UNIVERSITY OF TWENTE.

Local Bifurcations in Neural Field Equations

Yu.A. Kuznetsov (UT/UU)
Joint work with: S. van Gils, S. Janssens, and S. Visser

Reference:

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Overview

Neural field models as dynamical systems

Linearized problem and its spectral properties

Normalization on Center Manifold

Explicit example

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Neural field models as dynamical systems

Neural activity dynamics $V(t, \mathbf{r})$ in an open connected brain domain $\Omega \subset \mathbb{R}^3$ is modeled by

$$\frac{\partial V}{\partial t}(t, \mathbf{r}) = -\alpha V(t, \mathbf{r}) + \int_\Omega J(\mathbf{r}, \mathbf{r}') S(V(t - \tau(\mathbf{r}, \mathbf{r}'), \mathbf{r}'))\, d\mathbf{r}'$$

[Wilson & Cowan, 1972; Amari, 1977]

(H_J) The **connectivity kernel** $J \in C(\overline{\Omega} \times \overline{\Omega})$.

(H_S) The **synaptic activation function** $S \in C^\infty(\mathbb{R})$ and its kth derivative is bounded for every $k \in \mathbb{N}_0$.

(H_τ) The **delay function** $\tau \in C(\overline{\Omega} \times \overline{\Omega})$ is non-negative and not identically zero.

$$0 < h := \sup\{\tau(\mathbf{r}, \mathbf{r}') : \mathbf{r}, \mathbf{r}' \in \overline{\Omega}\} < \infty$$

Introduce

space	norm
$Y := C(\overline{\Omega})$	$\|y\| := \sup_{\mathbf{r}\in\Omega} \|y(\mathbf{r})\|$
$X := C([-h,0];Y)$	$\|\phi\| := \sup_{t\in[-h,0]} \|\phi(t,\cdot)\|$

Define the nonlinear operator $G : X \to Y$ by

$$G(\phi)(\mathbf{r}) := \int_{\overline{\Omega}} J(\mathbf{r},\mathbf{r}')S(\phi(-\tau(\mathbf{r},\mathbf{r}'),\mathbf{r}'))\,d\mathbf{r}' \quad \forall\,\phi \in X, \forall\,\mathbf{r} \in \overline{\Omega}$$

The operator $G : X \to Y$ is Fréchet differentiable with derivative $DG(\phi) \in \mathscr{L}(X,Y)$ in the point $\phi \in X$ given by

$$(DG(\phi)\psi)(\mathbf{r}) = \int_{\overline{\Omega}} J(\mathbf{r},\mathbf{r}')S'(\phi(-\tau(\mathbf{r},\mathbf{r}'),\mathbf{r}'))\psi(-\tau(\mathbf{r},\mathbf{r}'),\mathbf{r}')\,d\mathbf{r}'$$

for all $\psi \in X$ and all $\mathbf{r} \in \overline{\Omega}$.

The operator G is in $C^\infty(X,Y)$. For $k = 1,2,\dots$ its kth Fréchet derivative $D^k G(\phi) \in \mathscr{L}_k(X,Y)$ in the point $\phi \in X$ is given by

$$(D^k G(\phi)(\psi_1,\dots,\psi_k))(\mathbf{r}) = \int_{\overline{\Omega}} J(\mathbf{r},\mathbf{r}') S^{(k)}(\phi(-\tau(\mathbf{r},\mathbf{r}'),\mathbf{r}')) \prod_{i=1}^{k} \psi_i(-\tau(\mathbf{r},\mathbf{r}'),\mathbf{r}')\, d\mathbf{r}'$$

for $\psi_1,\dots,\psi_k \in X$ and $\mathbf{r} \in \overline{\Omega}$.

These derivatives will be used in the computations of the normal form coefficients.

Define the **history** at time $t \geq 0$ by

$$V_t(\theta) := V(t+\theta) \quad \forall t \geq 0,\, \theta \in [-h, 0]$$

Then studying of the neural field equation is equivalent to analyzing the following **Delay Differential Equation**:

$$\begin{cases} \dot{V}(t) = F(V_t) & t \geq 0 \\ V(t) = \phi(t) & t \in [-h, 0] \end{cases} \qquad \text{(DDE)}$$

where $V : [-h, \infty) \to Y$ and $\phi \in X$ is the initial condition, while $F : X \to Y$ is given by

$$F(\phi) := -\alpha\phi(0) + G(\phi) \qquad \forall \phi \in X$$

The operator F is globally Lipschitz continuous.

If $F \equiv 0$, then the solution semigroup corresponding to (DDE) is the **shift semigroup** T_0, defined as

$$(T_0(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) & -h \leq t+\theta \leq 0 \\ \phi(0) & 0 \leq t+\theta \end{cases}$$

for all $\phi \in X$, $t \geq 0$, $\theta \in [-h,0]$.

Let $X^{\odot} \subset X^*$ be the maximal subspace of strong continuity of T_0^*:

$$X^{\odot} = \overline{D(A_0^*)},$$

where A_0 is the infinitesimal generator of T_0. It holds

$$X^{\odot} = Y^* \times L^1([0,h];Y^*),$$

where the second factor is the space of Bochner integrable Y^*-valued functions on $[0,h]$ [Greiner & Van Neerven, 1992]

Let $T_0^{\odot}$ be the strongly continuous semigroup on $X^{\odot}$ obtained by restriction of T_0^* to $X^{\odot}$.

Its infinitesimal generator $A_0^{\odot}$ is

$$D(A_0^{\odot}) = \{\phi^{\odot} \in D(A_0^*) : A_0^*\phi^{\odot} \in X^{\odot}\}, \qquad A_0^{\odot}\phi^{\odot} = A_0^*\phi^{\odot}$$

Performing this construction once more, but now starting from the strongly continuous semigroup $T_0^{\odot}(t)$ on the Banach space $X^{\odot}$, we obtain the adjoint semigroup $T_0^{\odot\star}$ on the dual space $X^{\odot\star}$.

Inspired by Diekmann et al.[1995], we study the relationship between DDEs and Abstract Integral Equations.

The original space X is canonically embedded into $X^{\odot\star}$ via $j : X \to X^{\odot\star}$ given by

$$\langle \phi^{\odot}, j\phi \rangle := \langle \phi, \phi^{\odot} \rangle \quad \forall \phi \in X, \forall \phi^{\odot} \in X^{\odot}$$

Define $E : X \to X^{\odot\star}$ by

$$E(\phi) := (F(\phi), 0)$$

for all $\phi \in X$. Hence E maps into $Y \times \{0\}$ which is a closed subspace of $X^{\odot\star}$.

Consider the **Abstract Integral Equation**

$$u(t) = T_0(t)\phi + j^{-1}\left(\int_0^t T_0^{\odot\star}(t-s)E(u(s))\,ds\right) \quad \forall t \geq 0 \qquad \text{(AIE)}$$

where $\phi \in X$ is an initial condition, $u \in C([0,\infty);X)$ is the unknown and the convolution integral is of weak* Riemann type.
Then

(i) Suppose that $u \in C([0,\infty);X)$ satisfies (AIE). Define $x : [-h,\infty) \to Y$ by $x_0 = \phi$ and $x(t) = u(t)(0)$ for $t \geq 0$. Then x is a global solution of (DDE).

(ii) Conversely, suppose that x is a global solution of (DDE). Define $u : [0,\infty) \to X$ by $u(t) = x_t$. Then $u \in C([0,\infty);X)$ and u satisfies (AIE).

This implies that for any $\phi \in X$ problem (DDE) has a unique global solution.

Overview

Linearized problem and its spectral properties

Let $L := DG(\hat{\phi}) \in \mathscr{L}(X,Y)$ be the Fréchet derivative of G at the **equilibrium** $\hat{\phi} \in X$, i.e. $\hat{\phi}$ is independent of time (but possibly dependent on space) and

$$-\alpha\hat{\phi} + G(\hat{\phi}) = 0$$

The solution of the **linearized problem**

$$\begin{cases} \dot{x}(t) = -\alpha x(t) + Lx_t & t \geq 0 \\ x(t) = \phi(t) & t \in [-h, 0] \end{cases}$$

defines a strongly continuous semigroup T on X generated by $A : D(A) \subset X \to X$ where

$$D(A) = \{\phi \in X \,:\, \phi' \in X \text{ and } \phi'(0) = -\alpha\phi(0) + L\phi\}, \quad A\phi = \phi'$$

Since

$$D(A^*) = D(A_0^*),$$

the sun-duals of X with respect to T_0 and T are identical and may both be denoted by $X^{\odot}$. Moreover,

$$D(A^{\odot}) = \{\phi^{\odot} \in D(A^*) : A^*\phi^{\odot} \in X^{\odot}\}, \quad A^{\odot} = A^*$$

It also follows that if $\phi \in C^1([-h,0];Y)$ then $j\phi \in D(A^{\odot\star})$ and

$$A^{\odot\star}j\phi = (0,\phi') + (DF(\hat{\phi})\phi, 0).$$

Finally, all **spectra** coincide:

$$\sigma(A) = \sigma(A^*) = \sigma(A^{\odot}) = \sigma(A^{\odot\star})$$

For $f \in Y$ and $z \in \mathbb{C}$, let $(\varepsilon_z \otimes f) \in X$ be such that

$$(\varepsilon_z \otimes f)(\theta) = e^{\theta z} f$$

for $\theta \in [-h, 0]$. Define

$$\begin{aligned}
L_z &\in \mathscr{L}(Y), & L_z f &:= L(\varepsilon_z \otimes f) \\
H_z &\in \mathscr{L}(X), & (H_z \phi)(\theta) &:= \int_\theta^0 e^{z(\theta - s)} \phi(s)\, ds \\
S_z &\in \mathscr{L}(X, Y), & S_z \phi &:= \phi(0) + L H_z \phi
\end{aligned}$$

for all $f \in Y$, $\phi \in X$ and $\theta \in [-h, 0]$.

Introduce the **characteristic operator**

$$\Delta(z) := z + \alpha - L_z \in \mathscr{L}(Y)$$

Then $\phi \in \mathscr{R}(z - A)$ if and only if

$$\Delta(z)f = S_z\phi$$

has a solution $f \in Y$ and, moreover, $z \in \rho(A)$ if and only if f is also unique. If such is the case, then

$$R(z,A)\phi = (\varepsilon_z \otimes \Delta(z)^{-1}S_z\phi) + H_z\phi$$

Furthermore, $\lambda \in \sigma(A)$ if and only if $0 \in \sigma(\Delta(\lambda))$ and $\psi \in D(A)$ is an eigenvector corresponding to λ if and only if $\psi = \varepsilon_\lambda \otimes q$ where $q \in Y$ satisfies $\Delta(\lambda)q = 0$ [Engel & Nagel, 2000].

The set $\sigma(A) \setminus \{-\alpha\}$ consists of isolated eigenvalues of finite type.

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Normalization on Center Manifold

Assume $\hat{\phi} \equiv 0$. Suppose that A has $n_c \geq 1$ critical eigenvalues with $\operatorname{Re} \lambda = 0$. This implies the existence of an invariant **center manifold** $\mathscr{W}^c_{loc}$ on which

$$\dot{u}(t) = j^{-1}\left(A^{\odot\star} j u(t) + R(u(t))\right) \qquad \forall t \in \mathbb{R}$$

where

$$R(\phi) = E(\phi) - DE(0)\phi = \frac{1}{2}B(\phi,\phi) + \frac{1}{3!}C(\phi,\phi,\phi) + O(\|\phi\|^4)$$

The projection of $u(t)$ onto $X_0 = T_{\hat{\phi}}\mathscr{W}^c_{loc}$ satisfies the **normal form** in some coordinates $z \in \mathbb{R}^{n_c}$ on X_0,

$$\dot{z}(t) = \sum_{1 \leq |\nu| \leq 3} g_\nu z^\nu(t) + O(|z(t)|^4) \qquad \forall t \in \mathbb{R}$$

The invariance of $\mathscr{W}^c_{loc}$, which is given by the graph of

$$\mathscr{H}(z) = \sum_{1 \leq |\nu| \leq 3} \frac{1}{\nu!} h_\nu z^\nu + O(|z|^4),$$

implies the **homological equation** [Iooss & Adelmeyer, 1992]

$$A^{\odot\star} j\mathscr{H}(z) + R(\mathscr{H}(z)) = j(D\mathscr{H}(z)\dot{z})$$

Collecing the z^ν-terms leads to a linear equation

$$(\lambda - A^{\odot\star})\phi^{\odot\star} = \psi^{\odot\star}$$

Let $\lambda \in \mathbb{C} \setminus \{-\alpha\}$. Then this equation is solvable for $\phi^{\odot\star} \in D(A^{\odot\star})$ given $\psi^{\odot\star} \in X^{\odot\star}$ if and only if $\langle \phi^{\odot}, \psi^{\odot\star} \rangle = 0$ for all $\phi^{\odot} \in \mathscr{N}(\lambda - A^*)$ (**Fredholm Solvability**)

Andronov-Hopf Bifurcation $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$

Let ϕ and $\phi^{\odot}$ be complex eigenvectors of A and A^* corresponding to $\lambda_1 = i\omega_0$,

$$A\phi = i\omega_0\phi, \;\; A^*\phi^{\odot} = i\omega_0\phi^{\odot},$$

and satisfying $\langle \phi, \phi^{\odot} \rangle = 1$.

- Poincaré normal form [Arnold, 1972]:

$$\dot{z} = i\omega_0 z + g_{21} z|z|^2 + O(|z|^4), \;\; z \in \mathbb{C}$$

where $l_1 = \frac{1}{2\omega_0}\mathrm{Re}\, g_{21}$ is the **first Lyapunov coefficient**.

- Center manifold expansion:

$$\mathscr{H}(z,\bar{z}) = z\phi + \bar{z}\bar{\phi} + \sum_{2\leq j+k\leq 3} \frac{1}{j!k!} h_{jk} z^j \bar{z}^k + O(|z|^4)$$

The homological equation

$$A^{\odot\star} j \mathscr{H}(z,\bar{z}) + R(\mathscr{H}(z,\bar{z})) = j\left(D_z \mathscr{H}(z,\bar{z})\dot{z} + D_{\bar{z}} \mathscr{H}(z,\bar{z})\dot{\bar{z}}\right)$$

gives

$$jh_{20} = -(A^{\odot\star})^{-1} B(\phi,\overline{\phi})$$
$$jh_{11} = (2i\omega_0 - A^{\odot\star})^{-1} B(\phi,\phi)$$

as well as

$$(i\omega_0 I - A^{\odot\star}) jh_{21} = C(\phi,\phi,\overline{\phi}) + B(\overline{\phi},h_{20}) + 2B(\phi,h_{11}) - 2g_{21} j\phi$$

so that

$$g_{21} = \frac{1}{2}\langle \phi^{\odot}, C(\phi,\phi,\overline{\phi}) + B(\overline{\phi},h_{20}) + 2B(\phi,h_{11})\rangle$$

Evaluation of normal form coefficients

To compute $\psi^{\odot\star} = R(z, A^{\odot\star})(y, 0)$, we need to solve

$$(z - A^{\odot\star})\psi^{\odot\star} = (y, 0)$$

where $z \in \rho(A)$, $y \in Y$ and $\psi^{\odot\star} \in D(A^{\odot\star})$.

For each $y \in Y$ the function $\psi = \varepsilon_z \otimes \Delta(z)^{-1} y$ is the unique solution in $C^1([-h, 0]; Y)$ of the system

$$\begin{cases} z\psi(0) - DF(0)\psi = y \\ z\psi - \psi' = 0 \end{cases}$$

Then $\psi^{\odot\star} = j\psi$.

Let $P^{\odot}$ and $P^{\odot\star}$ be the spectral projections on $X^{\odot}$ and $X^{\odot\star}$ corresponding to a simple $\lambda \in \sigma(A)$. We want to evaluate $\langle \phi^{\odot}, \phi^{\odot\star} \rangle$ where $\phi^{\odot\star} = (y,0) \in Y \times \{0\} \subset X^{\odot\star}$.
Since the range of $P^{\odot\star}$ is spanned by $j\phi$ we have

$$P^{\odot\star}\phi^{\odot\star} = \kappa j\phi$$

for a certain $\kappa \in \mathbb{C}$. Furthermore,

$$\langle \phi^{\odot}, \phi^{\odot\star} \rangle = \langle P^{\odot}\phi^{\odot}, \phi^{\odot\star} \rangle = \langle \phi^{\odot}, P^{\odot\star}\phi^{\odot\star} \rangle = \kappa \langle \phi^{\odot}, j\phi \rangle = \kappa$$

Thus [Dunford & Schwartz, 1958]

$$P^{\odot\star}\phi^{\odot\star} = \frac{1}{2\pi i} \oint_{\partial C_\lambda} R(z, A^{\odot\star})\phi^{\odot\star}\, dz = \kappa j\phi$$

and the first component shows that κ can be found from

$$\frac{1}{2\pi i} \oint_{\partial C_\lambda} \Delta(z)^{-1} y\, dz = \kappa \phi(0)$$

Overview

Explicit example

Consider a homogeneous neural field with transmission delays due to a finite propagation speed of action potentials as well as a finite, fixed delay $\tau_0 \geq 0$ caused by synaptic processes. Space and time are each rescaled such that $\overline{\Omega} = [-1, 1]$ and the propagation speed is 1. This yields

$$\tau(r, r') = \tau_0 + |r - r'| \qquad \forall\, r, r' \in \overline{\Omega}$$

For the connectivity function we take a linear combination of $N \geq 1$ exponentials,

$$J(r, r') = \sum_{i=1}^{N} c_i e^{-\mu_i |r - r'|} \qquad \forall\, r, r' \in \overline{\Omega}$$

where

$$c_i \in \mathbb{C} \text{ with } c_i \neq 0, \qquad \mu_i \in \mathbb{C} \text{ with } \mu_i \neq \mu_j \text{ for } i \neq j$$

We study the stability of a spatially homogeneous steady state $\hat{\phi} \equiv 0$, requiring that $S(0) = 0$ and $S'(0) = 1$.
Let

$$k_i(\lambda) := \lambda + \mu_i \qquad \forall i = 1, \dots, N$$

and

$$\mathscr{S} := \{\lambda \in \mathbb{C} : \exists i,j \in \{1, \dots, N\}, i \neq j, \text{ s.t. } k_i^2(\lambda) = k_j^2(\lambda)\}.$$

Define for $\lambda \notin \mathscr{S}$ the **characteristic polynomial**

$$\mathscr{P}(\rho) := \frac{e^{\lambda \tau_0}(\lambda + \alpha)}{2} \prod_{j=1}^{N} (\rho^2 - k_j(\lambda)^2) + \sum_{i=1}^{N} c_i k_i(\lambda) \prod_{\substack{j=1 \\ j \neq i}}^{N} (\rho^2 - k_j(\lambda)^2)$$

and assume that it has $2N$ distinct roots, denoted by $\pm \rho_i(\lambda)$ for $i = 1, 2, \dots, N$, and such that

$$k_j(\lambda) \neq \pm \rho_i(\lambda) \qquad \forall i,j = 1, 2, \dots, N$$

Under above conditions, introduce

$$S(\lambda) := \begin{bmatrix} S_\lambda^- & S_\lambda^+ \\ S_\lambda^+ & S_\lambda^- \end{bmatrix}$$

where

$$[S_\lambda^-]_{j,i} := \frac{e^{\rho_i(\lambda)}}{\lambda + \mu_j - \rho_i(\lambda)}, \qquad [S_\lambda^+]_{j,i} := \frac{e^{-\rho_i(\lambda)}}{\lambda + \mu_j + \rho_i(\lambda)}$$

Then λ is an eigenvalue of A if and only if $\det S(\lambda) = 0$. The corresponding eigenfunction is $\varepsilon_\lambda \otimes q_\lambda$ with

$$q_\lambda(x) = \sum_{i=1}^{N} \left[\gamma_i e^{\rho_i(\lambda)x} + \gamma_{-i} e^{-\rho_i(\lambda)x}\right] \qquad \forall x \in [-1, 1]$$

where $\Gamma = [\gamma_1, \gamma_2, \ldots, \gamma_N, \gamma_{-1}, \gamma_{-2}, \ldots, \gamma_{-N}]$ is a solution to $S(\lambda)\Gamma = 0$.

Suppose that $z \in \rho(A)$ and the above conditions hold. Then the solution to the resolvent equation

$$(z-A)\psi = \phi, \quad \phi \in X,$$

is given by $\psi_z = \varepsilon_z \otimes q_z + H_z\phi$ with

$$q_z(x) = \frac{h_z(x)}{z+\alpha} + \sum_{i=1}^{N} \left[\gamma_{i,z}(x)e^{\rho_i(z)x} + \gamma_{-i,z}(x)e^{-\rho_i(z)x}\right] \qquad \forall x \in [-1,1]$$

where

$$h_z(x) := \phi(0,x) + \int_{-1}^{1}\int_{-\tau_0-|x-r|}^{0} J(x,r)e^{-z(\tau_0+s)-z|x-r|}\phi(s,r)\,ds\,dr$$

and

$$\Gamma_z = [\gamma_{1,z}, \gamma_{2,z}, \ldots, \gamma_{N,z}, \gamma_{-1,z}, \gamma_{-2,z}, \ldots, \gamma_{-N,z}]$$

can be found by matrix inversion and integration.

Numerical Example

Take

$$J(r,r') = \hat{c}_1 e^{-\mu_1|r-r'|} + \hat{c}_2 e^{-\mu_2|r-r'|} \qquad \forall r,r' \in [-1,1]$$

and

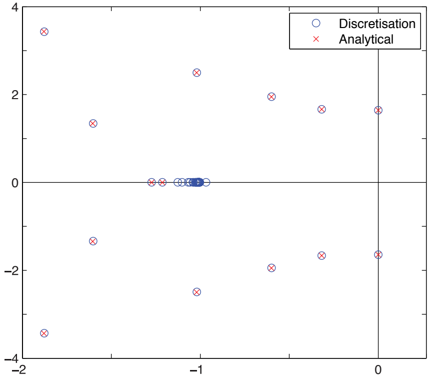
$$S(V) = \frac{1}{1+e^{-rV}} - \frac{1}{2} \qquad \forall V \in \mathbb{R}$$

Here $S'(0) = \frac{r}{4}$ and hence $c_i = \frac{r}{4}\hat{c}_i$ for $i = 1,2$.
Let

$$\alpha = 1,\ \tau_0 = 1,\ \hat{c}_1 = 3,\ \hat{c}_2 = -5.5,\ \mu_1 = 0.5,\ \mu_2 = 1.$$

Simulations can be performed by reduction to a classical DDE via the finite-difference approximation and its numerical integration [Faye & Faugeras, 2010].

Hopf bifurcation at $r = 4.220215$



The approximate eigenvalues were computed with DDE-BIFTOOL [Engelborghs et al., 2002].

$$\phi(t,x) = e^{\lambda t}\left[\gamma_1(e^{\rho_1 x} + e^{-\rho_1 x}) + \gamma_2(e^{\rho_2 x} + e^{-\rho_2 x})\right] \qquad \forall t \in [-h, 0]$$

where

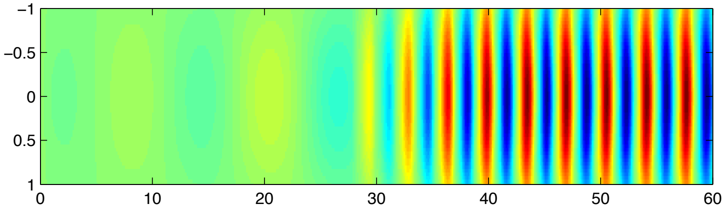
$$\rho_1 = 0.321607348361597 - 0.880461478656249i,$$
$$\rho_2 = 0.110838003673357 - 2.312123026384049i$$
$$\gamma_1 = -0.191821747840362 - 0.172140605861736i,$$
$$\gamma_2 = -0.080160108888561$$

corresponding to $\lambda = i\omega_0 = 1.644003102046893i$.

$$g_{21} = \frac{1}{2}\langle \phi^{\odot}, D^3E(0)(\phi, \phi, \overline{\phi})\rangle \approx -0.326 + 0.0389i$$

Thus, the first Lyapunov coefficient is $l_1 \approx -0.198 < 0$ indicating a **supercritical** Hopf bifurcation.

A transient to periodic inhomogeneous oscillations at $r = 6$:



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- The **double Hopf** case has been treated as well. We have observed multiple stable periodic inhomogeneous oscillations.
- The theory covers bifurcations from inhomogeneous equilibria, although we do not have relevant numerical examples yet.
- Unbounded domains ???
- Arbitrary Banach space Y [Thursday's talk by S. Janssens].