Homoclinic Bifurcations to Equilibria

I. Theory and applications

Alan Champneys University of Bristol

Department of Engineering Mathematics

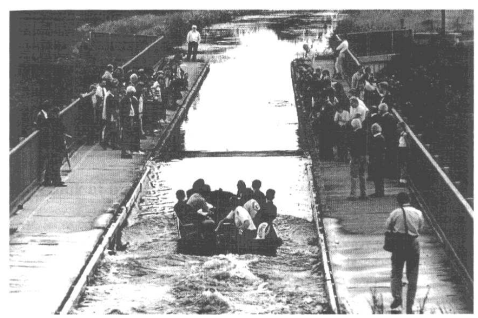
Outline: Lecture 1

1. Introduction: solitary waves & global bifurcations
2. Tame and chaotic homoclinic bifurcations to equilibria
   - Shil'nikov's theorems
   - application: excitable systems
3. Reversible and Hamiltonian systems
   - hyperbolic cases $\Rightarrow$ one codimension less
   - saddle-centre homoclinics
4. Simple strategies for continuation of homoclinic orbits in AUTO-07P.

1. solitary waves.

- 1834 J Scott-Russell observed barge on aqueduct

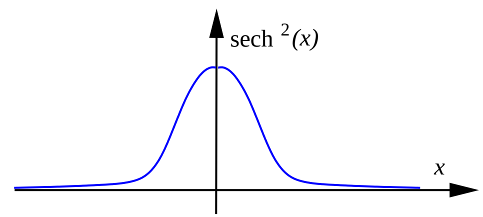
  . . . a boat drawn along a narrow channel . . . suddenly stopped . . . the mass of water in the channel . . . accumulated around the prow [and] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth heap of water, which continued its course along the channel [at 8 or 9 miles an hour for 1.5 miles] preserving its original feature some thirty feet long and a foot and a half high . . .



- Explanation: 1870s Bousinesq & Lord Rayleigh, theory of wall of water = 'solitary waves'

The 'soliton'

- 1895 Korteweg & de Vries derived KdV equation
  $\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}$ solution speed c
  $u(x,t) = (-c/2)\mathrm{sech}^2(\sqrt{c}/2)(x-ct)$



- 1960s Zabusky & Kruskal showed its 'completely' stable
  $\Rightarrow$ New name for particle-like solitary waves; *solitons*
- 1970s because KdV equation is *integrable* (Lax, Gardner, Zakarov . . . & the 'Clarkson mafia')

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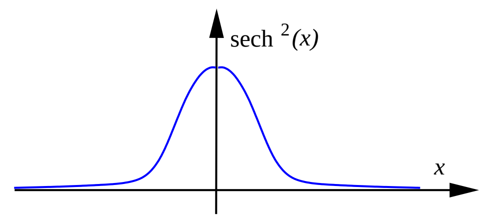

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- Nb. 'Solitary killer' waves e.g. Tsunami

Optical solitons

- Optical fibres; means of transatlantic communication. Pulses travel at speed of light c

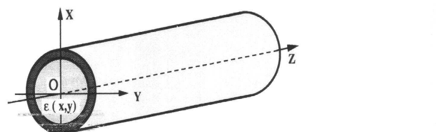

Fig.8.1. Sketch of an optical fiber with an optical index n(x, y), or a dielectric function ε (x,y) which varies in the transverse section

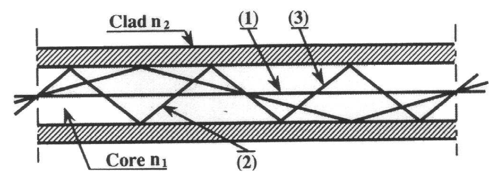

Fig.8.2. Representation of an optical fiber with different modes of propagation: (1) lowest-order mode; (2) middle-order mode; (3) high-order mode

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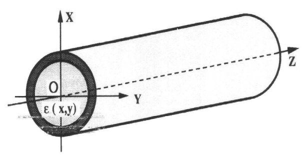

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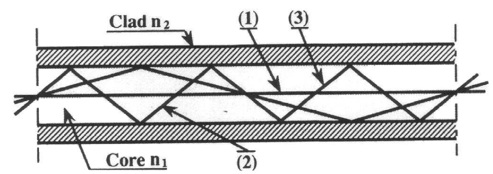

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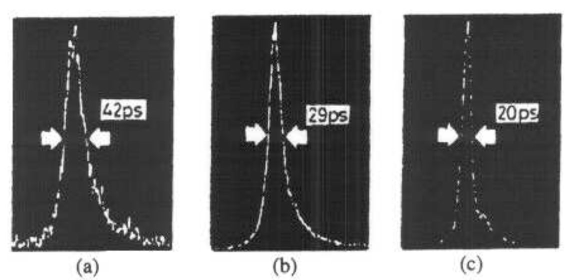
- Problem: denigration due to dispersion



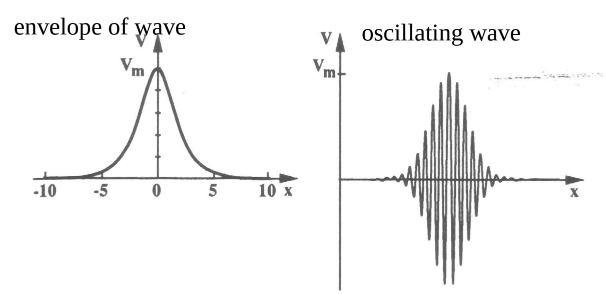
- One idea (Hasegawa & Tappert 1973) use natural Kerr nonlinearity to self-focus the light

⇒ Nonlinear Schrödinger (NLS) equation

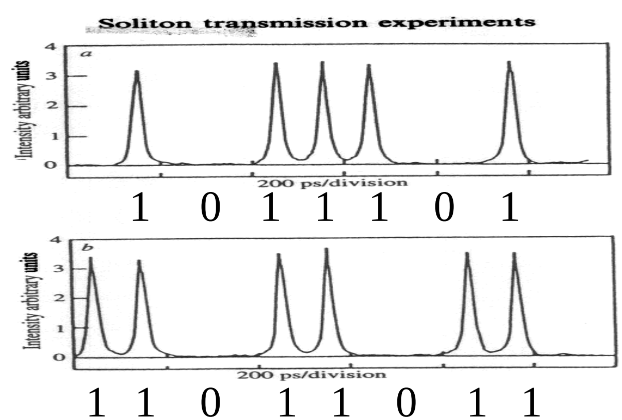
$$i\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + Q|v|^2 v = 0$$

also integrable & explicit ‘envelope of waves’ soliton

$$v(x,t) = V_m e^{ikt}\text{sech}(V_m\sqrt{|Q|/2}U(x-ct))$$



Use solitons as bits of information



. . . but

- KdV and NLS are leading-order approximations: ‘Most’ (probability 1) nonlinear wave equations NOT integrable
- can exist a ‘zoo of solitary waves’:

$$u(x,t) = U(x-ct) \Rightarrow \text{ ODE for } U(\xi)$$

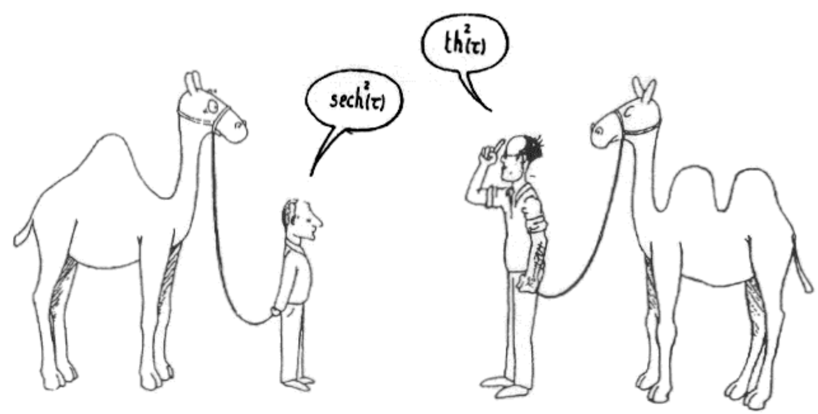


Fig. 1. Do these ‘animals’ belong to the same soliton family? (the drawing made by Marc Haelterman in 1989

- How big is the zoo? their dynamics? (not these lectures)

Another motivation: global bifurcation

- Ed Lorenz 1963 discovered chaos in simple system

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z\end{aligned}$$

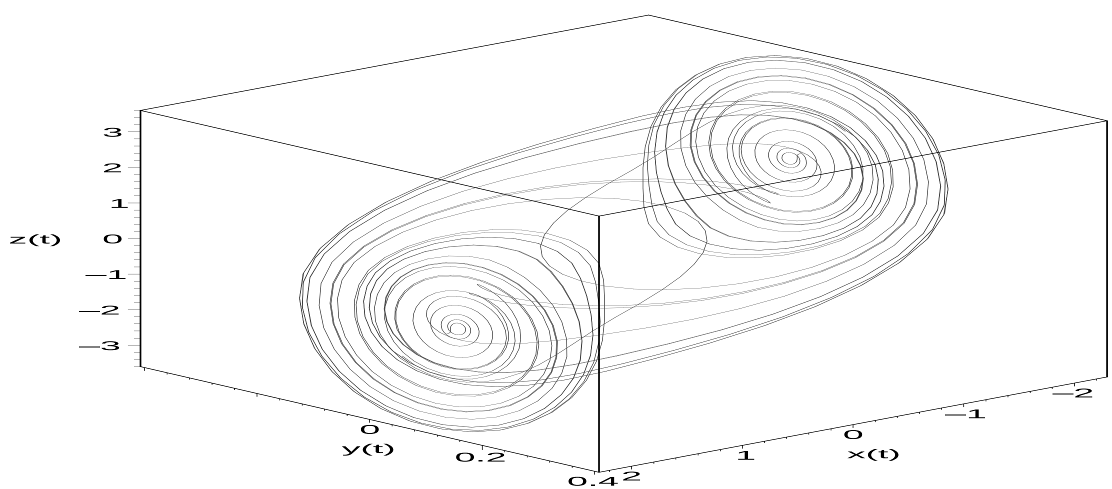
- $\sigma = 10$, $\beta = 8/3$, ρ increasing
- homoclinic bifurcation triggers chaos at $\rho \approx 24$ (Sparrow 1982)

Application: Chua's electronic circuit

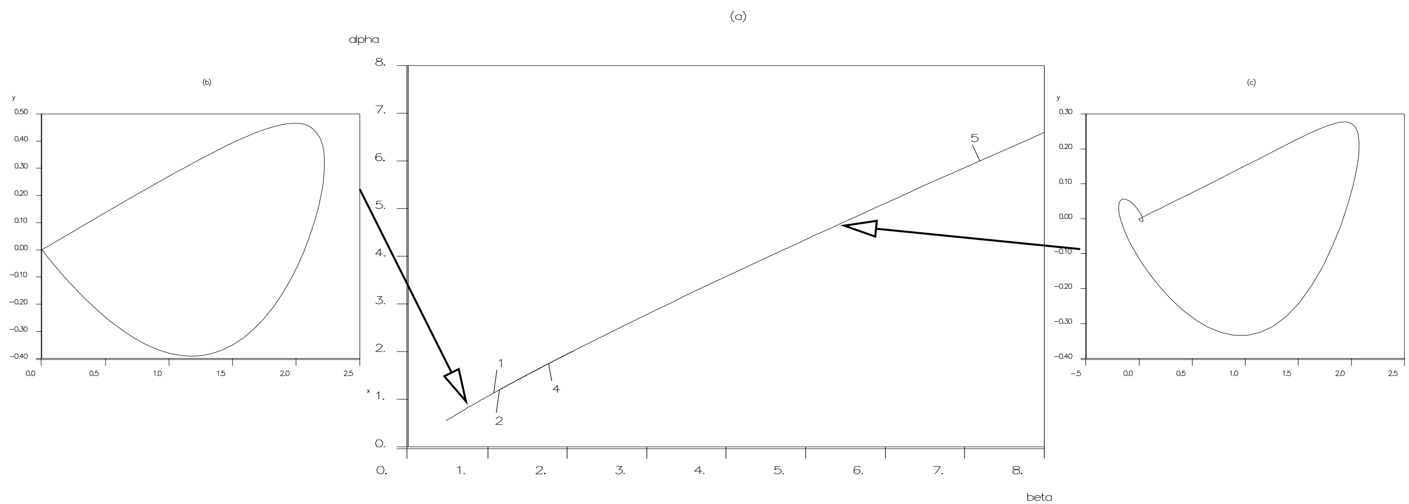
- smooth version (due to A. Khibnik 93)

$$\begin{aligned} \dot{U} &= \alpha(V - (1/6)(U - U^3)) \\ \dot{V} &= U - V + W \\ \dot{W} &= -\beta V \end{aligned}$$

- $(\alpha, \beta) = (0, 0)$: Z_2-symmetric Takens Bogdanov point at 0: $\sigma(0) = \{0, 0, -\lambda\} \Rightarrow$ tame homoclinic bifurcation.
- Large enough α, β 'double scroll' chaotic attractor



tame and chaotic homoclinic orbits



- 1. at $(\alpha, \beta) = (1.13515, 1.07379)$, neutral saddle;
- 2. at $(\alpha, \beta) = (1.20245, 1.14678)$, double real leading eigenvalue (with respect to stable eigenspace, which is non-determining);
- 4. at $(\alpha, \beta) = (1.74917, 1.76178)$, neutral saddle-focus;
  $\delta = 1 \Rightarrow$ transition to chaotic bifurcation
- 5. at $(\alpha, \beta) = (6.00000, 7.191375)$, neutrally-divergent saddle-focus
  $\delta = 1/2$
  $\Rightarrow$ div$Df(0)x < 0 \Rightarrow$ attracting (strange attractor)

Later we will study a similar simple electronic circuit due to Friere *et al* 1993

. . .

2. Homoclinic orbits to equilibria

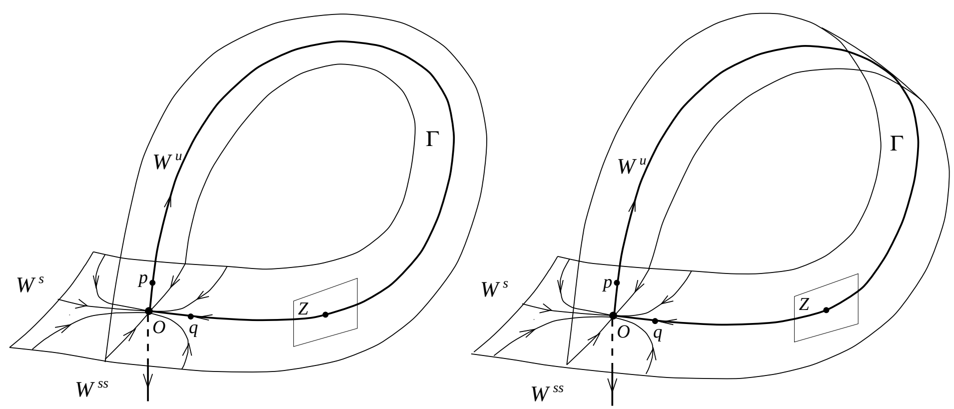
- Heteroclinic orbit Γ connecting equilibria $x_1, x_2 \in \mathbb{R}^n$

$$\dot{x}(t) = f(x(t), \alpha)$$

$$x(t) \to x_1,\ x_2 \quad \text{as } t \to \pm\infty.$$

'generic' system: no symmetry or first integrals

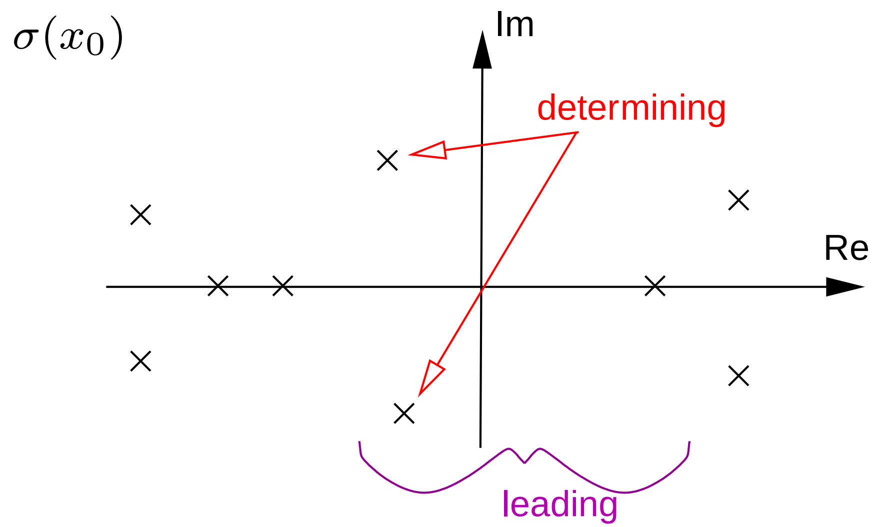
- Homoclinic orbit special case $x_1 = x_2 = x_0$
x_0 hyperbolic $\Rightarrow$ codim 1, i.e. at isolated $\alpha = \alpha_0$



(a) (b)

Homoclinic bifurcation as α varies

Suppose $\exists$ Hom orbit Γ at $\alpha = \alpha_0$. Linearisation at x_0:



Theorem 1 (Shil'nikov's tame homoclinic bifurcation)
Real determining eigenvalue $\Rightarrow$ unique periodic orbit destroyed at infinite period as $\alpha \to \alpha_0$.

Theorem 2 (Shil'nikov's chaotic homoclinic bifurcation)
Complex determining eigenvalue $\Rightarrow$ ∞-many high-period periodic orbits in neighbourhood of Γ and α_0.

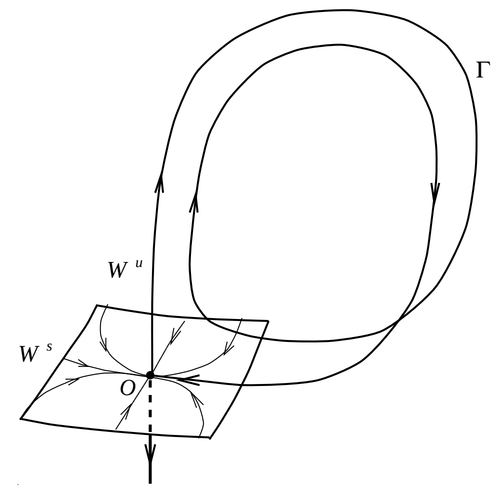
Exercise

Proof of tame homoclinic bifurcation in 2D system using Poincaré maps.
Consider

$$\begin{aligned}\dot{x} &= \lambda x + \text{ nonlinear} \\ \dot{y} &= -\mu y + \text{ nonlinear}\end{aligned}$$

Make assumption that at parameter value $\alpha = 0$, there is a homoclinic orbit that connects this equilibrium to itself.

- In chaotic case $\exists$ ∞-many N-pulse homoclinic orbits at nearby α-values. For each N



e.g. 2-pulse homoclinic orbits Gaspard

- more recently: Turaev , Sandstede 00 see (Shil'nikov et al 1992, 1998)

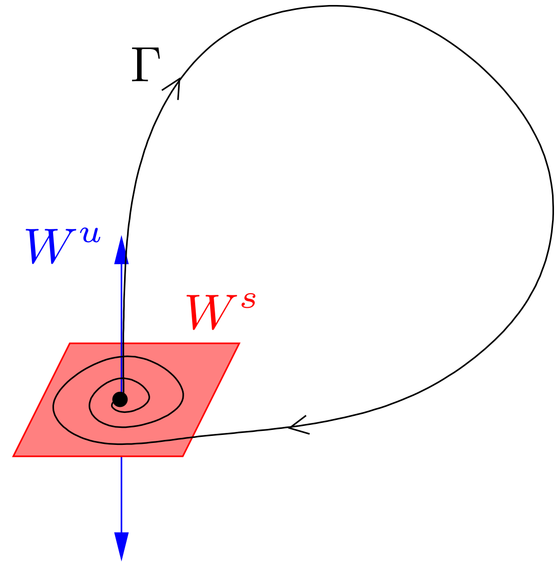
Theorem 3 (Homoclinic 'centre manifold' theorem) *There exists a C^0 manifold of dimension of the leading eigenspace in the neighbourhood of Γ that captures all nearby recurrent dynamics.*

Sketch proof of Shil'nikov chaotic case

saddle focus case in $\mathbb{R}^3$ (Glendinning & Sparrow 84)

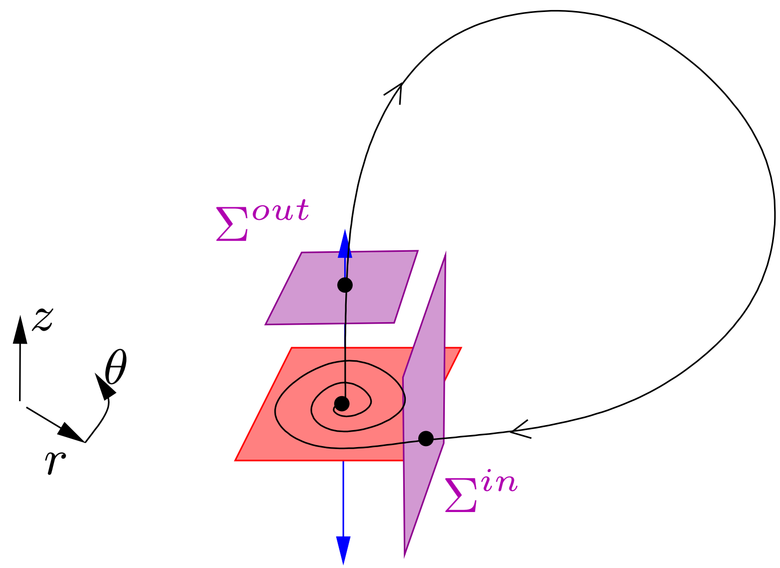
- assume $\exists\ \alpha_0 = 0$ at which hom orbit Γ to $x_0 = 0$ WLOG
- $\sigma(0) = \{-\mu \pm i\omega, \lambda\}$ (WLOG reverse time if nec.)
- Construct Poincaré map close to Γ in α and x
- Fixed points $\Rightarrow$ periodic orbits

construct Poincaré map



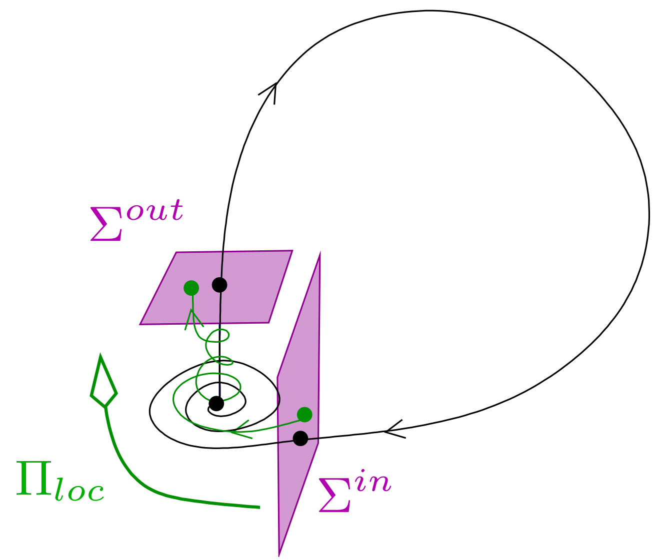
Step 1: Set up Poincaré sections

$$\Sigma^{in} = \{\theta = 0\}, \qquad \Sigma^{out} = \{z = h\}$$



Step 2 Linearise flow near 0 to compute $\Pi_{loc} : \Sigma^{in} \to \Sigma^{out}$

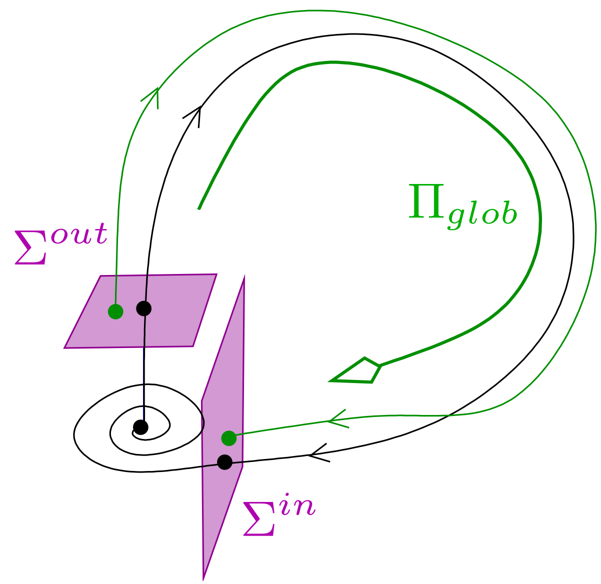
$$\begin{aligned} \dot{z} &= \lambda z \\ \dot{\theta} &= \omega \qquad\qquad + \text{h.o.t} \\ \dot{r} &= -\mu r \end{aligned}$$



Step 3 $z(T) = z_0 e^{\lambda T},\ r(T) = r_0 e^{-\mu T},\ \theta(T) = \theta_0 + \omega t$

'time of flight' $T = \frac{1}{\lambda} \ln\left(\frac{z_0}{h}\right)$. $\delta = \mu/\lambda < 1$ for chaotic case

$$\Rightarrow \quad \Pi_{loc} : (r, \theta, z) \mapsto \left(r\left(\frac{r}{h}\right)^{\delta}, \frac{\omega}{\lambda} \ln\left(\frac{h}{z_0}\right), h \right)$$

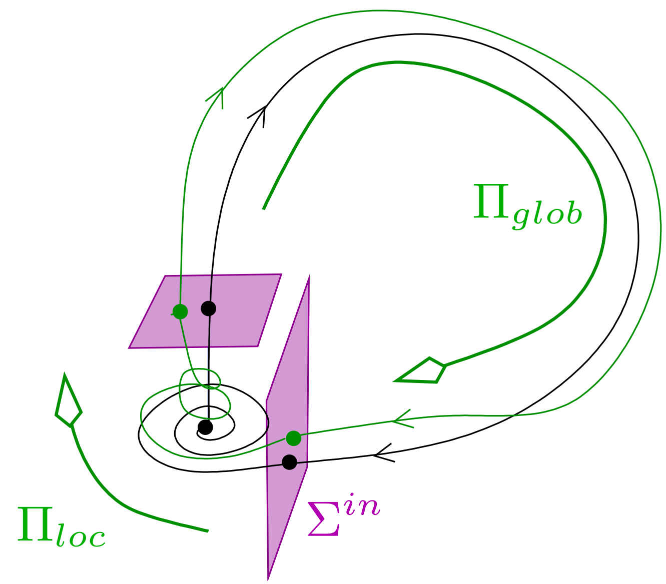


Step 4:

Computation of $\Pi_{glob} : \Sigma^{out} \to \Sigma^{in}$

Assume diffeomorphism; expand as Taylor series

$$\begin{pmatrix} r \\ \theta \\ h \end{pmatrix} \mapsto \begin{pmatrix} \overline{r} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \alpha + \begin{pmatrix} c & d \\ 0 & 0 \\ e & f \end{pmatrix} \begin{pmatrix} r\cos\theta \\ 0 \\ r\sin\theta \end{pmatrix} + \text{h.o.t}$$



Step 5 Poincaré map $\Pi : \Sigma^{in} \to \Sigma^{in} = \Pi_{glob} \circ \Pi_{loc}$

$$\begin{pmatrix} r \\ z \end{pmatrix} \mapsto \begin{pmatrix} \bar{r} \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \alpha + \begin{pmatrix} c_1 r z^\delta \cos(k_1 \ln z + \phi_1) \\ c_2 r z^\delta \cos(k_2 \ln z + \phi_2) \end{pmatrix}$$

dynamics of the Poincaré map

- search for fixed points: r-dynamics slaved to z
- $\Rightarrow$ 1D map for z (nb. period $\sim -\ln z$)

$$(z - b\alpha) = \Phi(z) = K z^{\delta} \cos(k \ln z + \phi) + \text{h.o.t}$$

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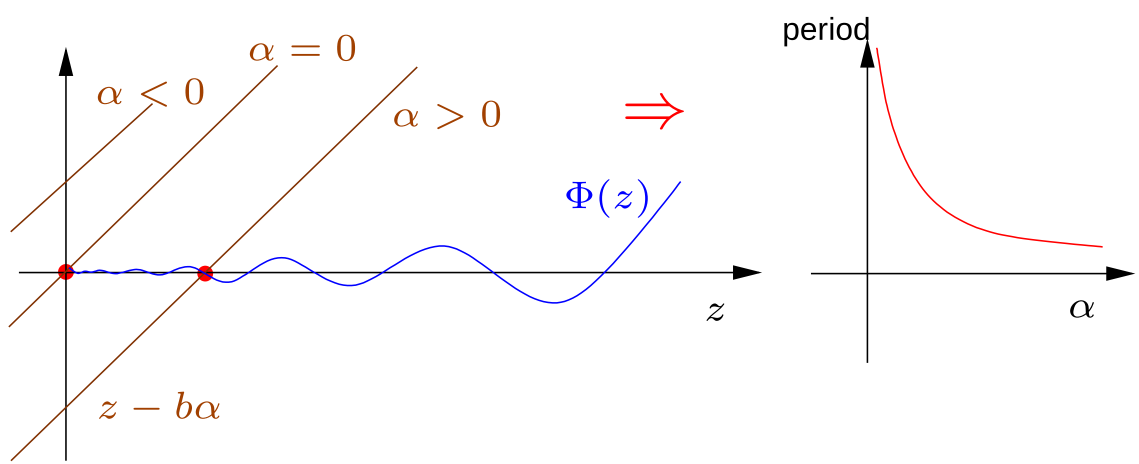
- $\delta > 1$ (tame case)



unique periodic orbit bifurcates

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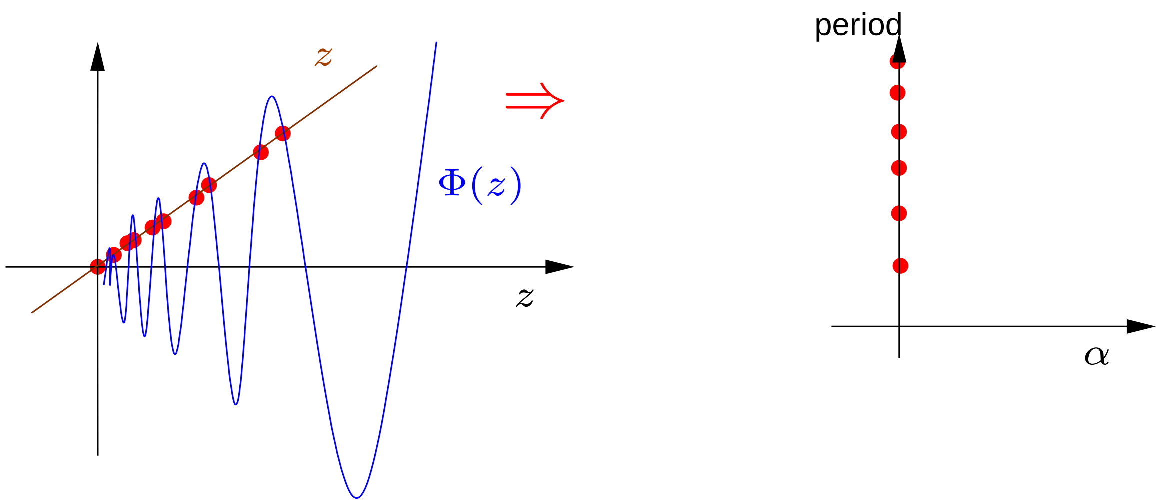

infinitely many periodic orbits for $\alpha = 0$

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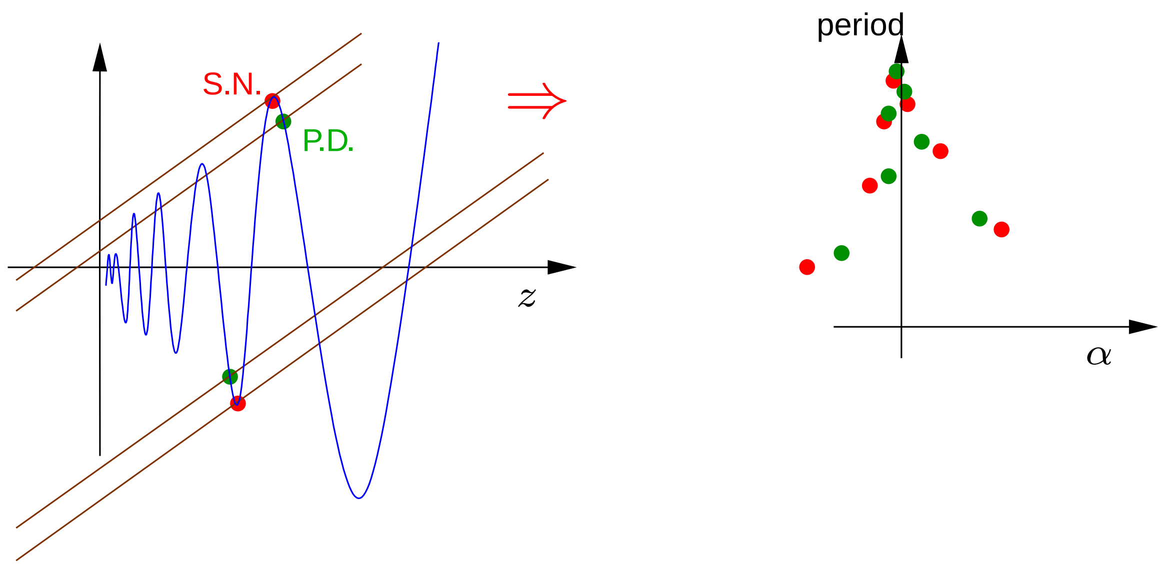

∞-many saddle-node & period-doubling as $\alpha \to \alpha_0$

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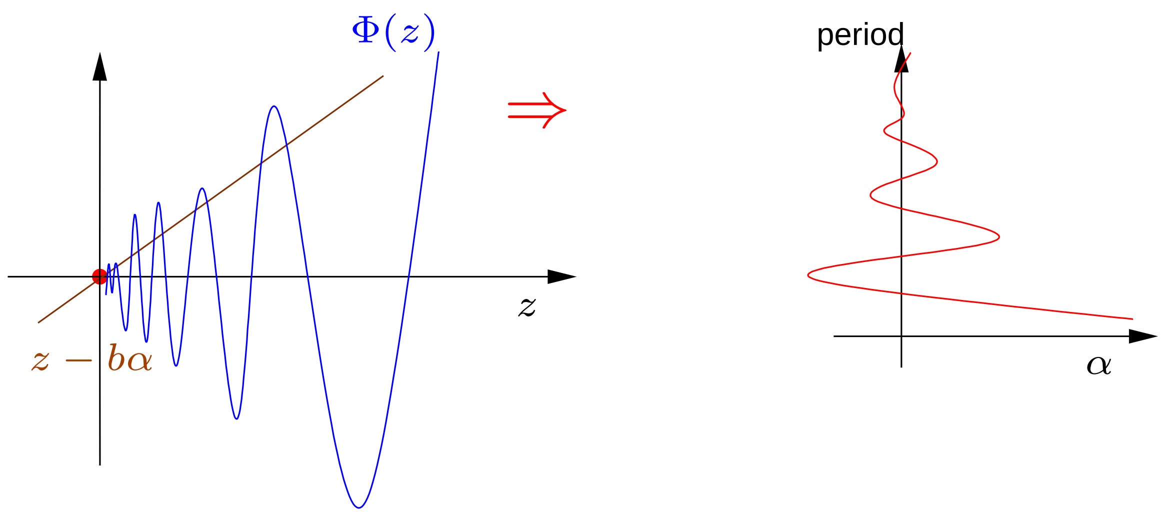
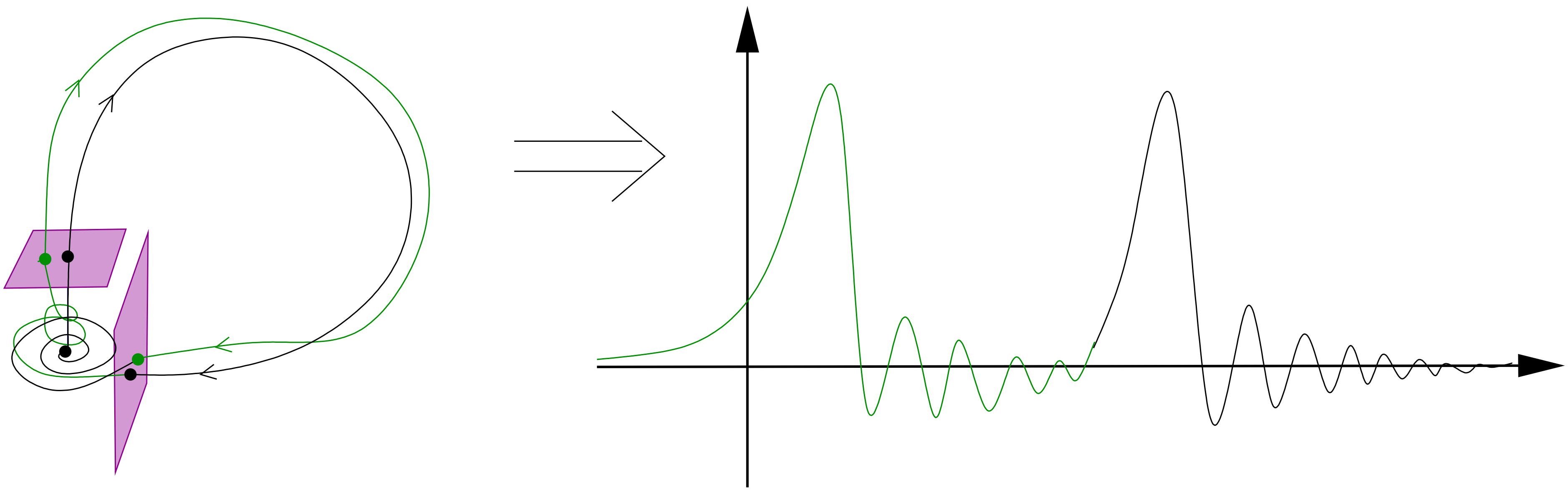


single 'wiggly curve' of periodic orbits. Also symbolic dynamics on ∞-many symbols

Multi-pulses

Infinitely many parameter values $\alpha_i^{(2)}$, $i = 1, \ldots \infty$, converging as $\alpha \to \alpha_0$ from both sides at which there exist 2-pulse homoclinic orbits.



$\ldots$ and N-pulses for all N $\ldots$

A word about rigour

Linearisation to compute Π_{loc} cannot be rigorously justified in general.

Hartman-Grobman Theorem gives only C^0 topological equivalence

Three rigorous approaches:

- Sometimes (non-resonance) can justify linearisation
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- Use normal vector to homoclinic centre manifold (adjoint) to project
  Lin's method or HLS: 'Homoclinic Lyanpunov-Schmidt'
  'Hale, Lin, Sandstede' (Lin 2008)

Application: excitable systems

- Small input $\Rightarrow$ gradual relaxation
  Large enough $\Rightarrow$ burst $+$ gradual relaxation

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- the ‘pendulum equation’ of excitable systems:
  Fitz-Hugh Nagumo (FHN) system (1961-2)

$$\begin{aligned} v_t &= Dv_{xx} + f_\alpha(v) - w + c \\ w_t &= \varepsilon(v - \gamma w) \end{aligned}$$

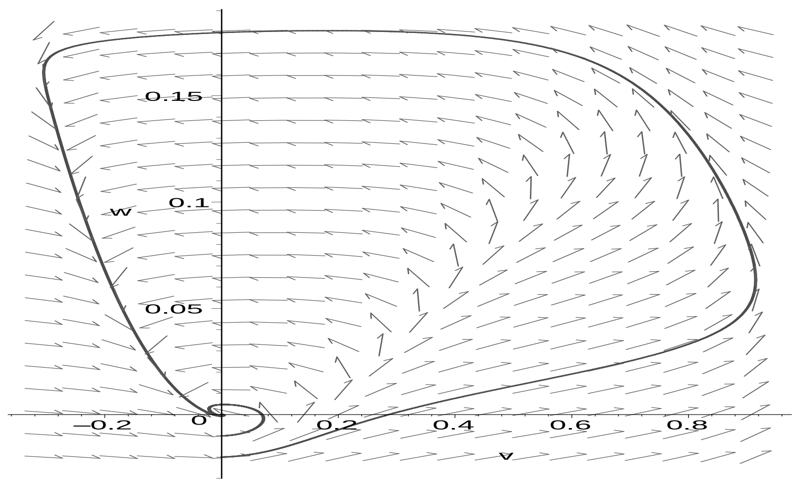
$f_\alpha(v) = v(v-1)(\alpha - v)$, e.g. $\alpha = 0.1$, $\gamma = 1.0$, $\varepsilon = 0.001$.

example FitzHugh Nagumo equations

A. spatially homogeneous ($D = 0$) dynamics

$$v_t = f_\alpha(v) - w + p$$
$$w_t = \varepsilon(v - \gamma w)$$

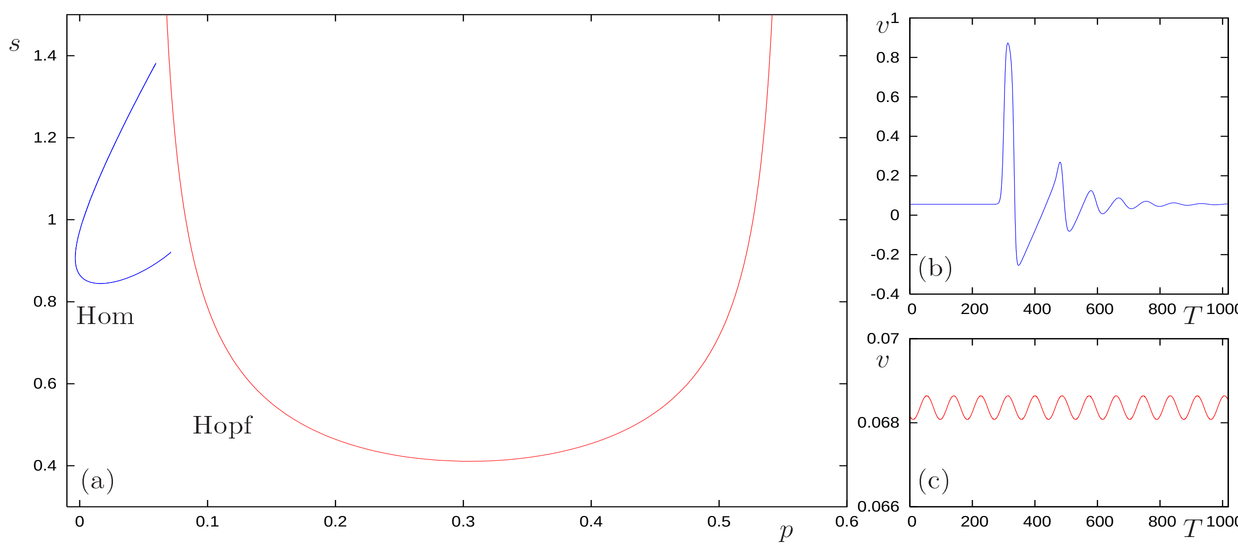
isoclines: $v = \gamma w$, $f_\alpha(v) + p = w$. $\Rightarrow$ excitable



B. travelling structures (for $D > 0$) $z = x + st$ ($\cdot = d/dz$)

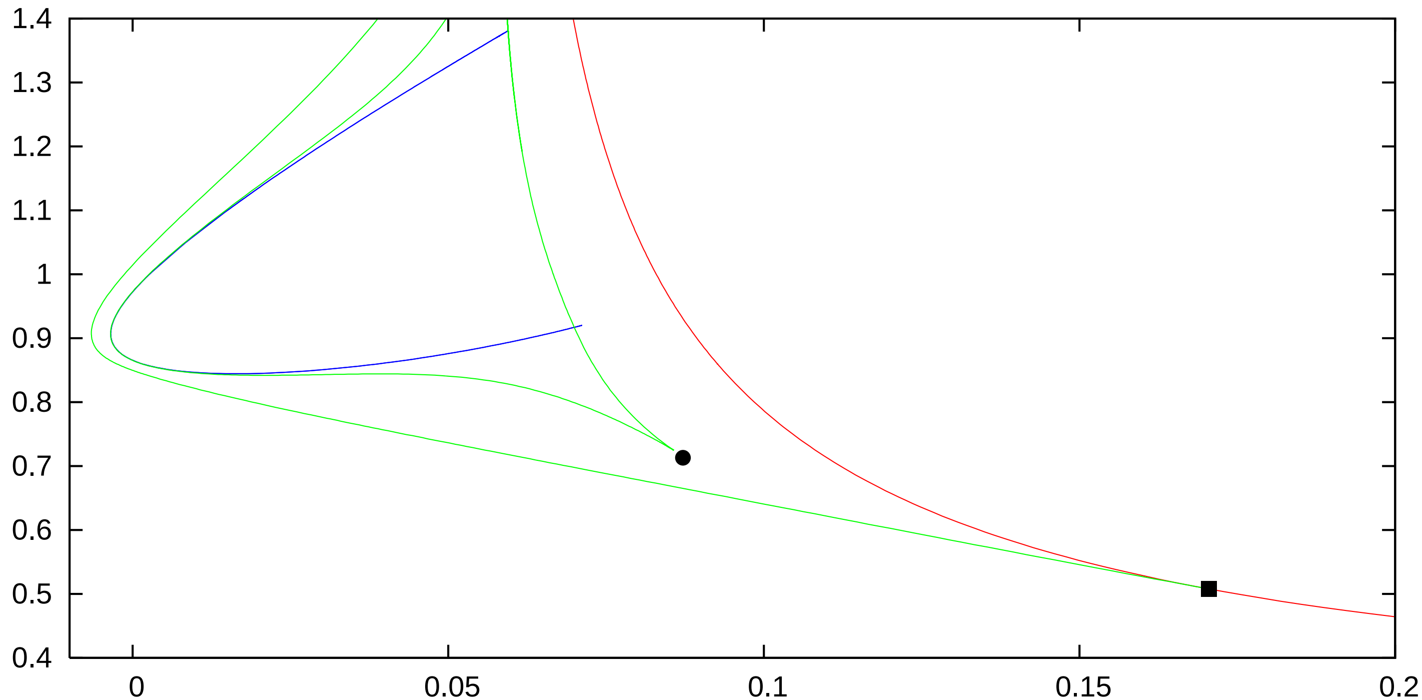
$$\begin{aligned} \ddot{v} - s\dot{v} &= -f_\alpha(v) + w - p \\ \dot{w} &= (\varepsilon/s)(v - \gamma w) \end{aligned}$$

node $\rightarrow$ saddle: 2 kinds of travelling wave:
periodic wave trains $\Leftarrow$ Hopf bifurcation
pulse solution $\Leftarrow$ homoclinic orbit to saddle



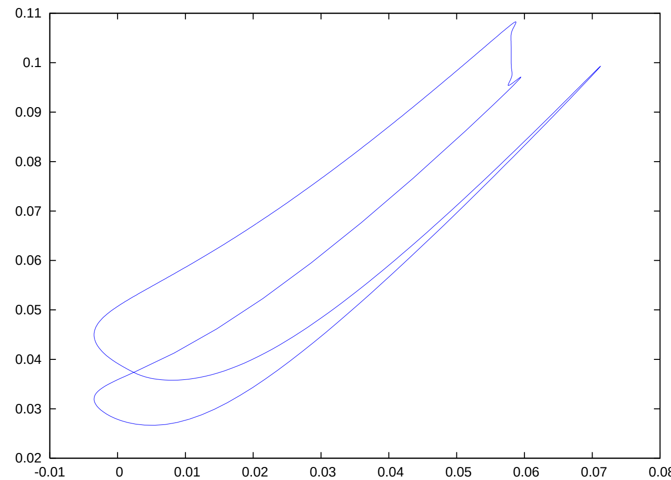
This C U structure common for excitable models

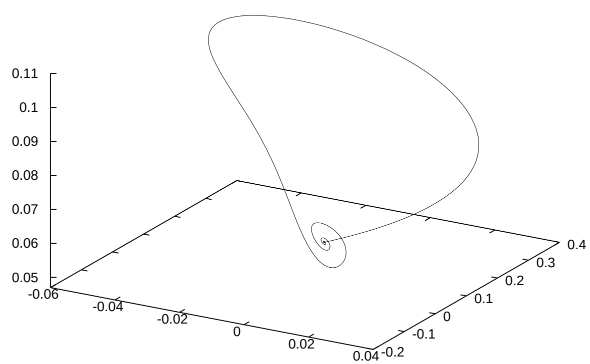
But how does the Hom curve end as it approaches Hopf?
More details: (p vs. wavespeed s)

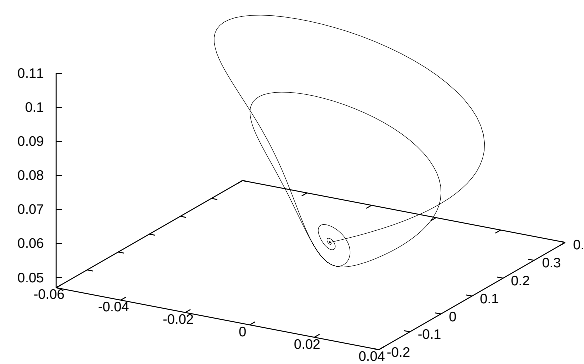


(see Champneys, Kirk et al 07)

The hom curve doubles back on itself(!) and gains an extra large loop in so doing (p vs. ‘norm’)







two homoclinics for $p \approx 0.06$, $s \approx 0.894386$.

example 2: 8-variable Ca^{2+} model

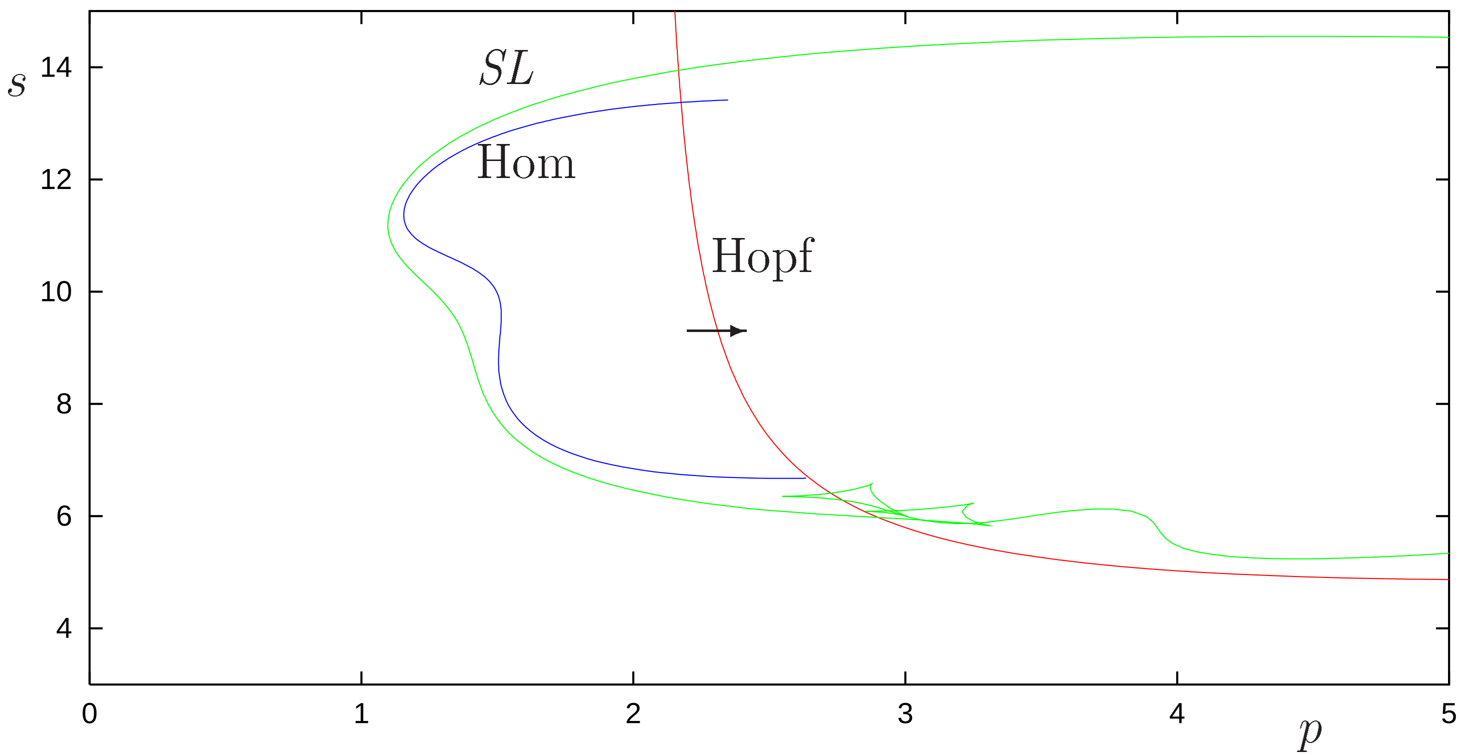
- Sneyd, Yule *et al* Calcium waves in pancreatic acinar cell

$$
\begin{aligned}
\frac{\partial c}{\partial t} &= D\frac{\partial^2 c}{\partial x^2} + k_1(G)(c - c_e) + J_1(c, G) \\
\frac{dc_e}{dt} &= -k_2(G)(c_e - c) + J_2(c, G) \\
\frac{dG}{dt} &= k_3(p, c)G
\end{aligned}
$$

$c(x, t)$ concentration of Ca^{2+}. $c_e(t)$ concentration in boundary. $G(t) \in \mathbb{R}^6$ receptor variables.

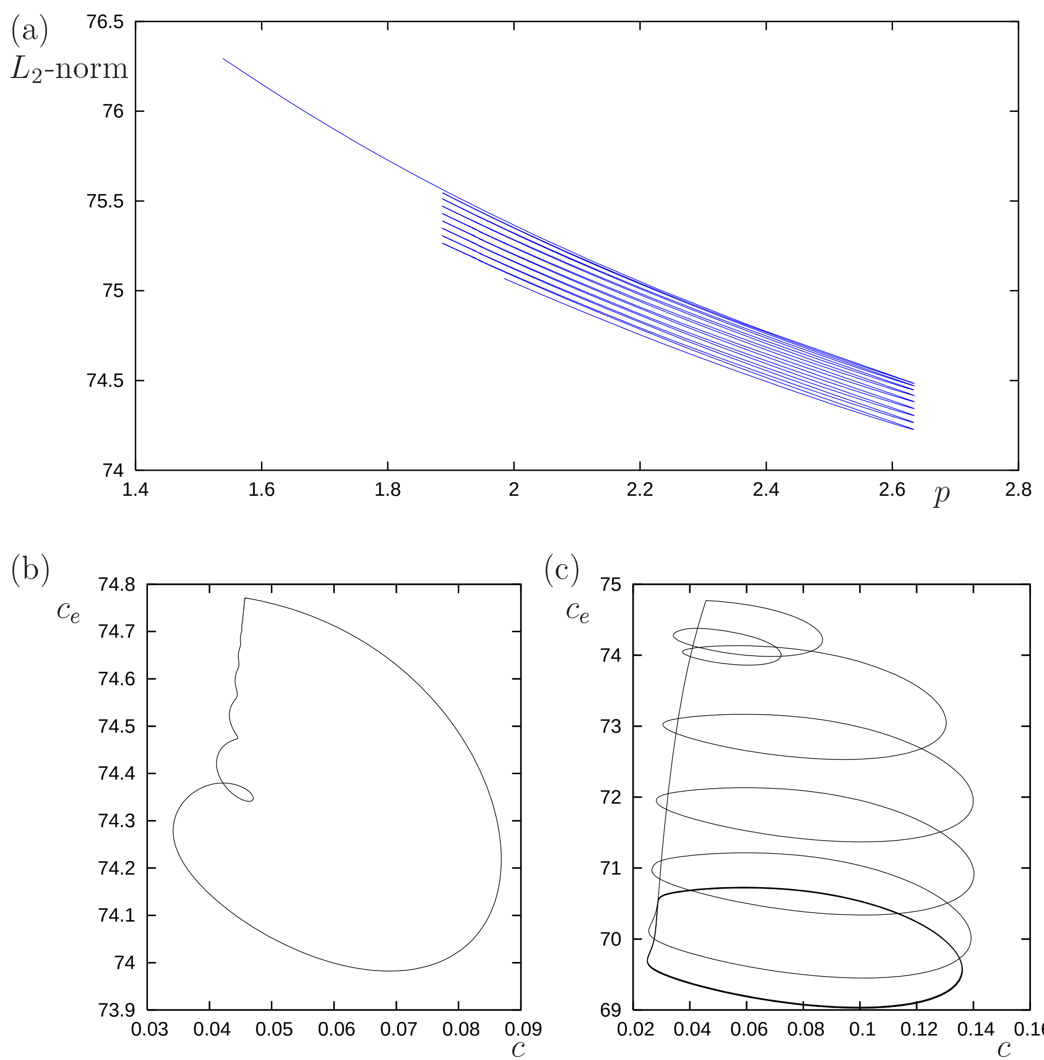
- Travelling waves $\xi = x - st \Rightarrow$ 9D ODE system
- Bif. pars: wavespeed s, IP$_3$ concentration p

Similar C-U bifn diagram.



at upper end: Hom curve passes straight through Hopf!?

And at lower end, the homoclinic curve doubles back on itself ∞ many times.



4. Reversible and Hamiltonian systems

- Reversible systems

$\dot{x} = f(x), x \in \mathbb{R}^{2n}, Rf(x) = -f(Rx), R^2 = \text{Id}, \mathcal{S} = \text{fix}(R) \cong \mathbb{R}^n.$

$\Rightarrow$ *symmetric* homoclinic orbits are codim 0 (Devaney)

$\gamma(t) \to x_0$ as $t \to \pm\infty, \gamma(0) \in \mathcal{S},$ where $f(x_0) = 0, x_0 \in \mathcal{S}.$

- Hamiltonian systems

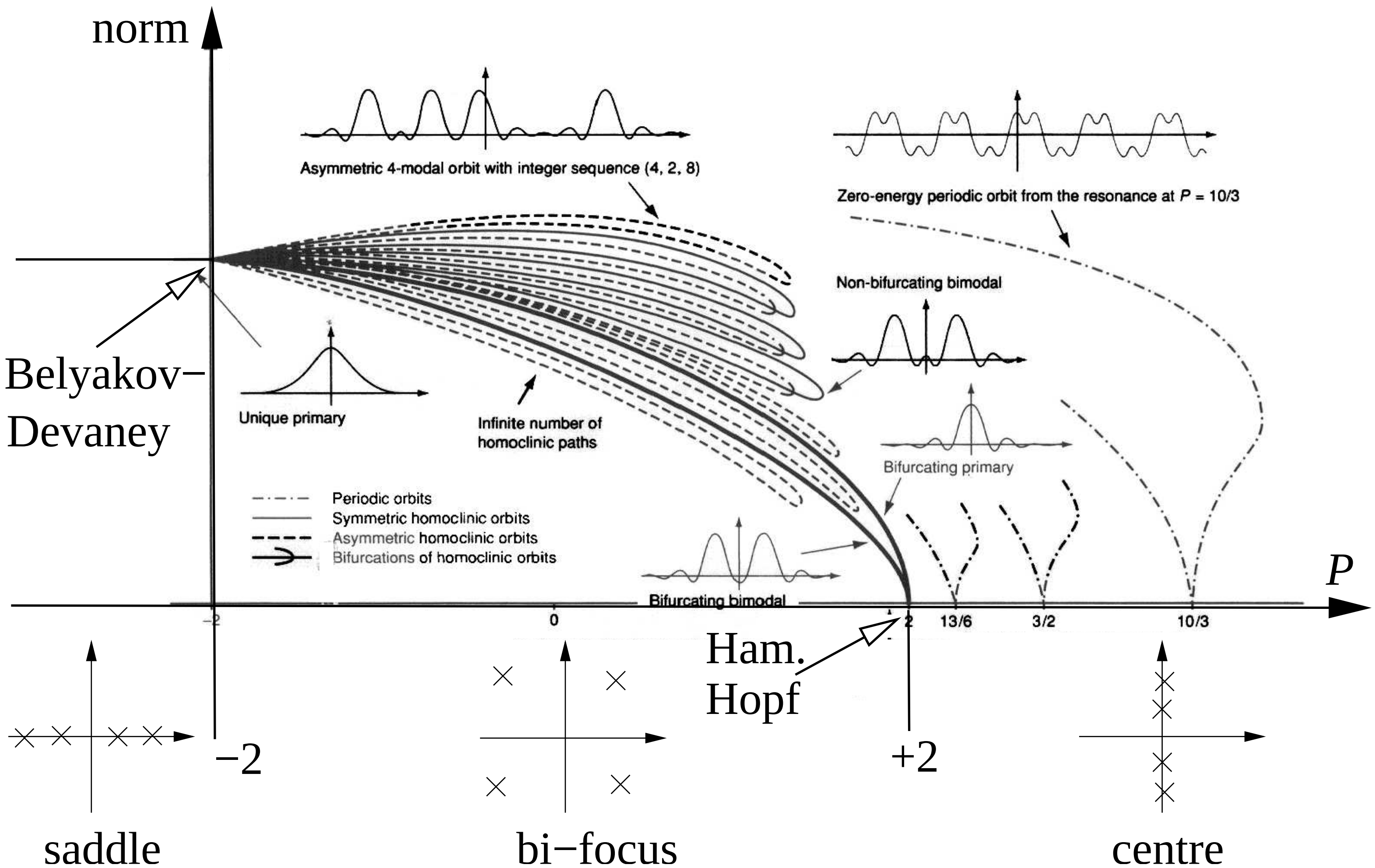
$\dot{x} = f(x) = J\nabla H(x), \quad x \in \mathbb{R}^{2n}, \qquad \Rightarrow H(x(t)) =$ const.

$\Rightarrow W^u$ & W^s live in $H^{-1}(x_0) \Rightarrow$ codim 0

- In either case, $\sigma(x_0)$ symmetric w.r.t. Im-axis
- Everything happens with one codimension less

a fourth-order example

Buffoni, C. & Toland 96 $u^{iv} + Pu'' + u - u^2 = 0$

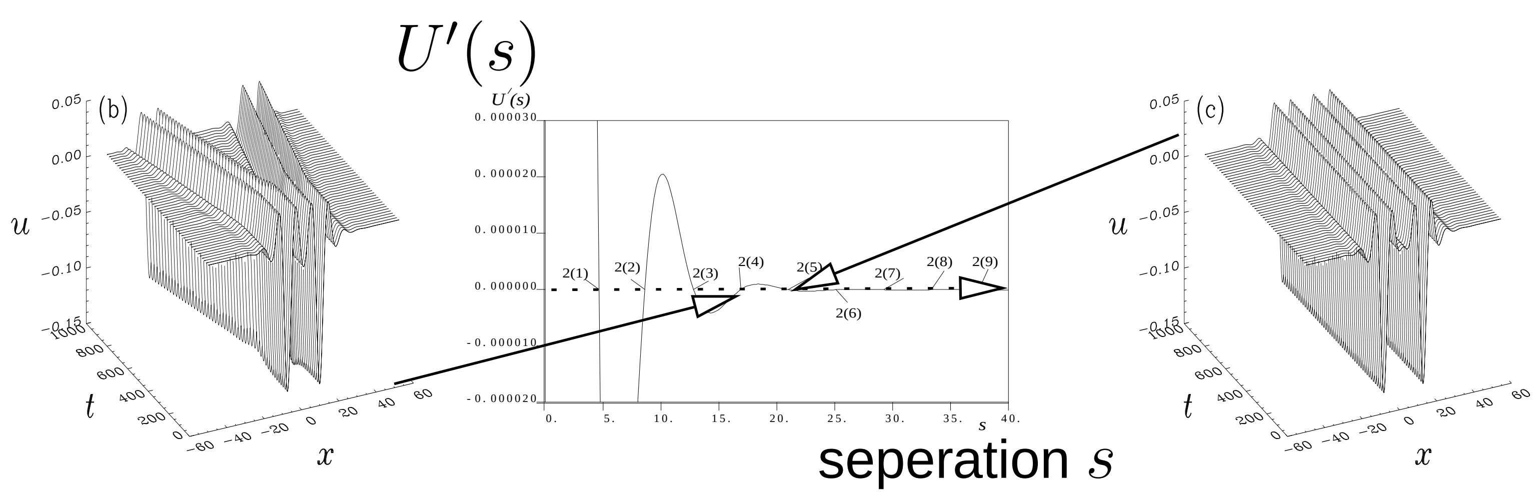


Application 1: wobbly bridges

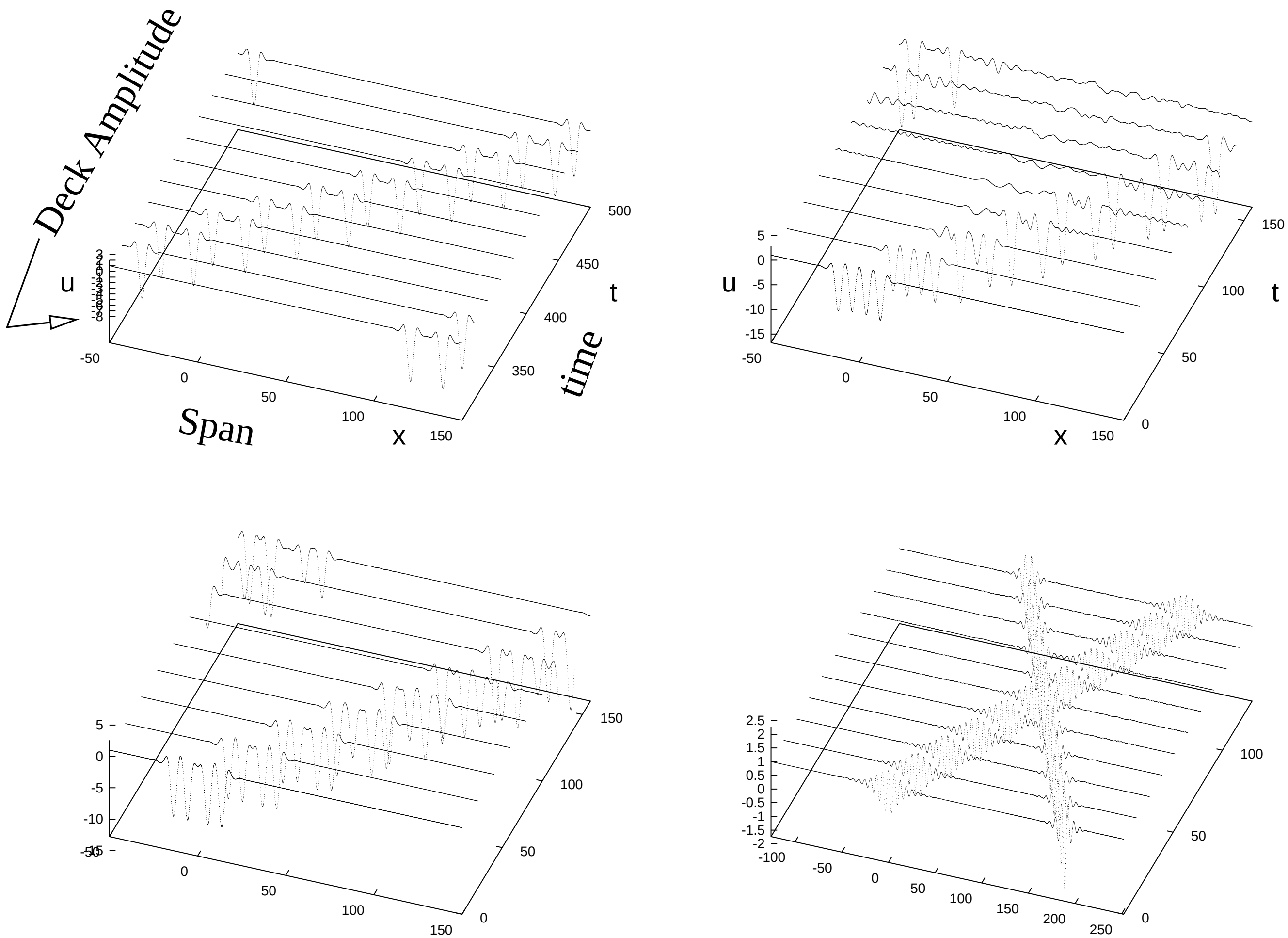
- Goldern gate bridge 1938, Chief Engineer R.G. Cone

  a wind of unusual high velocity was blowing . . . normal to the axis of the bridge . . . The suspended structure of the bridge was undulating vertically in a wave-like motion of considerable amplitude, . . . a running wave similar to that made by cracking a whip. The truss would be quiet for a second, and then in the distance one could see a running wave of several nodes approaching . . .

- C. & McKenna 96 asymmetric beam model
- Stable multi-humped solitary waves (Buryak & C 96)



- Nb. R.G. Cone was sacked for ‘disloyalty to the bridge’

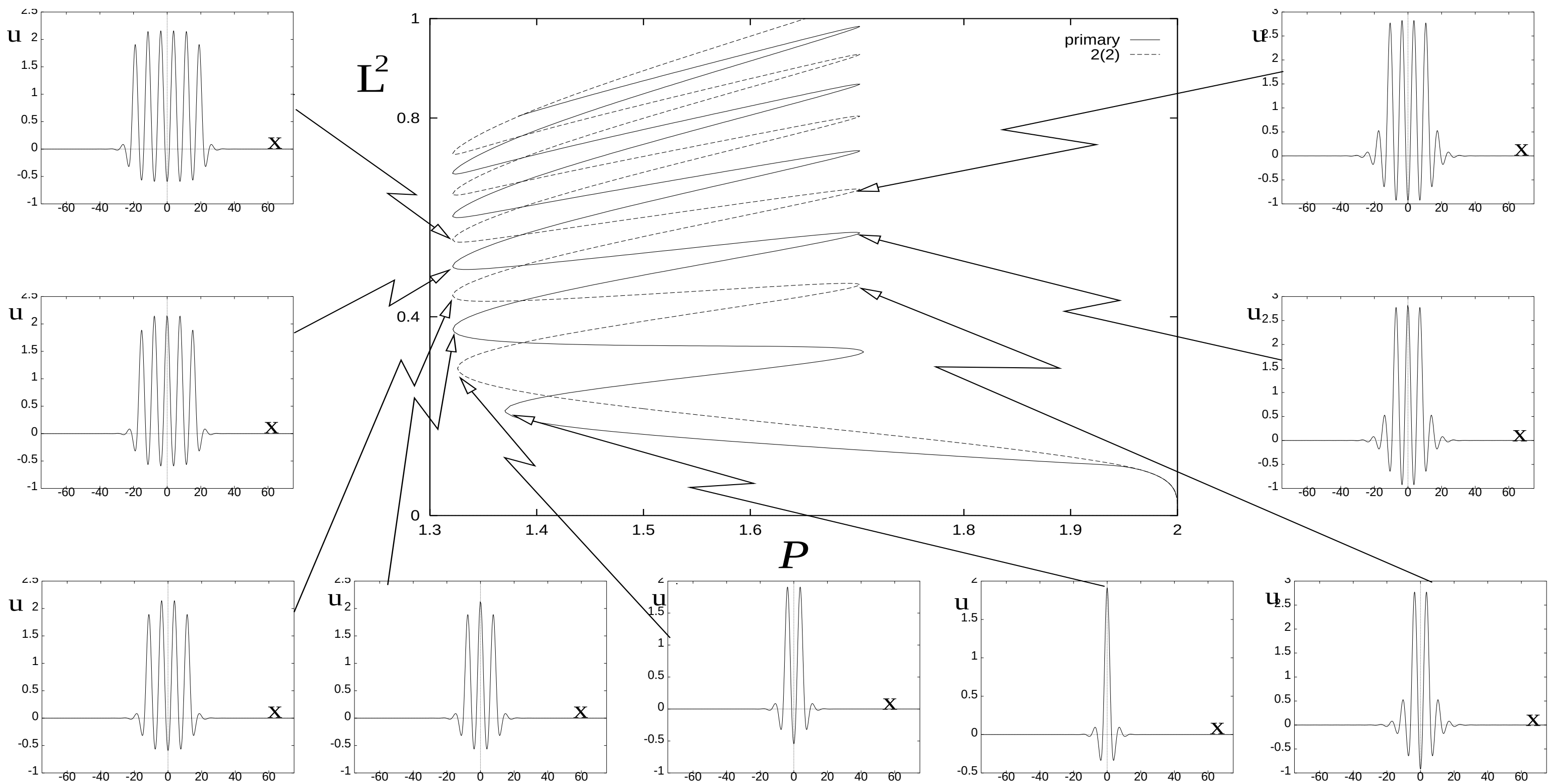


. . . add competing nonlinearity

(Woods & C. 99)

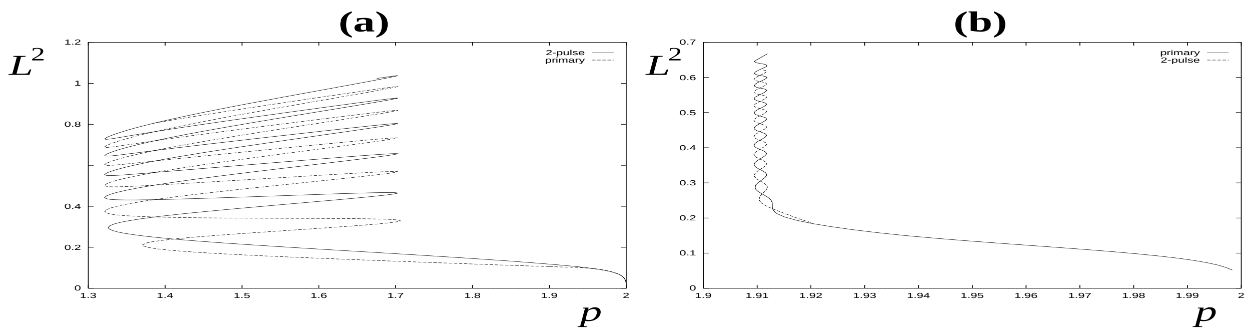
$$u_{xxxx} + Pu_{xx} + u - u^2 + bu^3 = 0$$

$\frac{2}{9} < b < \frac{38}{27}$. $b = 0.29$:

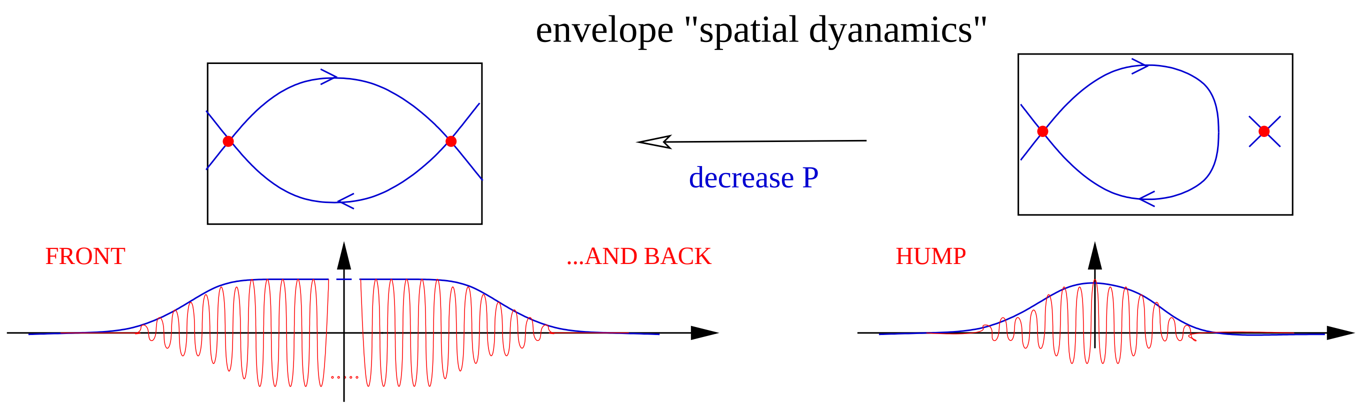


a small amplitude limit

- $b \to (38/27) \Rightarrow$ narrow snake



- For $P \approx 2$, $b \approx (38/27)$ normal form theory



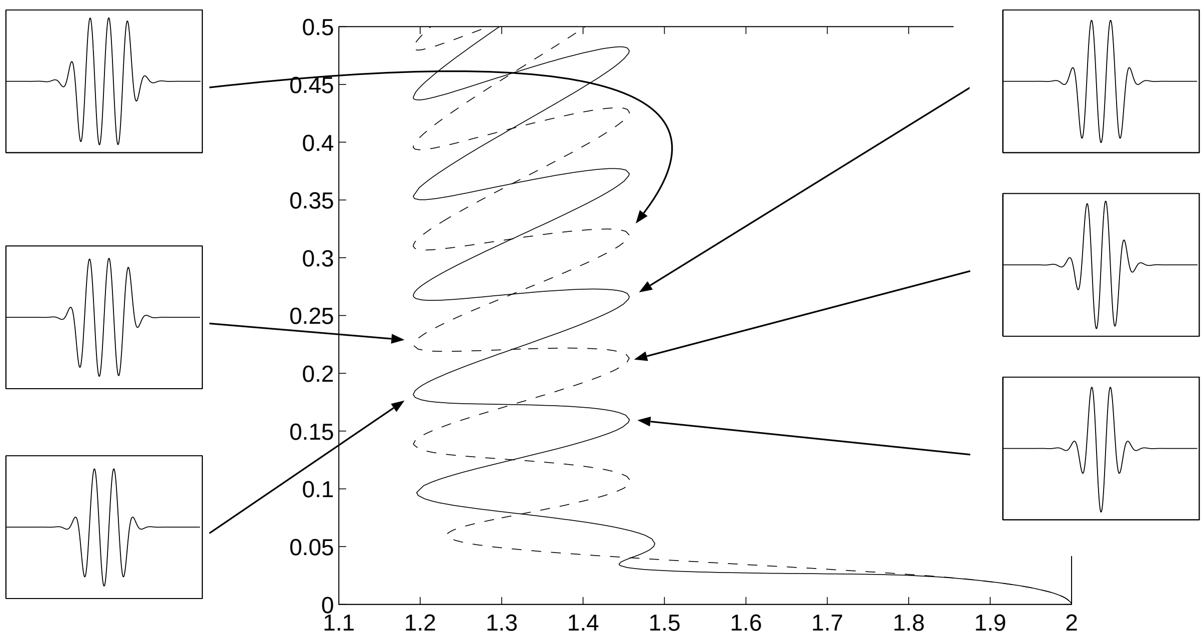
- But misses beyond-all-orders effects.

a related problem

Similar results for

$$u_{xxxx} + P u_{xx} + u - \alpha u^3 + u^5 = 0$$

$\alpha = 3/10$ (Hunt et al 00)

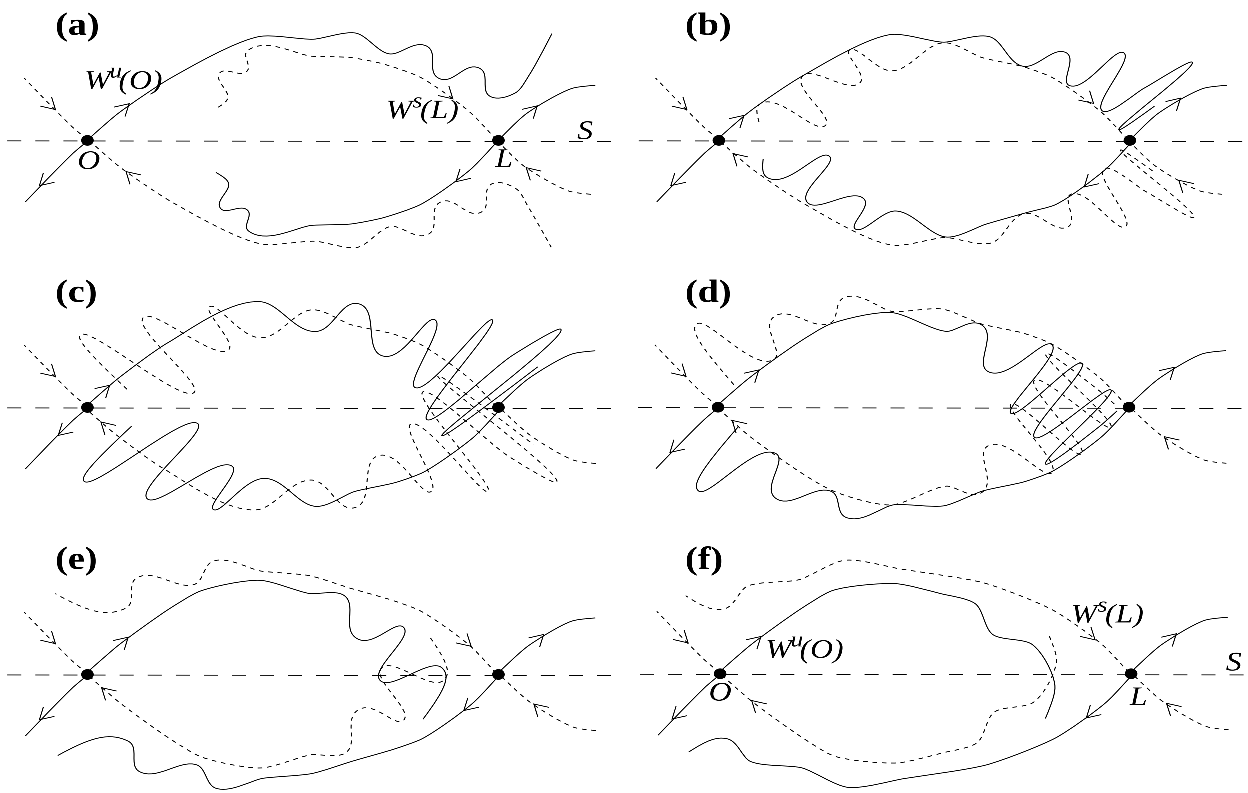


See also Burke & Knobloch 07

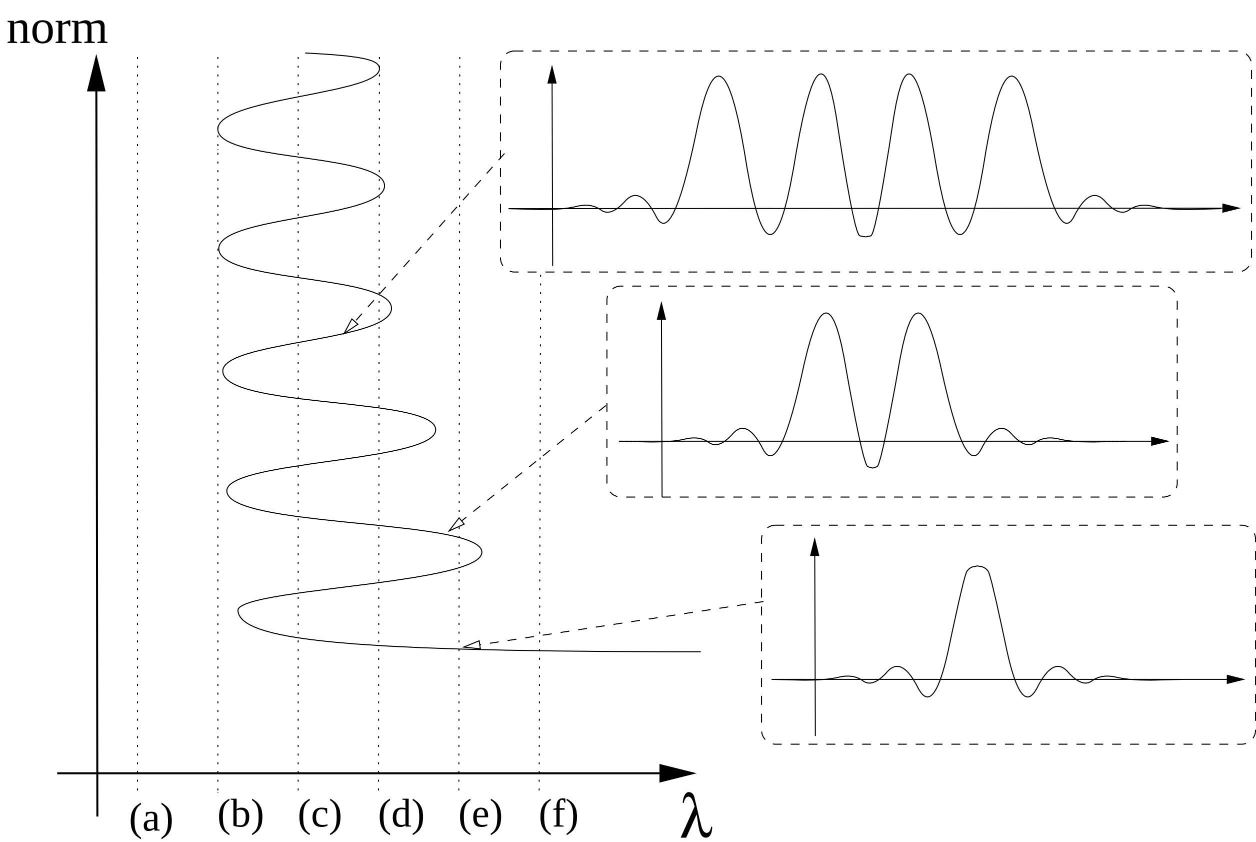
spatial dynamics explanation of snake

Woods & C. 99 (cf. Coullet et al 2000)

Hamiltonian case. Take 2D Poincaré section within $\{H = 0\}$:



cf. Poincaré “heteroclinic tangle”



See Burke & Knobloch 06, Beck et al 09 for application to localised patterns in Swift-Hohenberg eqns. (snakes and ladders!)

4. Computing homoclinic/heteroclinics

- 3 simple special cases approaches in AUTO

- compute a periodic orbit to large period
- case of 1D unstable manifold
- reversible case

Next lecture: HomCont — general AUTO-07P method for computing homoclinic orbits & detecting codimension-two points

Computing large-period periodic orbits

- AUTO solves for periodic orbits via boundary-value problem

$$\dot{x} = Tf(x,\alpha), \quad x(0) = x(1), \quad x(0) \int_0^1 (u(t)^T \dot{\tilde{u}})dt$$

for $x(t)$ in $\mathbb{R}^n$, parameter $\alpha \in \mathbb{R}$, $T \in \mathbb{R}$.

- Homoclinic bifurcation $\Rightarrow T \to \infty$
- To compute homoclinic; fix T (large), solve for $\alpha_1, \alpha_2 \in \mathbb{R}^2$.
- Can show not the optimal choice (see next lecture)
- this afternoon: AUTO demo **pp2**.

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1D unstable manifold homoclinics case

- Suppose $A = Df(x_0, 0)$ has $n_s = 1$ unstable eigenvalue λ (and $n_u = n - 1$ stable eigs)
- Let $Av_1 = \lambda v_1$, $A^T w_1 = \lambda w_1$.
- Compute boundary value problem

$$\begin{aligned} \dot{x} &= 2Tf(x, \alpha) \\ x(0) &= x_0 + \varepsilon v_1 \\ 0 &= w_1^T (x(1) - x_0) \end{aligned}$$

- can show convergence as $2T \to \infty$ (see next lecture)
- $\Rightarrow$ continuation problem with $n + 1$ boundary conditions for $n + 2$ unknowns $x(t)$ α_1, α_2.

Reversible case

Consider reversible homoclinic x(t)

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^{2n}, \quad x(0) \in \text{fix}(R), \qquad x(\pm\infty) \to x_0$$

Truncate to $[-T, 0]$ and solve the two-point BVP with n B.C.s:

$$\begin{aligned} \dot{x} &= f(x, \alpha) \\ L_u x(-T) - x_0 &= 0 \\ D\text{fix}(R)^{\perp} x(0) &= 0 \end{aligned}$$

where L_u is projection onto unstable eigenspace of A (using stable eigenspace of A^T
e.g. for 4^{th}-order example $x = (u, u', u'', u''')$,
fix $R = (u, 0, u'', 0)$. D fix $(R)^{\perp} = (0, 1, 0, 1)$.
$\Longrightarrow$ computes a curve of solutions in one parameter α_1

What we have learnt so far:

- Homoclinic orbits to equilibria can be tame or chaotic
- Chaotic case leads to birth of multi-pulses
- Everything depends on linearisation (+ twistedness - see next lecture)
- Topological ideas can be posed rigorously analytically
- Hamiltonian (and reversible) case drops a co-dimension
- Many applications

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