PRACTICUM I

SUMMER SCHOOL UTRECHT:

APPLIED BIFURCATION THEORY

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1. Numerical simulation of DDEs with MATLAB

In this section we show how to approximate solutions DDEs numerically with the MATLAB function `dde23`. Explicit Runge-Kutta triples are a standard way to solve the ODE problem $\dot{x}(t) = f(t, x)$ on $[a, b]$ with given $x(a)$. They can be extended to solve DDEs. Indeed, `dde23` is closely related to the ODE solver `ode23` from the MATLAB ODE Suite [5], which implements the BS(2,3) triple [1].

A typical call to the function `dde23` has the form

```
sol = dde23(ddefun,lags,history,tspan)
```

where

ddefun Function handle that evaluates the right side of the differential equation

$$\dot{x}(t) = f(x(t - \tau_0), x(t - \tau_1), \ldots, x(t - \tau_m)).$$

The function must have the form

```
dydt = ddefun(t,x,z)
```

where `t` corresponds to the current `t`, `x` is a column vector that approximates `x(t)`, and `z(:,j)` approximates `x(t-τj)` for delay `τj = lags(j)`. The output is a column vector corresponding to

$$f(x(t - \tau_0), x(t - \tau_1), \ldots, x(t - \tau_m)).$$

lags Vector of constant delays $\tau_1, \ldots, \tau_m$.

history Specify *history*, i.e. the initial solution segment, in one of three ways:

- A function of `t` such that `x = history(t)` returns the solution `x(t)` for $t \leq t_0$ as a column vector
- A constant column vector, if `x(t)` is constant
- The solution `sol` from a previous integration, if this call continues that integration

tspan Interval of integration from `t0=tspan(1)`to `tf=tspan(end)` with `t0 < tf`.

The structure sol returned by `dde23` contains the following fields.

sol.x Mesh selected by `dde23`

sol.y Approximation to $x(t)$ at the mesh points in `sol.x`

sol.yp Approximation to $\dot{x}(t)$ at the mesh points in `sol.x`

Note that MATLAB uses x as the structure key for the mesh for the independent variable.

Example

Consider the scalar DDE

$$\dot{x}(t) = x(t-1)\,(x(t)-1)\,, \qquad t \geq 0, \tag{1}$$

with some initial function

$$x(t) = \phi(t), \qquad -1 \leq t \leq 0.$$

We create two files `history_ex1.m` and `ddefun_ex1.m` containing the initial function and the DDE respectively.

```
% filename: history_ex1.m
function y = history_ex1(t)
    y = cos(t);
end

% filename: ddefun_ex1.m
function  f = ddefun_ex1(xx)
    % (Using notation of DDEBifTool):
    % xx(1,1) = x(t), x(1,2) = x(t-lags(1))
    f = xx(1,2)*(xx(1,1)-1);
end
```

In the function `history_ex1.m` we have set the initial function to $\phi(t) = \phi_1(t) = \cos(t)$. Then the code

```
tspan=[0;4];
hspan=-1:0.01:0;
lags=1;

% solve the DDE
sol = dde23(@(t,y,Z)ddefun_ex1([y,Z]),lags,@history_ex1,tspan);

% plot solution
plot([hspan, sol.x],[history_ex1(hspan), sol.y])

xlabel('t')
ylabel('x')
ylim([0,1.1])
```

solves the DDE on the interval $0 \leq t \leq 4$. Here we use an anonymous function in the first argument of the function `dde23` to put the arguments in the correct form for the function `ddefun`, see above. This will also allow us to incorporate parameters, which we will freuqently do. In Figure 1 we have plotted the solution. In the same figure we also plotted the solutions for different initial functions $\phi_2(t) = 1 - t$ and $\phi_3(t) = \exp(t)$. There we see that solutions on the interval $0 \leq t \leq 4$ coincide. Indeed, if the history function satisfies $\phi(0) = 1$, then the DDE (1) becomes

$$\dot{x}(0) = 0,$$

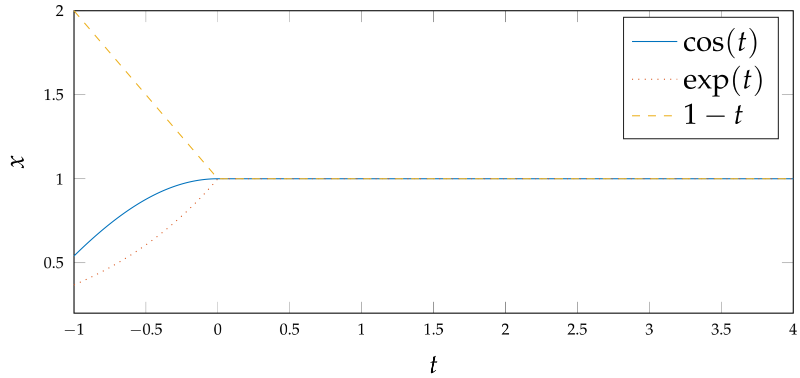


Figure 1: Solutions to the DDE (1). The different history functions $\phi_1(t) = \cos(t)$, $\phi_2(t) = e^t$ and $\phi_3(t) = 1 - t$ result in the same solution for $t \geq 0$.

defining an equilibrium. We notice that different initial states ϕ may be mapped to the same state $x_t(\cdot, \phi)$ at some future time $t > 0$, where $x_t(\cdot, \phi) = x_\phi(t + \cdot)$, $\cdot \in [-1, 0]$ is the history segment of the solution x at time t with initial state ϕ.

2. Example with Hopf bifurcation

Consider the model

$$\dot{x}(t) = -\left(\frac{\pi\sqrt{3}}{9} + \mu\right)[x(t-1) + x(t-2)]\left(1 - x^2(t)\right), \tag{2}$$

with two delays, see [2] and [4]. The DDE is an idealized form of a model for the populations of the field vole (Microtus agrestis) [6]. At the critical value $\mu = 0$ there is a Hopf bifurcation, which we will first find analytically and then we confirm our findings numerically with `dde23`.

Analytical derivation of Hopf bifurcation

The linearization of (2) at $x = 0$ yields

$$\dot{x}(t) = -\left(\frac{\pi\sqrt{3}}{9} + \mu\right)[x(t-1) + x(t-2)].$$

Substituting $x(t) = e^{\lambda t}$ gives

$$\lambda = -\left(\frac{\pi\sqrt{3}}{9} + \mu\right)\left[e^{-\lambda} + e^{-2\lambda)}\right], \tag{3}$$

which, as in ODEs, is called the characteristic equation and $\lambda \in \mathbb{C}$ satisfying (3) an eigenvalue. Let $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$. To prove that at $\mu = 0$ equation (2) undergoes a *Hopf* bifurcation we must show that there exists an eigenvalue such that:

- $\lambda(0) = i\omega(0) = \pm i\omega_0, \omega_0 \in \mathbb{R}_{>0}$, so that $\alpha(0) = 0$
- $\alpha'(0) \neq 0$ (the transversality hypothesis)
- the fist Lyapunov coefficient is non-vanishing

The first two conditions will be verified here. The third condition will be treated in Practicum II.

For equation (3) to have purely imaginary root $\lambda = i\omega, \omega > 0$ at $\mu = 0$, it must be that

$$0 = \cos\omega + \cos 2\omega = 2\cos\frac{\omega}{2}\cos\frac{3\omega}{2}, \tag{4}$$

$$9\omega = \sqrt{3}\pi(\sin\omega + \sin 2\omega) = 2\sqrt{3}\pi\cos\frac{\omega}{2}\sin\frac{3\omega}{2}. \tag{5}$$

Note that that eigenvalues of *real* systems come in conjugate pairs, so we can restrict to the case that $\omega > 0$. If $\cos\frac{\omega}{2} = 0$, then (3) is not satisfied. Let

$$\omega_k = \frac{2}{3}\left(\frac{\pi}{2} + k\pi\right), \qquad k \in \mathbb{Z},$$

then equation (4) is satisfied. Substituting into equation (5) yields

$$9\omega_k = \sqrt{3}\pi(\sin\omega_k + \sin 2\omega_k),$$

which gives $k = 0, -1$. It follows that $\omega = \dfrac{\pi}{3}$.

We will now show that there are no eigenvalues with positive real part at $\mu = 0$. Suppose that $\lambda = \alpha + i\omega$, with $\alpha, \omega \geq 0$. Substituting into equation (3) at $\mu = 0$ and separating the real and imaginary parts yields the equations

$$\alpha = -\frac{\pi}{3\sqrt{3}}\left(e^{-\alpha}\cos\omega + e^{-2\alpha}\cos 2\omega\right), \tag{6}$$

$$\omega = \frac{\pi}{3\sqrt{3}}\left(e^{-\alpha}\sin\omega + e^{-2\alpha}\sin 2\omega\right). \tag{7}$$

Since $\alpha > 0$ the second equation implies that $|\omega| \leq \frac{2\pi}{3\sqrt{3}}$. The first equation yields

$$-\frac{3\sqrt{3}\alpha}{\pi\cos\omega} = e^{-\alpha}\left(1 + \frac{\cos 2\omega}{\cos\omega}e^{-\alpha}\right). \tag{8}$$

Since

$$\frac{\cos 2\omega}{\cos\omega} \geq -1, \qquad \omega \in [0, \frac{\pi}{3}]$$

the right hand side of equation (8) is positive there. It follows that we only have to look for solutions of equations (6)-(7) for

$$\omega \in \left(\frac{\pi}{3}, \frac{2\pi}{3\sqrt{3}}\right].$$

But

$$G(\omega) := \omega - \frac{\pi}{3\sqrt{3}}\left(e^{-\alpha}\sin\omega + e^{-2\alpha}\sin 2\omega\right)$$

has a positive derivative for $\omega \in (\frac{\pi}{3}, \frac{2\pi}{3\sqrt{3}}]$, see Appendix A for details. Furthermore, $G(\frac{\pi}{3}) > 0$. Hence the characteristic equation (3) has no roots with positive real parts for $\mu = 0$.

Implicit differentiation of (3) yields

$$\lambda' = -\left[e^{-\lambda} + e^{-2\lambda)}\right] + \left(\frac{\pi\sqrt{3}}{9} + \mu\right)\left[e^{-\lambda} + 2e^{-2\lambda)}\right]\lambda',$$

so that

$$\lambda' = -\frac{e^{-\lambda} + e^{-2\lambda}}{1 - \left(\frac{\pi\sqrt{3}}{9} + \mu\right)[e^{-\lambda} + 2e^{-2\lambda}]}$$

Hence

$$\lambda'(0) = \frac{27\sqrt{3}\pi}{6\pi\sqrt{3}+14\pi^2+54} + i\frac{9\left(6\sqrt{3}+\pi\right)}{6\pi\sqrt{3}+14\pi^2+54}.$$

Thus $\alpha'(0) \neq 0$ and the transversality hypothesis is satisfied.

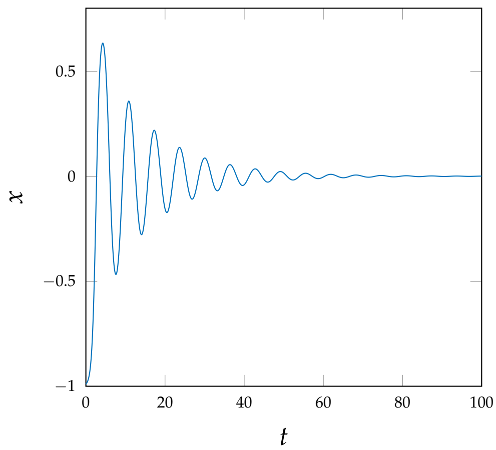
We conclude that the pair of purely imaginary eigenvalues passes form the left half-plane to the right half-plane as μ changes from negative to positive. The equilibrium $x = 0$ loses its stability at $\mu = 0$ and a unique limit cycle bifurcates from it, see Figure 2.

Numerical illustration of Hopf bifurcation

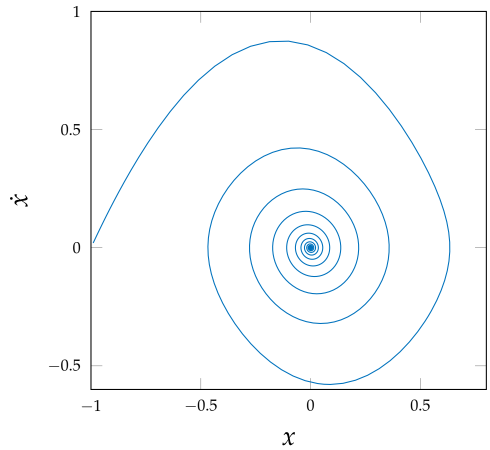
The code in the Listing below generates the plots in Figure 2.

```
%% Simulation of Hopf bifurcation
%
% Simulation using MATLAB dde23 near a Hopf bifurction in an idealized form
% of a model for the populations of the field vole (Microtus agrestis)
%
%% Differential equations
%
% From: Stirzaker, D
% On a population model
% Mathematical Biosciences, 23(3):329--336, 1975
%
% $$\dot{x}(t)=-\left(\dfrac{\pi\sqrt{3}}{9}+\mu\right)\left[x(t-1)+x(t-2)\...
    right]\left(1-x^{2}(t)\right)$$
%
%% Simulation of Hopf bifurcation
% Set default text interpreter for axis to LaTeX
set(groot, 'defaultTextInterpreter', 'LaTeX');
% Define the field vole model
sys_rhs = @(xx,mu) -(pi*sqrt(3)/9+mu)*(xx(1,2)+xx(1,3))*(1-xx(1,1)^2);
% Just before the hopf bifurcation
mu=-0.1;
history1=-0.99;
sol1 = dde23(@(t,y,Z)sys_rhs([y,Z],mu),[1 2],history1,[0 100]);

% time series plot
figure(1); clf
subplot(3,2,1)
plot(sol1.x,sol1.y)
title('$\mu=-0.1$')
xlabel('$t$')
ylabel('$x$')

% plot in phase-plane
```

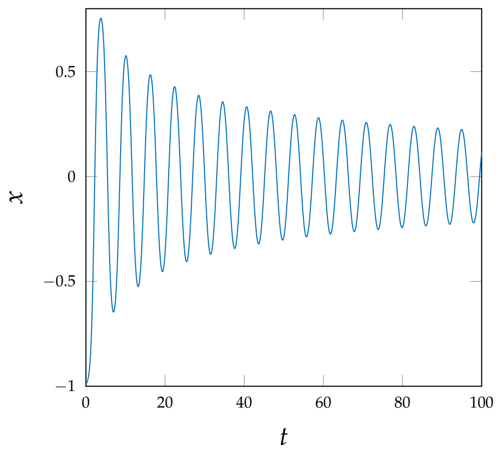
```
subplot(3,2,2)
plot(sol1.y,sol1.yp)
title('$\mu=-0.1$')
xlabel('$x$')
ylabel('$\dot{x}$')

% at critical value
mu=0;
history1=-0.99;
sol1 = dde23(@(t,y,Z)sys_rhs([y,Z],mu),[1 2],history1,[0 100]);

% time series plot
subplot(3,2,3)
plot(sol1.x,sol1.y)
title('$\mu=0$')
xlabel('$t$')
ylabel('$x$')

% plot in phase-plane
subplot(3,2,4)
plot(sol1.y,sol1.yp)
title('$\mu=0$')
xlabel('$x$')
ylabel('$\dot x$')

% just after the hopf bifurcation
mu=0.1;
options = ddeset('MaxStep',0.1);
sol1 = dde23(@(t,y,Z)sys_rhs([y,Z],mu),[1 2],-0.99,[0 100],options);
sol2 = dde23(@(t,y,Z)sys_rhs([y,Z],mu),[1 2], 0.01,[0 100],options);

% time series plot
subplot(3,2,5)
plot(sol1.x,sol1.y)
hold on
plot(sol2.x,sol2.y)
title('$\mu=0.1$')
xlabel('$t$')
ylabel('$x$')

% plot in phase-plane
subplot(3,2,6)
plot(sol1.y,sol1.yp)
hold on
plot(sol2.y,sol2.yp)
title('$\mu=0.1$')
xlabel('$x$')
ylabel('$\dot x$')
```



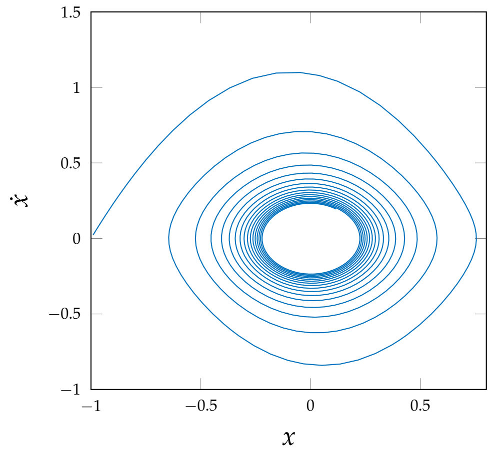
(a) $\mu = -0.1$



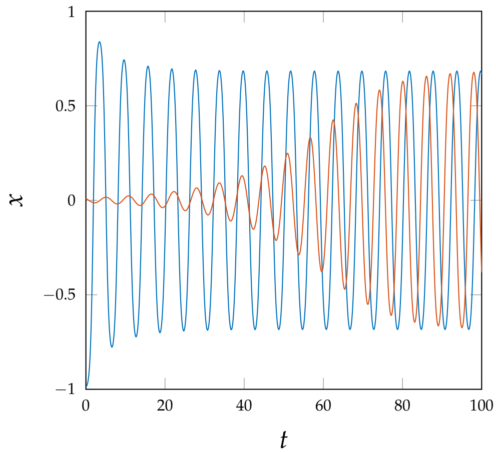
(b) $\mu = -0.1$



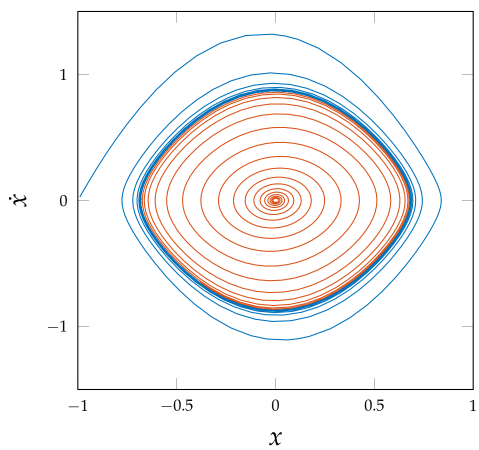
(c) $\mu = 0$



(d) $\mu = 0$



(e) $\mu = 0.1$



(f) $\mu = 0.1$

Figure 2: Simulation using `dde23` near a Hopf bifurcation in the field vole model (2).

3. Exercise

In biology one of the earliest mathematical models involving time lags is due to Hutchinson [3]. He considered a population of a single species obeying a logistic growth law with a constant growth rate r modified by a maximum population factor evaluated at an earlier time:

$$\frac{dN(t)}{dt} = r\left[1 - N(t-T)/K\right] N(t), \qquad r, T > 0. \tag{9}$$

Here K is the maximum population size that can be sustained. Using the substitution

$$t = Ts, \qquad x(s) = \frac{N(Ts) - K}{K},$$

then (9) becomes

$$\frac{dx(s)}{ds} = -\left(rT\right) x(s-1)\left[1 + x(s)\right],$$

which we rewrite as

$$\dot{x}(t) = \left(-\frac{\pi}{2} + \mu\right) x(t-1)\left[1 + x(t)\right]. \tag{10}$$

1. Show that the linearization of (10) at the steady state $x(t) = 0$ has two imaginary eigenvalues at $\mu = 0$. Hint: Use the substitution $x(t) = e^{\lambda t}$. This leads to the characteristic equation
$$\lambda + (\frac{\pi}{2} - \mu)e^{-\lambda} = 0. \tag{11}$$

2. Show that the characteristic equation (11) for $\mu = 0$ has no roots with positive real parts.

3. Verify the transversality condition
$$\operatorname{Re} \left.\frac{d\lambda}{d\mu}\right|_{\mu=0} \neq 0.$$

4. Create similar images as in Figure 2 for the DDE (10) with `dde23`.

A. Appendix

The following notes are due to Sebastiaan Janssens[1]. Consider G on the interval $[\omega_1, \omega_2]$ with $\omega_1 < \omega_2$ as in the text. Differentiation gives

$$G'(\omega) = 1 - \frac{\pi}{3\sqrt{3}} e^{-\alpha} \cos\omega \left(1 + 2e^{-\alpha} \frac{\cos 2\omega}{\cos\omega}\right).$$

We note that $\cos\omega > 0$ and $\frac{\cos 2\omega}{\cos\omega} \leq -1$ on $[\omega_1, \omega_2]$. This implies that

$$G'(\omega) \geq 1 - \frac{\pi}{3\sqrt{3}} e^{-\alpha}(1 - 2e^{-\alpha}) \cos\omega \qquad \text{on } [\omega_1, \omega_2].$$

We also note that $e^{-\alpha}(1 - 2e^{-\alpha}) \leq \frac{1}{8}$ for $\alpha \geq 0$. Together, this gives the lower bound

$$G'(\omega) \geq 1 - \frac{\pi\sqrt{3}}{144} > 0 \qquad \text{on } [\omega_1, \omega_2]$$

which is actually uniform in $\alpha > 0$.

[1] https://sebastiaanjanssens.nl

References

[1] P. Bogacki and L. F. Shampine. A 3 (2) pair of Runge-Kutta formulas. *Applied Mathematics Letters*, 2(4):321–325, 1989.

[2] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan. *Theory and applications of Hopf bifurcation*, volume 41 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1981.

[3] G. E. Hutchinson. Circular causal systems in ecology. *Annals of the New York Academy of Sciences*, 50(4):221–246, 1948.

[4] J. L. Kaplan and J. A. Yorke. Ordinary differential equations which yield periodic solutions of differential delay equations. *J. Math. Anal. Appl.*, 48:317–324, 1974.

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[6] D. Stirzaker. On a population model. *Mathematical Biosciences*, 23(3):329–336, 1975.