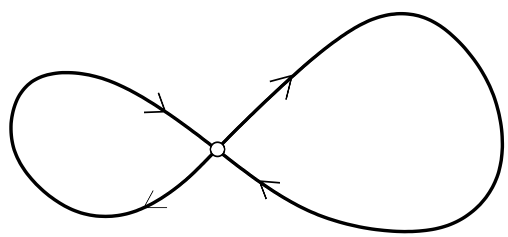


Other codim 2 cases:

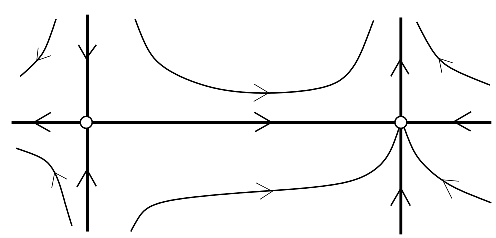
- Heteroclinic cycles



- "Figure of 8"

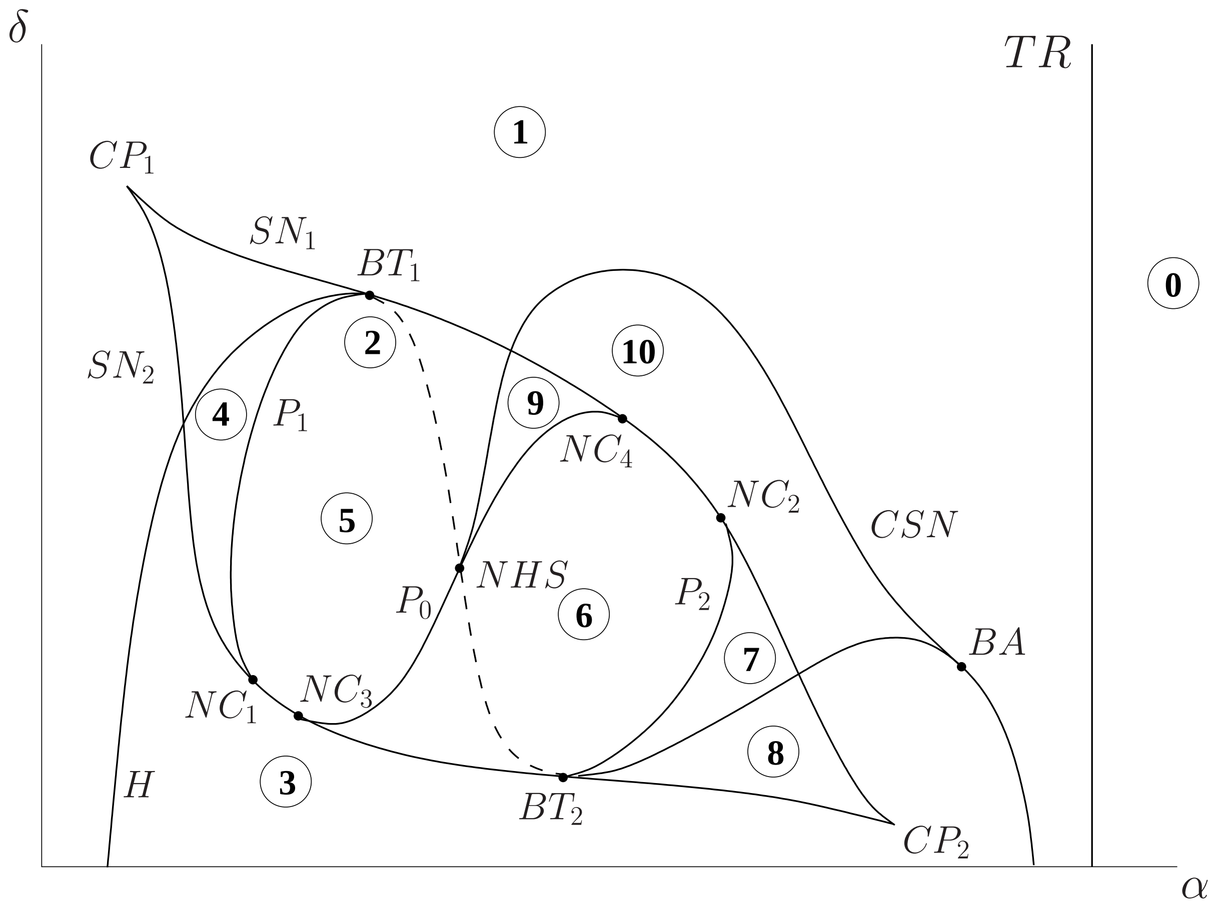


- Saddle-to-saddle-node heteroclinic orbits



Example: Bazykin's prey-predator model

$$\begin{cases} \dot{x}_1 &= x_1 - \dfrac{x_1 x_2}{1+\alpha x_1} - \varepsilon x_1^2, \\ \dot{x}_2 &= -\gamma x_2 + \dfrac{x_1 x_2}{1+\alpha x_1} - \delta x_2^2 \ . \end{cases}$$



Generic phase portraits: