# Initialization and Continuation of Homoclinic Orbits to Equilibria in MATLAB

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July 7, 2010

## Contents

## 1. Continuation of codim 1 homoclinic orbits


- Doedel, E.J. and Friedman, M.J. 1989. Numerical computation of heteroclinic orbits. *J. Comput. Appl. Math.* **26**, 1-2, 155-170.

- Beyn, W.J. 1990. The numerical computation of connecting orbits in dynamical systems. *IMA J. Numer. Anal.* **10**, 3, 379-405.

- Champneys, A.R., Kuznetsov, Yu.A., and Sandstede, B. 1996. A numerical toolbox for homoclinic bifurcation analysis. *Int. J. Bifurcation Chaos* **6**, 5, 867-887.

## Homoclinic orbits

- Consider a **family of smooth ODEs**

$$\dot{x} = f(x,\alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,$$

  having a hyperbolic equilibrium $x_0$ with eigenvalues

$$\Re(\mu_{n_S}) \leq \ldots \leq \Re(\mu_1) < 0 < \Re(\lambda_1) \leq \ldots \leq \Re(\lambda_{n_U})$$

  of $A(x_0,\alpha) = f_x(x_0,\alpha)$.

- **Homoclinic orbit** $\Gamma = W^S(x_0) \cap W^U(x_0)$ has codim 1.

- **Homoclinic solution problem**:

$$\begin{cases} f(x_0,\alpha) &= 0, \\ \dot{x}(t) - f(x(t),\alpha) &= 0, \\ \lim\limits_{t\to\pm\infty} x(t) - x_0 &= 0, \ t \in \mathbb{R}, \\ \int_{-\infty}^{\infty} \dot{\tilde{x}}(t)^{\mathsf{T}}(x(t) - \tilde{x}(t))dt &= 0, \end{cases}$$

  where $\tilde{x}$ is a reference solution.

## Defining BVP

- Truncate with the **projection boundary conditions**:

$$
\begin{cases}
f(x_0,\alpha) & = & 0, \\
\dot{x}(t) - f(x(t),\alpha) & = & 0, \; t \in [-T,T] \\
\langle x(-T) - x_0, q_{0,n_U+i}\rangle & = & 0, \; i = 1,2,\ldots,n_S \\
\langle x(+T) - x_0, q_{1,n_S+i}\rangle & = & 0, \; i = 1,2,\ldots,n_U \\
\int_{-T}^{T} \dot{\tilde{x}}(t)^{\mathsf{T}}(x(t)-y(t))dt & = & 0,
\end{cases}
$$

  where the columns of

$$Q^{U^\perp} = [q_{0,n_U+1},\ldots,q_{0,n_U+n_S}] \quad \text{and} \quad Q^{U^\perp} = [q_{1,n_S+1},\ldots,q_{1,n_S+n_U}]$$

  span the orthogonal complements to $T_{x_0}W^U(x_0)$ and $T_{x_0}W^S(x_0)$, resp.

- **Theorem** [Beyn] *There is a locally unique solution to the truncated BVP for a regular homoclinic orbit with the* $(x(\cdot),\alpha)$*-error that is* $O(e^{-2\omega T})$*, where* $\omega = \min(|\mu|,\lambda)$ *and* $(\mu,\lambda)$ *satisfy* $\Re(\mu_1) < \mu < 0 < \lambda < \Re(\lambda_1)$*.*

## 2. Continuation of invariant subspaces


- Dieci, L., and Eirola, T. 1999. On smooth decompositions of matrices. *SIAM J. Matrix Anal. Appl.* **20**, 3, 800-819.

- Dieci, L., and Friedman, M.J. 2001. Continuation of invariant subspaces. *Numer. Linear Algebra Appl.* **8**, 317-327.

- Demmel, J.W., Dieci, L., and Friedman, M.J. 2001. Computing connecting orbits via an improved algorithm for continuing invariant subspaces. *SIAM J. Sci. Comput.* **22**, 1, 81-94.

- Bindel, D., Demmel, J., and Friedman, M. 2003. Continuation of invariant subspaces for large bifurcation problems. *SIAM J. Sci. Comput.* **30**, 2, 637-656.

## Smooth Schur Block Factorization

**Theorem** *Any paramter-dependent matrix* $A(s) \in \mathbb{R}^{n\times n}$ *can be written as*

$$A(s) = Q(s) \begin{bmatrix} R_{11}(s) & R_{12}(s) \\ 0 & R_{22}(s) \end{bmatrix} Q^{\mathsf{T}}(s),$$

*where* $Q(s) = [Q_1(s) \quad Q_2(s)]$ *such that*

- $Q(s)$ *is orthogonal, i.e.* $Q^{\mathsf{T}}(s)Q(s) = I_n$*;*
- *the columns of* $Q_1(s) \in \mathbb{R}^{n\times m}$ *span an eigenspace* $\mathcal{E}(s)$ *of* $A(s)$ *corresponding to its* $m$ *selected eigenvalues;*
- *the columns of* $Q_2(s) \in \mathbb{R}^{n\times(n-m)}$ *span the orthogonal complement* $\mathcal{E}^{\perp}(s)$*.*
- *the eigenvalues of* $R_{11}(s) \in \mathbb{R}^{m\times m}$ *are the selected* $m$ *eigenvalues of* $A(s)$*, while the eigenvalues of* $R_{22}(s) \in \mathbb{R}^{(n-m)\times(n-m)}$ *are the remaning* $(n-m)$ *eigenvalues of* $A(s)$*;*
- $Q_i(s)$ *and* $R_{ij}(s)$ *have the same smoothness as* $A(s)$*.*

Then holds the **invariant subspace relation**:

$$Q_2^{\mathsf{T}}(s)A(s)Q_1(s) = 0.$$

**CIS-algorithm [Dieci & Friedman, 2001]**

- Define

$$\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix} = Q^{\top}(0)A(s)Q(0)$$

for small $|s|$, where $T_{11}(s) \in \mathbb{R}^{m\times m}$.

- Compute $Y \in \mathbb{R}^{(n-m)\times m}$ satisfying the **Riccati matrix equation**

$$YT_{11}(s) - T_{22}(s)Y + YT_{12}(s)Y = T_{21}(s).$$

- Then $Q(s) = Q(0)U(s)$ where

$$U(s) = [U_1(s) \quad U_2(s)]$$

with

$$U_1(s) = \begin{pmatrix} I_m \\ Y \end{pmatrix} (I_{n-m}+Y^{\top}Y)^{-\frac{1}{2}}, \ \ U_2(s) = \begin{pmatrix} -Y^{\top} \\ I_{n-m} \end{pmatrix} (I_{n-m}+YY^{\top})^{-\frac{1}{2}},$$

- The columns of

$$Q_1(s) = Q(0)U_1(s)$$

and

$$Q_2(s) = Q(0)U_2(s)$$

form **orthogonal** bases in $\mathcal{E}(s)$ and $\mathcal{E}^{\perp}(s)$.

- The columns of

$$Q(0)\begin{bmatrix} I_m \\ Y(s) \end{bmatrix},$$

and

$$Q(0)\begin{bmatrix} -Y(s)^{\mathsf{T}} \\ I_{n-m} \end{bmatrix}$$

form bases in $\mathcal{E}(s)$ and $\mathcal{E}^{\perp}(s)$, which are in general **non-orthogonal**.

## 3. Continuation of homoclinic orbits in MATCONT

- **Basic defining BVP**:

$$\left\{\begin{array}{rcl} \dot{x}(t) - 2Tf(x(t),\alpha) & = & 0, \\ f(x_0,\alpha) & = & 0, \\ \int_0^1 \dot{\tilde{x}}(t)^\top (x(t) - \tilde{x}(t))dt & = & 0, \\ \langle x(0) - x_0, q_{0,n_U+i}\rangle & = & 0,\ \ i = 1,2,\ldots,n_S \\ \langle x(1) - x_0, q_{1,n_S+i}\rangle & = & 0,\ \ i = 1,2,\ldots,n_U \\ T_{22U}Y_U - Y_U T_{11U} + T_{21U} - Y_U T_{12U} Y_U & = & 0, \\ T_{22S}Y_S - Y_S T_{11S} + T_{21S} - Y_S T_{12S} Y_S & = & 0, \\ \|x(0) - x_0\| - \epsilon_0 & = & 0, \\ \|x(1) - x_0\| - \epsilon_1 & = & 0, \end{array}\right.$$

where

$$\begin{array}{rcl} \left[q_{0,n_U+1}\ q_{0,n_U+2}\ \cdots\ q_{0,n_U+n_S}\right] & = & Q_U(0)\begin{bmatrix} -Y_U^\top \\ I_{n_S} \end{bmatrix} \\ \left[q_{1,n_S+1}\ q_{1,n_S+2}\ \cdots\ q_{1,n_S+n_U}\right] & = & Q_S(0)\begin{bmatrix} -Y_S^\top \\ I_{n_U} \end{bmatrix}. \end{array}$$

- **Active:** $\alpha_1, \alpha_2$, and two out of three **homoclinic parameters** $T, \epsilon_0, \epsilon_1$.

## 4. Initialization by homotopy


- E.J. Doedel, M.J. Friedman, and A.C. Monteiro. 1994. On locating connecting orbits. *Appl. Math. Comput.* **65**, 231–239.

- E.J. Doedel, M.J. Friedman, and B.I. Kunin. 1997. Successive continuation for locating connecting orbits. *Numer. Algorithms* **14** , 103–124.

- Champneys, A.R., and Kuznetsov, Yu.A. 1994. Numerical detection and continuation of codimension-two homoclinic bifurcations. *Int. J. Bifurcation Chaos* **4**, 795-822.

## Locating a connecting orbit, $\alpha$ is fixed

- *Step 1*: Integrate an orbit from

$$x(0) = x_0^{(0)} + \epsilon_0 (c_1 q_{0,1}^{(0)} + c_2 q_{0,2}^{(0)}),$$

where $c_2 = 0$ if $\lambda_1$ is real, and monitor $\epsilon_1$.

- *Step $k$*: For $k = 2, \ldots, n_U$ continue a solution to

$$\begin{cases} \dot{x} - 2Tf(x,\alpha) &=& 0, \\ \epsilon_0 c_i - \langle x(0) - x_0^{(0)}, q_{0,i}^{(0)} \rangle &=& 0, \quad i = 1, ..., n_U, \\ \tau_i - \frac{1}{\epsilon_1} \langle x(1) - x_0^{(0)}, q_{1,n_S+i}^{(0)} \rangle &=& 0, \quad i = 1, ..., n_U, \\ \langle x(0) - x_0^{(0)}, q_{0,n_U+i}^{(0)} \rangle &=& 0, \quad i = 1, ..., n_S, \\ \|x(0) - x_0^{(0)}\| - \epsilon_0 &=& 0, \\ \|x(1) - x_0^{(0)}\| - \epsilon_1 &=& 0, \end{cases}$$

to locate a zero of, say, $\tau_{k-1}$ (while $\tau_1, ..., \tau_{k-2} = 0$ are fixed).
**Active:** $c_1, ..., c_k, \tau_{k-1}, ..., \tau_{n_U}, \epsilon_1$

## Locating a connecting orbit, $\alpha$ varies

- *Step* $n_U + 1$: Continue a solution to

$$\left\{ \begin{array}{rcl} \dot{x} - 2Tf(x,\alpha) & = & 0, \\ f(x_0,\alpha) & = & 0, \\ \langle x(0) - x_0, q_{0,n_U+i} \rangle & = & 0, \quad i = 1, ..., n_S, \\ \tau_i - \frac{1}{\epsilon_1}\langle x(1) - x_0, q_{1,n_S+i} \rangle & = & 0, \quad i = 1, ..., n_U, \\ T_{22U}Y_U - Y_U T_{11U} + T_{21U} - Y_U T_{12U} Y_U & = & 0, \\ T_{22S}Y_S - Y_S T_{11S} + T_{21S} - Y_S T_{12S} Y_S & = & 0, \\ \|x(0) - x_0\| - \epsilon_0 & = & 0, \\ \|x(1) - x_0\| - \epsilon_1 & = & 0, \end{array} \right.$$

to locate a zero of $\tau_{n_U}$ (while $\tau_1, \ldots, \tau_{n_U-1} = 0$ are fixed).

**Active:** $\alpha_1, \tau_{n_U}, \epsilon_1$.

**Increasing accuracy of the connecting orbit, $\alpha$ varies**

- *Step* $n_U + 2$: Continue a solution to

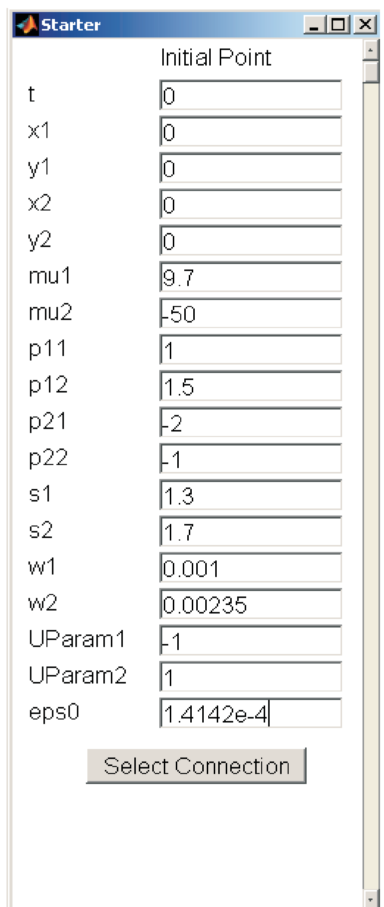
$$\left\{ \begin{array}{rcll} \dot{x} - 2Tf(x,\alpha) & = & 0, & \\ f(x_0,\alpha) & = & 0, & \\ \langle x(0) - x_0, q_{0,n_U+i} \rangle & = & 0, & i = 1,...,n_S, \\ \langle x(1) - x_0, q_{1,n_S+i} \rangle & = & 0, & i = 1,...,n_U, \\ T_{22U}Y_U - Y_U T_{11U} + T_{21U} - Y_U T_{12U} Y_U & = & 0, & \\ T_{22S}Y_S - Y_S T_{11S} + T_{21S} - Y_S T_{12S} Y_S & = & 0, & \\ \|x(0) - x_0\| - \epsilon_0 & = & 0, & \\ \|x(1) - x_0\| - \epsilon_1 & = & 0, & \end{array} \right.$$

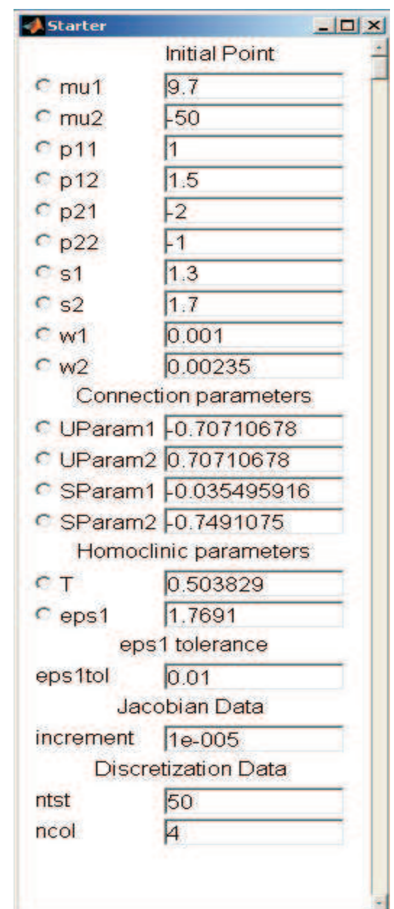
in the direction of decreasing $\epsilon_1$ until this distance is 'small'.

**Active:** $\alpha_1, T, \epsilon_1$.

# Implementation in MATCONT

$c_j \equiv$ `UParam1, UParam2, ...` $\tau_j \equiv$ `SParam1, SParam2, ...`



## 5. Example: Lorenz System (dim $W^u = 1$)

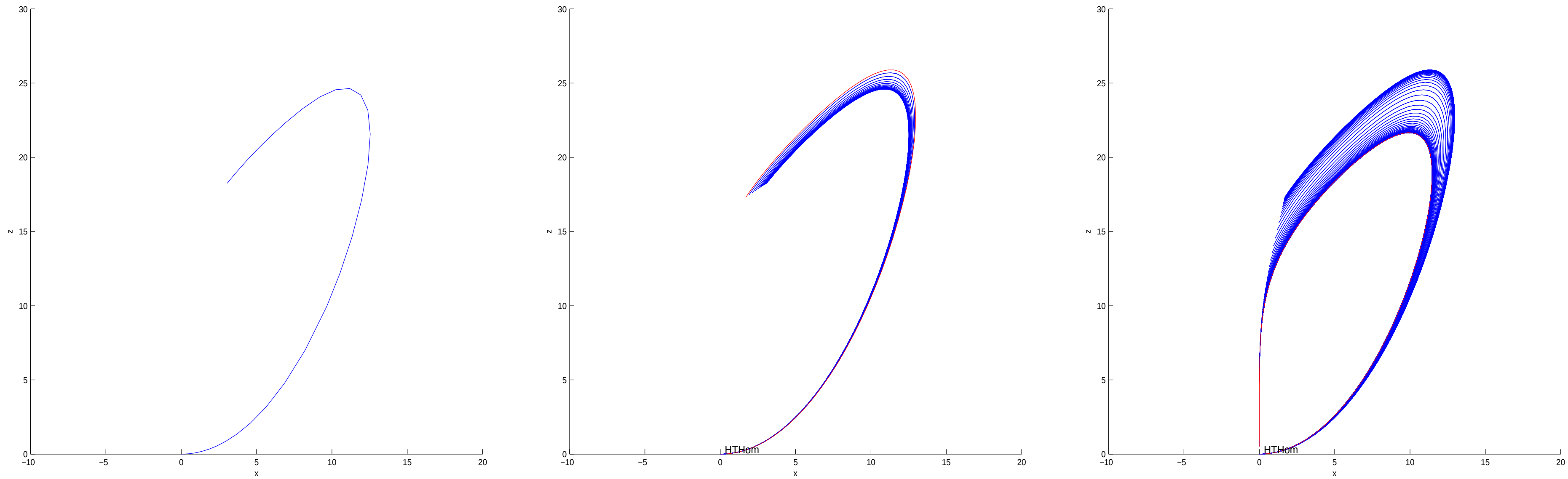
- Lorenz system:

$$\begin{cases} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3, \\ \dot{x}_3 &= x_1x_2 - bx_3, \end{cases}$$

  with the standard value $b = \dfrac{8}{3}$.

- Petrovskaya, N.V., and Judovich, V.I. 1980. Homoclinic loops of the Salzman-Lorenz system. In *Methods of Qualitative Theory of Differential Equations*, Gorkii State University, Gorkii, pp. 73-83 [In Russian]

# Homotopy for the $(1,0)$-homoclinic orbit



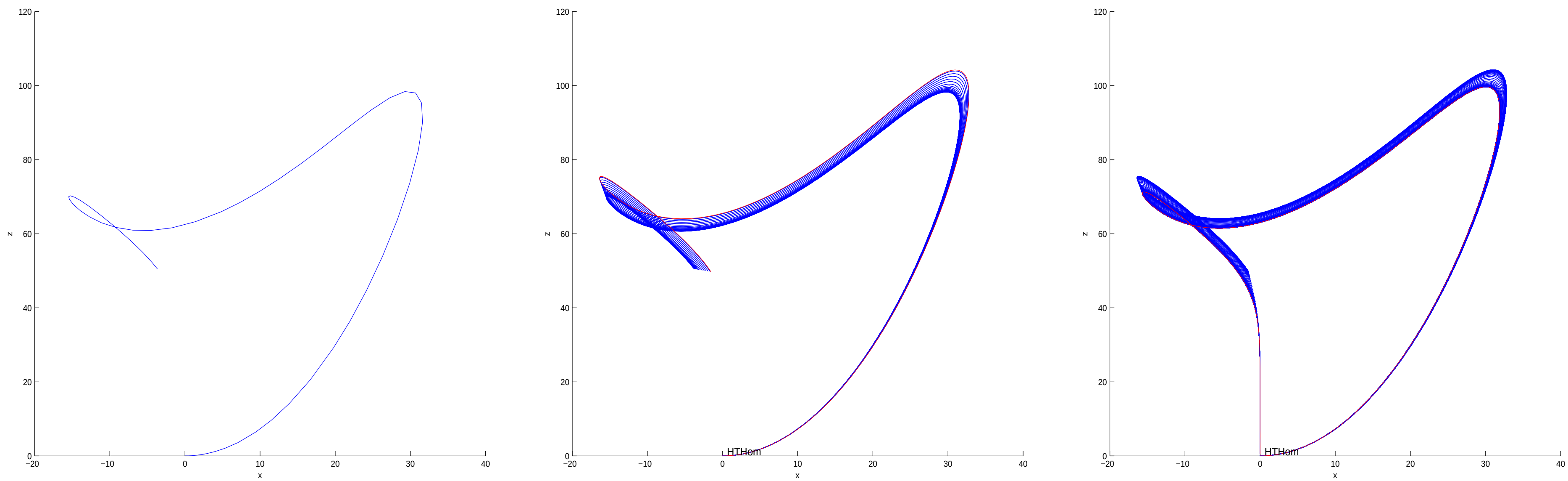

(a) (b) (c)

(a) Integration over $T = 1.3$ starting at $\epsilon_0 = 0.01$ for $\sigma = 10, r = 15.5$.

(b) Continuation in $(r, \tau_1, \epsilon_1)$ until $\tau_1 = 0$ at $r = 16.1793$.

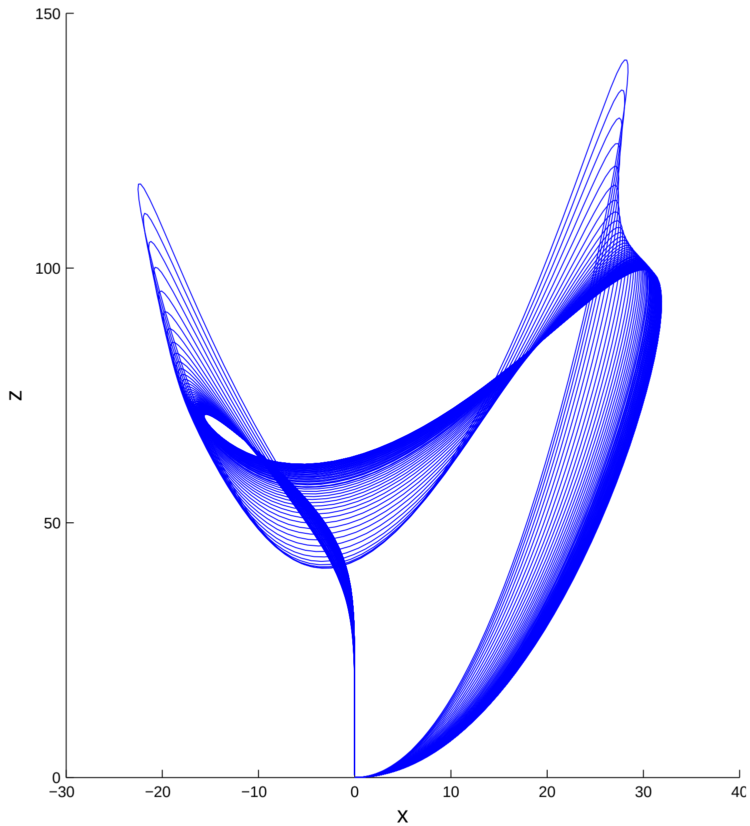
(c) Continuation in $(r, T, \epsilon_1)$ until $\epsilon_1 = 0.5$ at $r = 13.9266$.

# Homotopy for the $(1, 1)$-homoclinic orbit



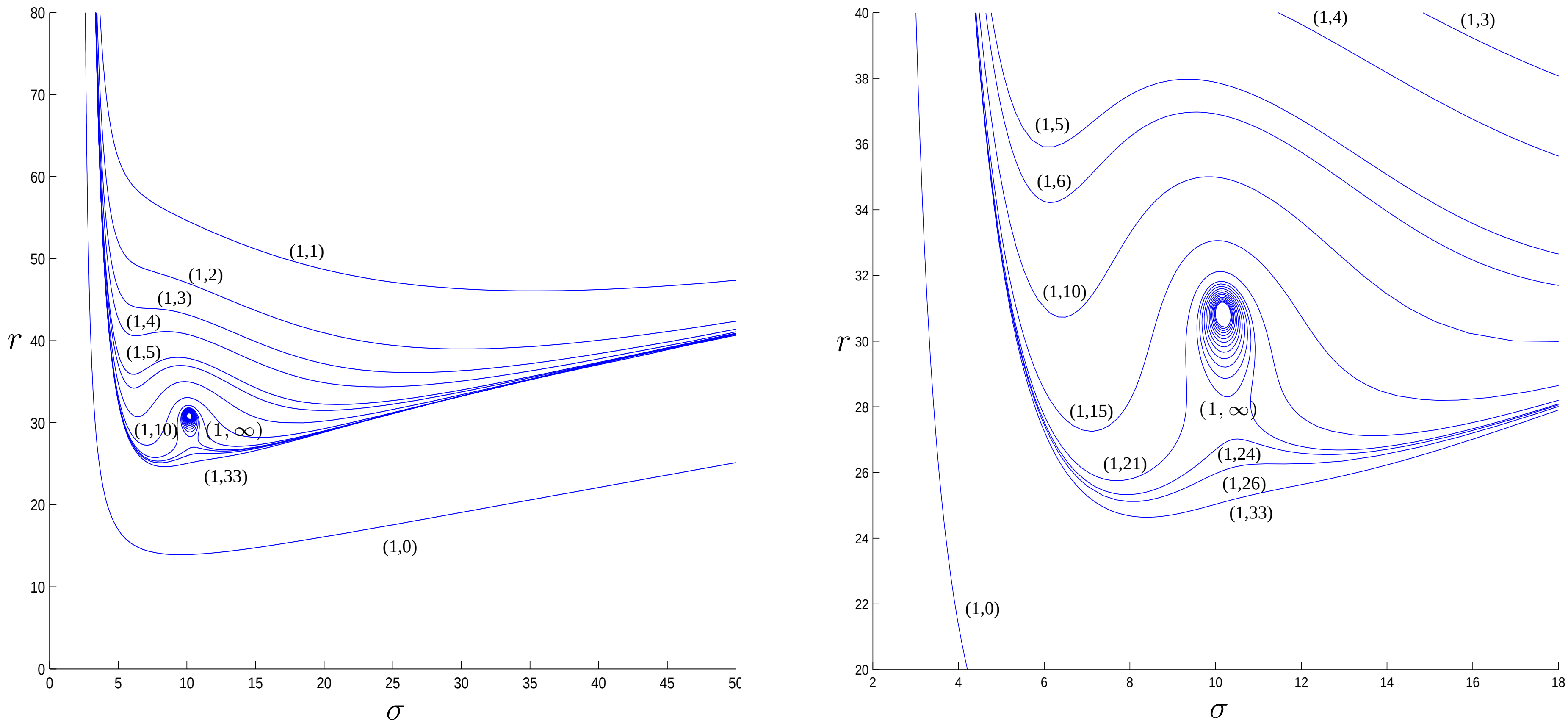

(a) (b) (c)

## A family of $(1, 1)$-homoclinic orbits



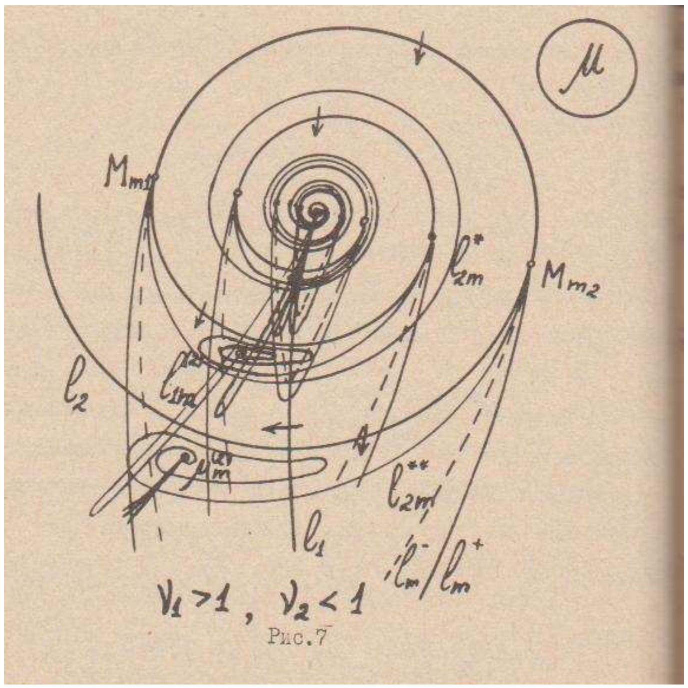

Continuation in $(\sigma, r, T, \epsilon_1)$.

# Homoclinic bifurcation curves



## Historic remark on $T$-points


1. Bykov, V.V. 1978. On the structure of a neighborhood of a separatrix contour with a saddle-focus. In *Methods of Qualitative Theory of Differential Equations*, Gorkii State University, Gorkii, pp. 3-32 [In Russian]




2. Bykov, V.V. 1980. Bifurcations of dynamical systems close to systems with a separatrix contour containing a saddle-focus. *In Methods of Qualitative Theory of Differential Equations*, Gorkii State University, Gorkii, pp. 44-72 [In Russian]

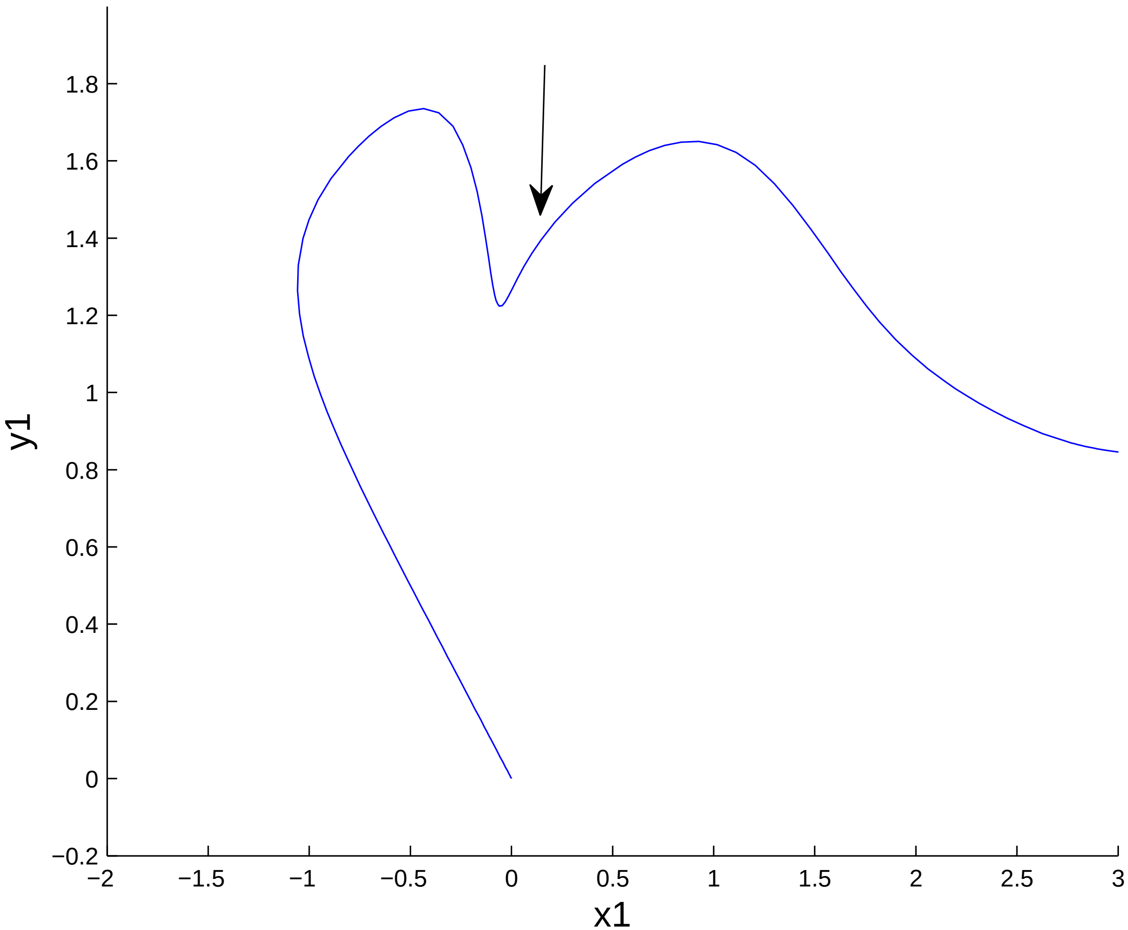
**Example: Perturbed Hopf-Hopf normal form** ($\dim W^u = 2$)

- The system:

$$\begin{cases} \dot{x_1} = x_1(\mu_1 + p_{11}(x_1^2 + y_1^2) + p_{12}(x_2^2 + y_2^2) + s_1(x_2^2 + y_2^2)^2) - y_1\omega_1 + \mathbf{3y_1^6} \\ \dot{y_1} = y_1(\mu_1 + p_{11}(x_1^2 + y_1^2) + p_{12}(x_2^2 + y_2^2) + s_1(x_2^2 + y_2^2)^2) + x_1\omega_1 \mathbf{-2x_2^6} \\ \dot{x_2} = x_2(\mu_2 + p_{21}(x_1^2 + y_1^2) + p_{22}(x_2^2 + y_2^2) + s_2(x_1^2 + y_1^2)^2) - y_2\omega_2 \mathbf{-7y_1^6} \\ \dot{y_2} = y_2(\mu_2 + p_{21}(x_1^2 + y_1^2) + p_{22}(x_2^2 + y_2^2) + s_2(x_1^2 + y_1^2)^2) + x_2\omega_2 \mathbf{+x_1^6}. \end{cases}$$

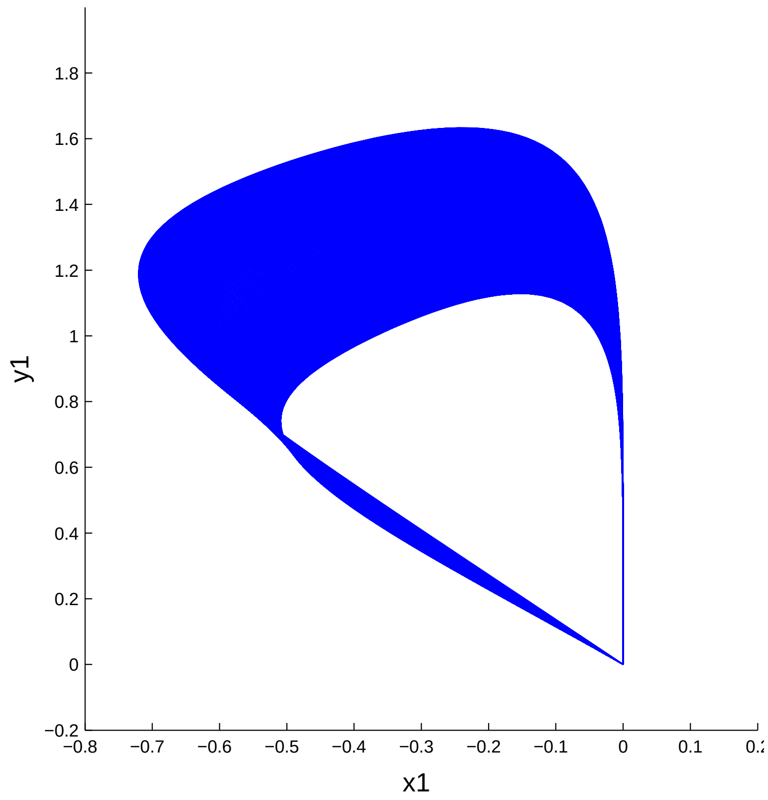
- Parameter values: $\mu_1 = 9.7, \mu_2 = -50, p_{11} = 1, p_{12} = 1.5, p_{21} = -2, p_{22} = -1, s_1 = 1.3, s_2 = 1.7, \omega_1 = 0.001, \omega_2 = 0.00235.$

## Initial orbit obtained by integration



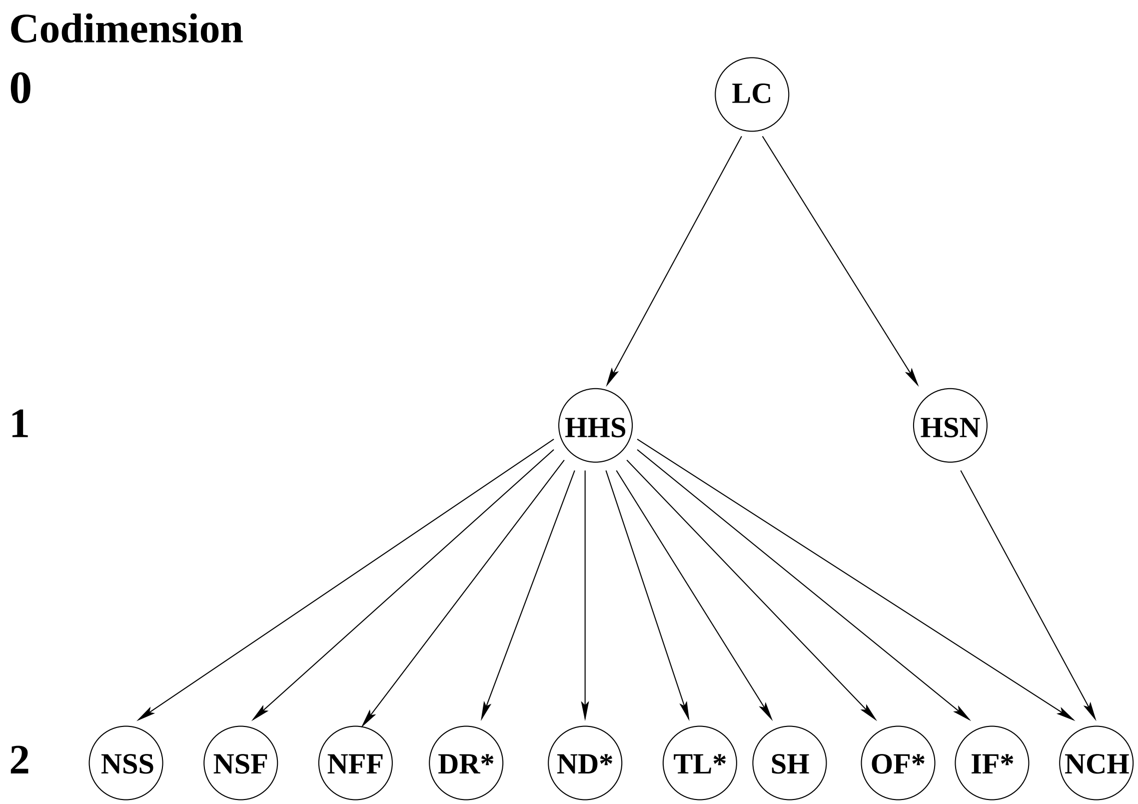

Then make $\tau_1 = \tau_2 = 0$ by homotopy and decrease $\epsilon_1$ to $\approx 10^{-5}$.

# A family of focus-focus homoclinic orbits




Active parameters $(\mu_1, \mu_2, T, \epsilon_1)$.

# 6. Detection of codim 2 homoclinic bifurcations




Here $*$ stands for S or U.

| Type of object | Label |
|---|---|
| Limit cycle | LC |
| Homoclinic to Hyperbolic Saddle | HHS |
| Homoclinic to Saddle-Node | HSN |
| Neutral saddle | NSS |
| Neutral saddle-focus | NSF |
| Neutral Bi-Focus | NFF |
| Shilnikov-Hopf | SH |
| Double Real Stable leading eigenvalue | DRS |
| Double Real Unstable leading eigenvalue | DRU |
| Neutrally-Divergent saddle-focus (Stable) | NDS |
| Neutrally-Divergent saddle-focus (Unstable) | NDU |
| Three Leading eigenvalues (Stable) | TLS |
| Three Leading eigenvalues (Unstable) | TLU |
| Orbit-Flip with respect to the Stable manifold | OFS |
| Orbit-Flip with respect to the Unstable manifold | OFU |
| Inclination-Flip with respect to the Stable manifold | IFS |
| Inclination-Flip with respect to the Unstable manifold | IFU |
| Non-Central Homoclinic to saddle-node | NCH |

**Orbit flips**

$$A^{\mathsf{T}}(x_0, \alpha_0)\ p_1^s = \mu_1\ p_1^s,\ \ A^{\mathsf{T}}(x_0, \alpha_0)\ p_1^u = \lambda_1\ p_1^u.$$

- Orbit-flip with respect to the stable manifold

$$\psi = \begin{cases} e^{-\Re(\mu_1)T}\langle \Re(p_1^s), x(1) - x_0 \rangle \\ e^{-\Re(\mu_1)T}\langle \Im(p_1^s), x(1) - x_0 \rangle \end{cases}$$

- Orbit-flip with respect to the unstable manifold

$$\psi = \begin{cases} e^{\Re(\lambda_1)T}\langle \Re(p_1^u), x(0) - x_0 \rangle \\ e^{\Re(\lambda_1)T}\langle \Im(p_1^u), x(0) - x_0 \rangle \end{cases}$$

## Inclination flips

$$A(x_0,\alpha_0)\ q_1^s = \mu_1\ q_1^s,\ \ A(x_0,\alpha_0)\ q_1^u = \lambda_1\ q_1^u.$$

- Inclination-flip with respect to the stable manifold

$$\psi = \begin{cases} e^{-\Re(\mu_1)T}\langle \Re(q_1^s), \phi(0)\rangle \\ e^{-\Re(\mu_1)T}\langle \Im(q_1^s), \phi(0)\rangle \end{cases}$$

- Inclination-flip with respect to the unstable manifold

$$\psi = \begin{cases} e^{\Re(\lambda_1)T}\langle \Re(q_1^u), \phi(1)\rangle \\ e^{\Re(\lambda_1)T}\langle \Im(q_1^u), \phi(1)\rangle \end{cases}$$

  where $\phi(t) \perp (T_{x(t)}W^U(x_0) + T_{x(t)}W^S(x_0))$.

In MATCONT a new method to compute $\phi(0)$ and $\phi(1)$ is implemented.

The function $\phi \in \mathcal{C}^1([0,1],\mathbb{R}^n)$ is the solution to the **adjoint system**:

$$\left\{ \begin{array}{rcl} \dot{\phi}(t) + 2\,T\,A^{\mathsf{T}}(x(t),\alpha_0)\,\phi(t) & = & 0 \\ Q^{S,\mathsf{T}}\phi(1) & = & 0 \\ Q^{U,\mathsf{T}}\phi(0) & = & 0 \\ \int_0^1 \tilde{\phi}(t)^{\mathsf{T}}[\phi(t) - \tilde{\phi}(t)]dt & = & 0, \end{array} \right. \tag{1}$$

where the columns of $Q^S$ and $Q^U$ span the stable and the unstable eigenspaces of $A(x_0,\alpha_0)$, resp.

**Theorem** *If $\phi$ is a solution to (1) and $\zeta_1 \in \mathbb{R}^{n_U}$, $\zeta_2 \in \mathbb{R}^{n_S}$, then*

$$\begin{pmatrix} \phi(t) \\ \zeta_1 \\ \zeta_2 \end{pmatrix} \perp \mathit{Range} \begin{pmatrix} D - 2\,T\,A(x(t),\alpha_0) \\ Q^{S^{\perp},\mathsf{T}}\ \delta(1) \\ Q^{U^{\perp},\mathsf{T}}\ \delta(0) \end{pmatrix} \Longleftrightarrow \left\{ \begin{array}{l} Q^{S^{\perp}}\zeta_1 = -\phi(1) \\ Q^{U^{\perp}}\zeta_2 = \phi(0). \end{array} \right. \tag{2}$$

*Here $D$ and $\delta$ are the differentiation and the evaluation operators, resp.*

$Q^{S^{\perp}}$ and $Q^{U^{\perp}}$ are known from CIS, $\zeta_1$ and $\zeta_2$ are computable via bordering a (sub)matrix of the discretized basic BVP that is also known.

## 6. Open problems

**Theoretical**:

- Dynamical implications of orbit and inclinations flips with complex leading eigenvalues.

- Bifurcation of Three Leading Eigenvalues (codim 2).

**Numerical:**

- Starting homoclinic orbits from codim 2 bifurcations of equilibria (only from BT is implemented; ZH and HH remain unsupported).

- Continuation of homoclinic orbits to limit cycles (no robust $n$-dimensional algorithm; a generalization of CIS to eigenspaces of differential operators is needed).