Continuation of cycle-to-cycle connections in 3D ODEs

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joint work with E.J. Doedel, B.W. Kooi, and G.A.K. van Voorn

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- Truncated BVP's with projection BC's
- The defining BVP in 3D
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Previous works

- W.-J. Beyn [1994] “On well-posed problems for connecting orbits in dynamical systems.” In *Chaotic Numerics (Geelong, 1993)*, volume 172 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 131–168.
- T. Pampel [2001] “Numerical approximation of connecting orbits with asymptotic rate,” *Numer. Math.* **90**, 309–348.
- L. Dieci and J. Rebaza [2004] “Point-to-periodic and periodic-to-periodic connections,” *BIT Numerical Mathematics* **44**, 41–62.
- L. Dieci and J. Rebaza [2004] “Erratum: “Point-to-periodic and periodic-to-periodic connections”,” *BIT Numerical Mathematics* **44**, 617–618.

2. Truncated BVP's with projection BC's

- Some notations
- Isolated families of connecting orbits
- Truncated BVP
- Error estimate

Some notations

- Consider the (local) flow φ^t generated by a smooth ODE

$$\frac{du}{dt} = f(u, \alpha), \quad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n.$$

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- Let O^+ be a hyperbolic *limit cycle* with $\dim W_+^s = m_s^+$.

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- Let O^- be a hyperbolic *limit cycle* with $\dim W^u_- = m^-_u$.
- Let O^+ be a hyperbolic *limit cycle* with $\dim W^s_+ = m^+_s$.
- Let $x^\pm(t)$ be periodic solutions (with minimal periods $T^\pm$) corresponding to $O^\pm$ and

$$M^\pm = D_x \varphi^{T^\pm}(x)\Big|_{x=x^\pm(0)} \quad (monodromy\ matrices).$$

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- Then $m^+_s = n^+_s + 1$ and $m^-_u = n^-_u + 1$, where n^+_s and n^-_u are the numbers of eigenvalues of $M^\pm$ satisfying $|\mu| < 1$ and $|\mu| > 1$, resp.

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Isolated families of connecting orbits

- Necessary condition: $p = n - m_s^+ - m_u^- + 2$ (Beyn, 1994).

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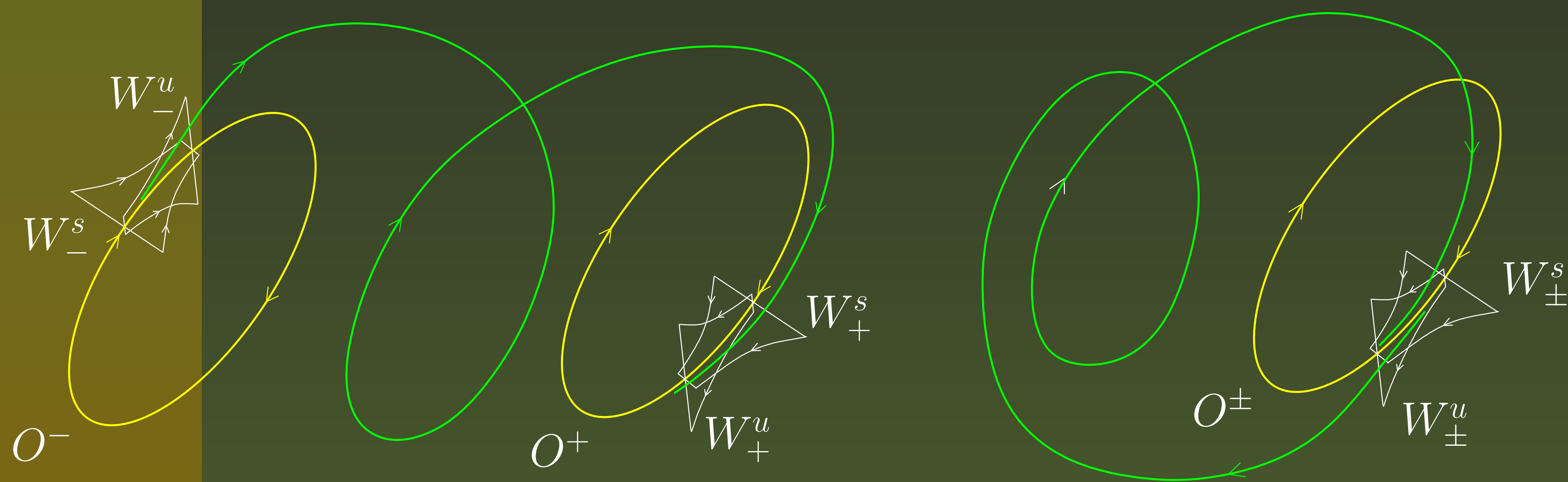

(a) heteroclinic (b) homoclinic

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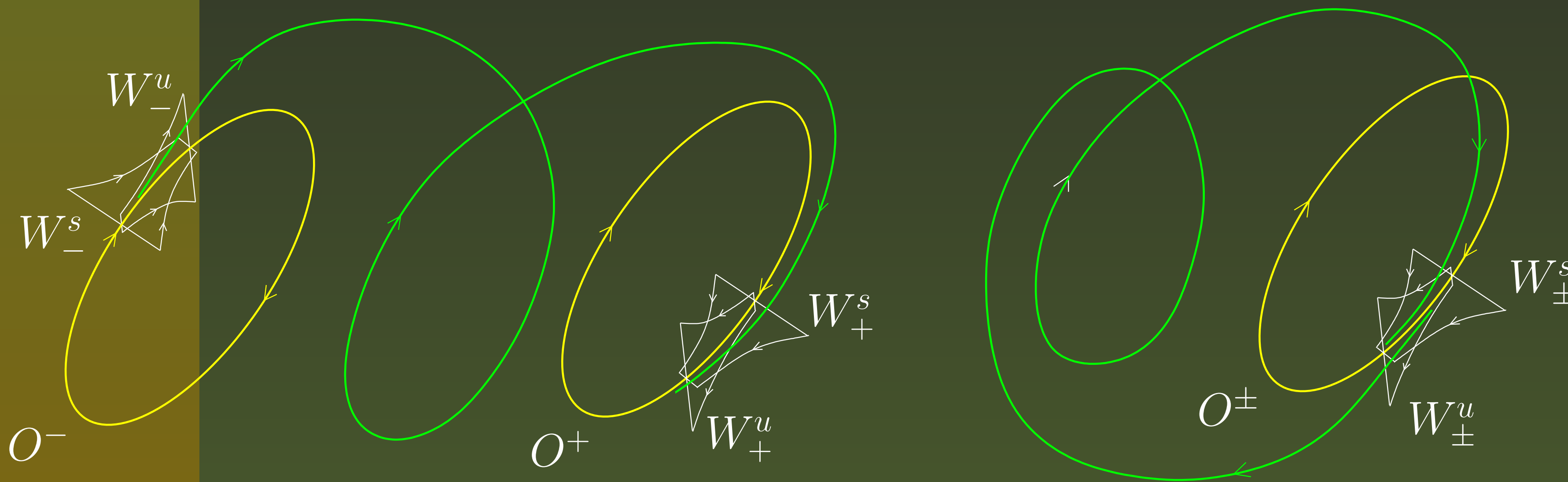

(a) heteroclinic (b) homoclinic

- L.P. Shilnikov [1967] "On a Poincaré-Birkhoff problem," *Math. USSR-Sb.* **3**, 353-371.

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Truncated BVP

- The connecting solution $u(t)$ is *truncated* to an interval $[\tau_-, \tau_+]$.

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- The points $u(\tau_+)$ and $u(\tau_-)$ are required to belong to the linear subspaces that are tangent to the stable and unstable invariant manifolds of O^+ and O^-, respectively:

$$\begin{cases} L^+(u(\tau_+) - x^+(0)) & = & 0, \\ L^-(u(\tau_-) - x^-(0)) & = & 0. \end{cases}$$

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- Generically, the truncated BVP composed of the ODE, the above *projection BC's*, and a *phase condition* on u, has a unique solution family $(\hat{u}, \hat{\alpha})$, provided that the ODE has a connecting solution family satisfying the pahase condition and Beyn's equality.

Error estimate

If u is a generic connecting solution to the ODE at parameter value α, then the following estimate holds:

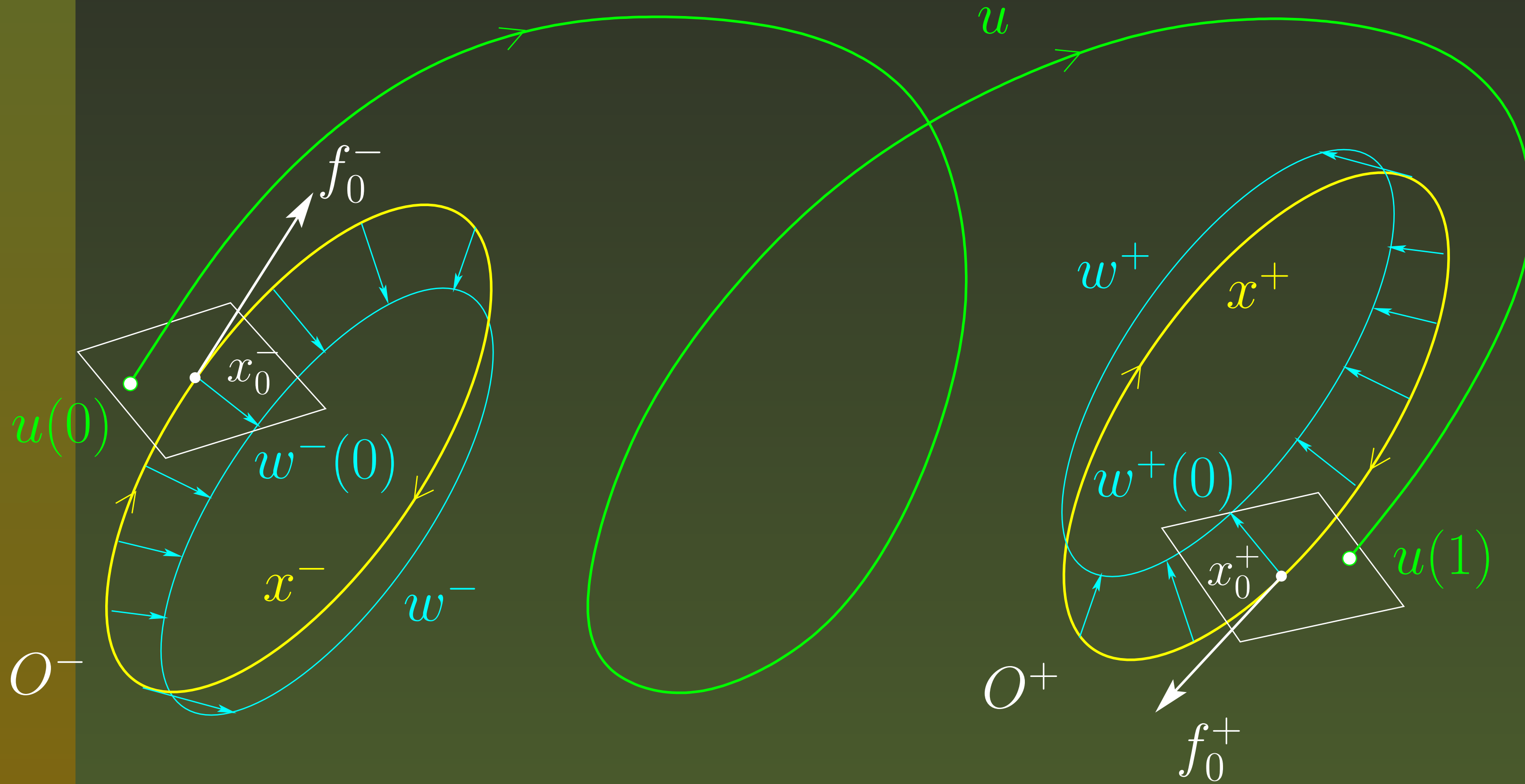
$$\|(u|_{[\tau_-,\tau_+]},\alpha) - (\hat{u},\hat{\alpha})\| \leq C\mathrm{e}^{-2\min(\mu_-|\tau_-|,\mu_+|\tau_+|)},$$

where

- $\|\cdot\|$ is an appropriate norm in the space $C^1([\tau_-,\tau_+],\mathbb{R}^n)\times\mathbb{R}^p$,
- $u|_{[\tau_-,\tau_+]}$ is the restriction of u to the truncation interval,
- $\mu_\pm$ are determined by the eigenvalues of the monodromy matrices $M^\pm$.

(Pampel, 2001; Dieci and Rebaza, 2004)

3. The defining BVP in 3D



It has cycle- and connection-related parts.

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Cycle-related equations

- Periodic solutions:

$$\left\{ \begin{array}{rcl} \dot{x}^{\pm} - f(x^{\pm}, \alpha) & = & 0 \, , \\ x^{\pm}(0) - x^{\pm}(T^{\pm}) & = & 0 \, . \end{array} \right.$$

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- Adjoint eigenfunctions: $\mu^{+} = \frac{1}{\mu_u^{+}}\,,\ \mu^{-} = \frac{1}{\mu_s^{-}}\,.$

$$\left\{\begin{array}{rcl} \dot{w}^{\pm} + f_u^{\mathrm{T}}(x^{\pm},\alpha)w^{\pm} &=& 0\,, \\ w^{\pm}(T^{\pm}) - \mu^{\pm}w^{\pm}(0) &=& 0\,, \\ \langle w^{\pm}(0), w^{\pm}(0)\rangle - 1 &=& 0\,, \end{array}\right.$$

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- Projection BC: $\langle w^{\pm}(0), u(\tau_{\pm}) - x^{\pm}(0) \rangle = 0.$

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Connection-related equations

- The equation for the connection:

$$\dot{u} - f(u, \alpha) = 0 \ .$$

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- We need the base points $x^{\pm}(0)$ to move freely and independently upon each other along the corresponding cycles $O^{\pm}$.

Connection-related equations

- The equation for the connection:

$$\dot{u} - f(u, \alpha) = 0 \; .$$

- We need the base points $x^{\pm}(0)$ to move freely and independently upon each other along the corresponding cycles $O^{\pm}$.
- We require the end-point of the connection to belong to a plane orthogonal to the vector $f(x^{+}(0), \alpha)$, and the starting point of the connection to belong to a plane orthogonal to the vector $f(x^{-}(0), \alpha)$:

$$\langle f(x^{\pm}(0), \alpha), u(\tau_{\pm}) - x^{\pm}(0) \rangle = 0 \; .$$

The defining BVP in 3D: $\lambda^{\pm} = \ln|\mu^{\pm}|, \quad s^{\pm} = \mathrm{sign}\mu^{\pm}$

$$
\left\{
\begin{array}{rcl}
\dot{x}^{\pm} - T^{\pm} f(x^{\pm}, \alpha) & = & 0, \\
x^{\pm}(0) - x^{\pm}(1) & = & 0, \\
\dot{w}^{\pm} + T^{\pm} f_u^{\mathrm{T}}(x^{\pm}, \alpha) w^{\pm} + \lambda^{\pm} w^{\pm} & = & 0, \\
w^{\pm}(1) - s^{\pm} w^{\pm}(0) & = & 0, \\
\langle w^{\pm}(0), w^{\pm}(0) \rangle - 1 & = & 0, \\
\dot{u} - T f(u, \alpha) & = & 0, \\
\langle f(x^{+}(0), \alpha), u(1) - x^{+}(0) \rangle & = & 0, \\
\langle f(x^{-}(0), \alpha), u(0) - x^{-}(0) \rangle & = & 0, \\
\langle w^{+}(0), u(1) - x^{+}(0) \rangle & = & 0, \\
\langle w^{-}(0), u(0) - x^{-}(0) \rangle & = & 0, \\
\|u(0) - x^{-}(0)\|^2 - \varepsilon^2 & = & 0.
\end{array}
\right.
$$

4. Finding starting solutions with homopoty

- Adjoint scaled eigenfunctions.
- Homotopy to connection.

References to homotopy techniques for point-to-point connections:

- E.J. Doedel, M.J. Friedman, and A.C. Monteiro [1994] “On locating connecting orbits", *Appl. Math. Comput.* **65**, 231–239.
- E.J. Doedel, M.J. Friedman, and B.I. Kunin [1997] “Successive continuation for locating connecting orbits", *Numer. Algorithms* **14** , 103–124.

Adjoint scaled eigenfunctions

- For fixed α and any λ, $x^{\pm}(\tau) = x^{\pm}_{old}(\tau), w^{\pm}(\tau) \equiv 0$, and $h^{\pm} = 0$ satisfy

$$\left\{ \begin{array}{rcl} \dot{x}^{\pm} - T^{\pm} f(x^{\pm}, \alpha) & = & 0, \\ x^{\pm}(0) - x^{\pm}(0) & = & 0, \\ \int_0^1 \langle \dot{x}^{\pm}_{old}(\tau), x^{\pm}(\tau) \rangle & = & 0, \\ \dot{w}^{\pm} + T^{\pm} f_u^{\mathrm{T}}(x^{\pm}, \alpha) w^{\pm} + \lambda w^{\pm} & = & 0, \\ w^{\pm}(1) - s^{\pm} w^{\pm}(0) & = & 0, \\ \langle w^{\pm}(0), w^{\pm}(0) \rangle - h^{\pm} & = & 0, \end{array} \right.$$

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- A *branch point* at $\lambda_1^{\pm}$ corresponds to the adjoint multiplier $\mu^{\pm} = s^{\pm} e^{\lambda_1^{\pm}}$. Branch switching and continuation towards $h^{\pm} = 1$ gives the eigenfunction $w^{\pm}$.

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Homotopy to connection in (T, h_{jk})

$$
\left\{
\begin{array}{rcl}
\dot{x}^{\pm} - T^{\pm} f(x^{\pm}, \alpha) & = & 0, \\
x^{\pm}(0) - x^{\pm}(1) & = & 0, \\
\Phi^{\pm}[x^{\pm}] & = & 0, \\
\dot{w}^{\pm} + T^{\pm} f_u^{\mathrm{T}}(x^{\pm}, \alpha) w^{\pm} + \lambda^{\pm} w^{\pm} & = & 0, \\
w^{\pm}(1) - s^{\pm} w^{\pm}(0) & = & 0, \\
\langle w^{\pm}(0), w^{\pm}(0) \rangle - 1 & = & 0, \\
\dot{u} - T f(u, \alpha) & = & 0, \\
\langle f(x^{+}(0), \alpha), u(1) - x^{+}(0) \rangle - h_{11} & = & 0, \\
\langle f(x^{-}(0), \alpha), u(0) - x^{-}(0) \rangle - h_{12} & = & 0, \\
\langle w^{+}(0), u(1) - x^{+}(0) \rangle - h_{21} & = & 0, \\
\langle w^{-}(0), u(0) - x^{-}(0) \rangle - h_{22} & = & 0.
\end{array}
\right.
$$

5. Implementation in AUTO

$$
\begin{aligned}
\dot{U}(\tau) - F(U(\tau), \beta) &= 0, \quad \tau \in [0, 1], \\
b(U(0), U(1), \beta) &= 0, \\
\int_0^1 q(U(\tau), \beta) d\tau &= 0,
\end{aligned}
$$

where

$$
U(\cdot), F(\cdot, \cdot) \in \mathbb{R}^{n_d},\ b(\cdot, \cdot) \in \mathbb{R}^{n_{bc}},\ q(\cdot, \cdot) \in \mathbb{R}^{n_{ic}},\ \beta \in \mathbb{R}^{n_{fp}},
$$

The number n_{fp} of *free parameters* β is

$$
n_{fp} = n_{bc} + n_{ic} - n_d + 1.
$$

In our primary BVPs: $n_d = 15,\ n_{ic} = 0$, and $n_{bc} = 19$ so that $n_{fp} = 5$.

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6. Example: Poincaré homoclinic structure in ecology

- The standard tri-trophic food chain model:

$$\begin{cases} \dot{x}_1 &= x_1(1 - x_1) - \dfrac{a_1 x_1 x_2}{1 + b_1 x_1}, \\ \dot{x}_2 &= \dfrac{a_1 x_1 x_2}{1 + b_1 x_1} - \dfrac{a_2 x_2 x_3}{1 + b_1 x_2} - d_1 x_2, \\ \dot{x}_3 &= \dfrac{a_2 x_2 x_3}{1 + b_1 x_2} - d_2 x_3, \end{cases}$$

with $a_1 = 5,\ a_2 = 0.1,\ b_1 = 3$, and $b_2 = 2$.

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- M.P. Boer, B.W. Kooi, and S.A.L.M. Kooijman [1999] "Homoclinic and heteroclinic orbits to a cycle in a tri-trophic food chain," *J. Math. Biol.* **39**, 19–38.
  Yu.A. Kuznetsov, O. De Feo, and S. Rinaldi [2001] "Belayakov homoclinic bifurcations in a tritrophic food chain model," *SIAM J. Appl. Math.* **62**, 462–487.

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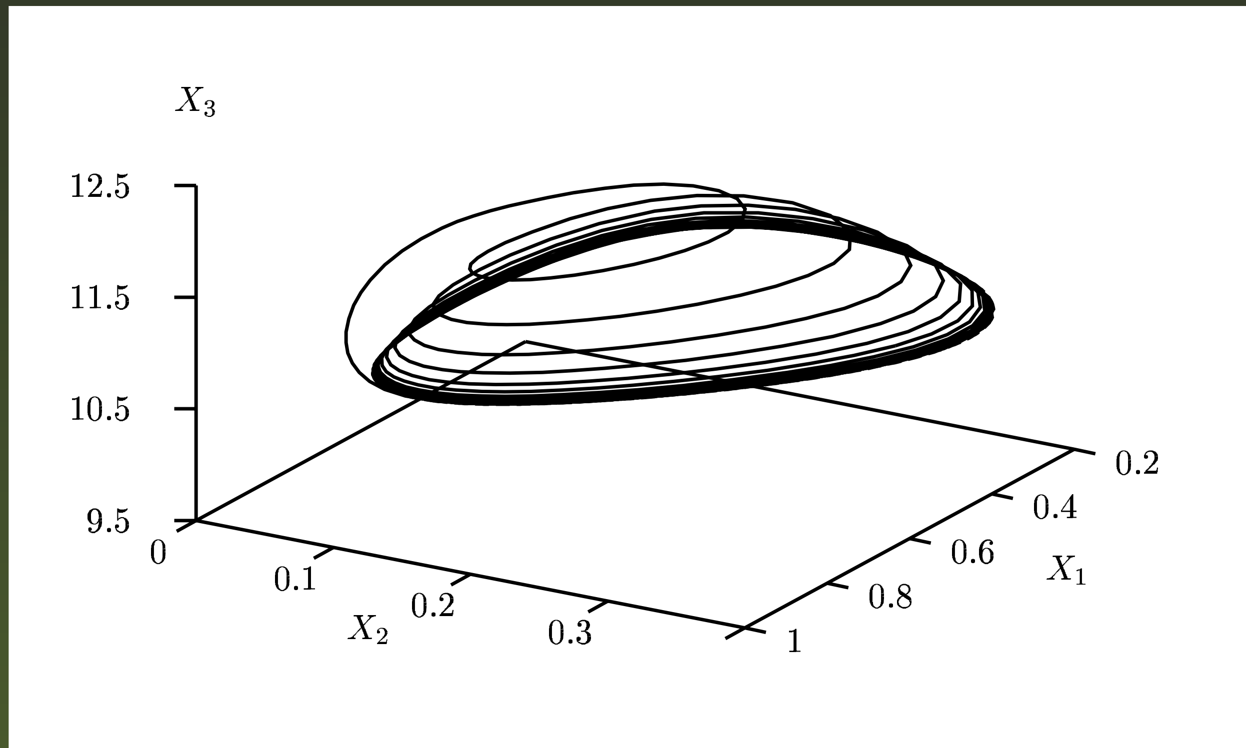
Continuation of cycle-to-cycle connections

- Homoclinic orbit to the cycle at $(d_1, d_2) = (0.25, 0.0125)$:



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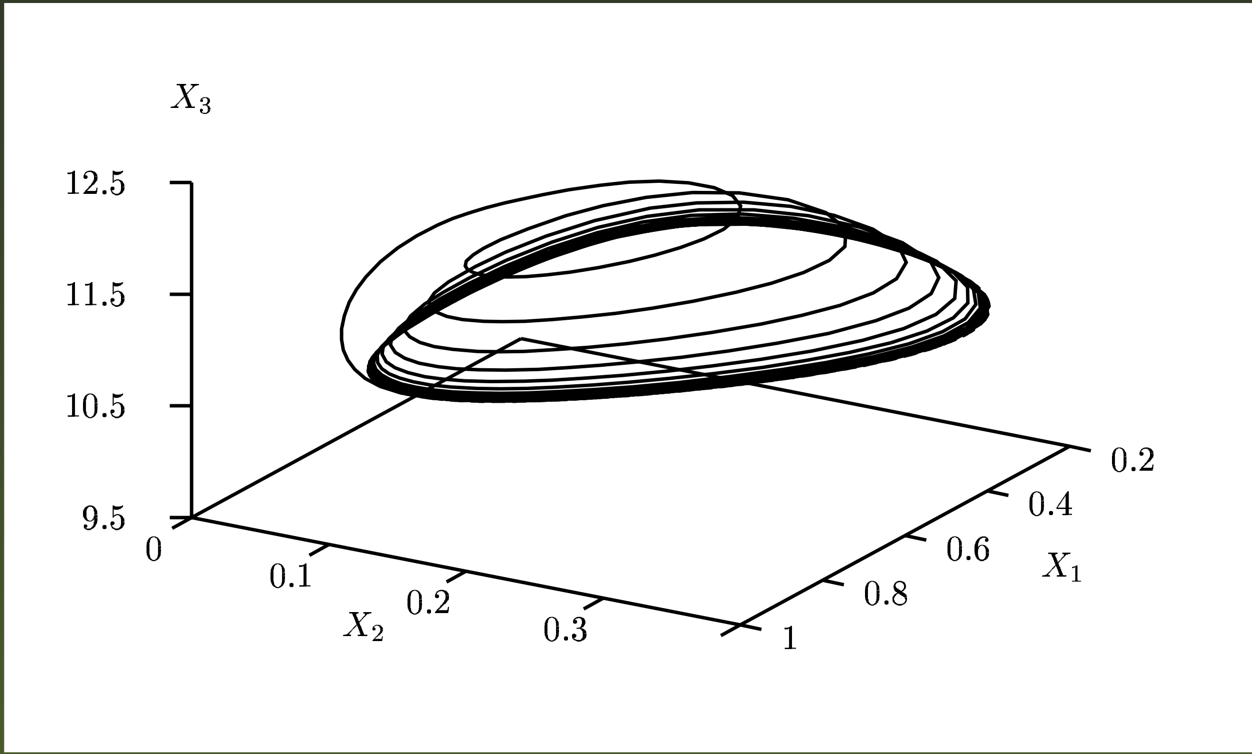

- Limit points: $d_1 = 0.2809078$ and $d_1 = 0.2305987$.

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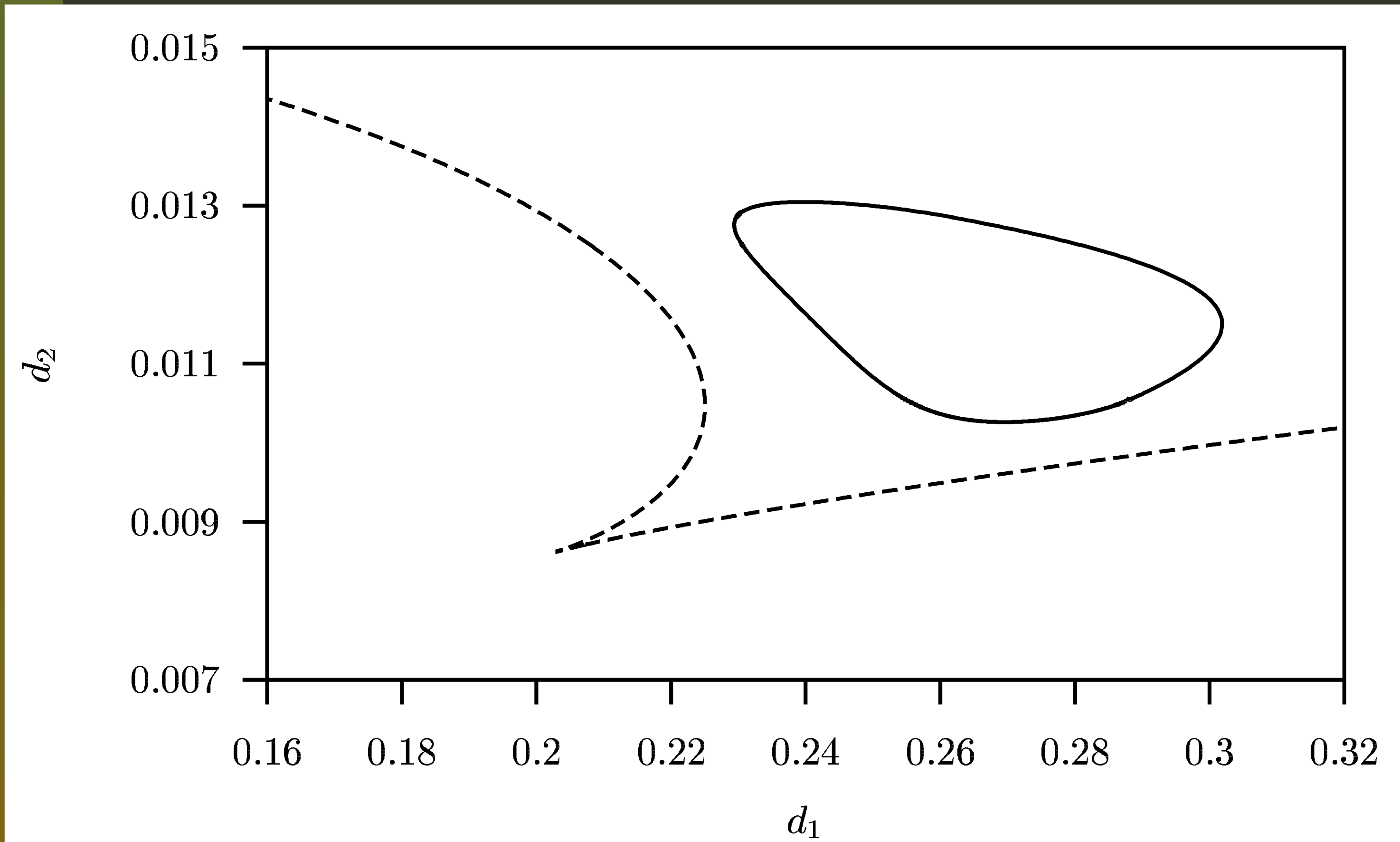
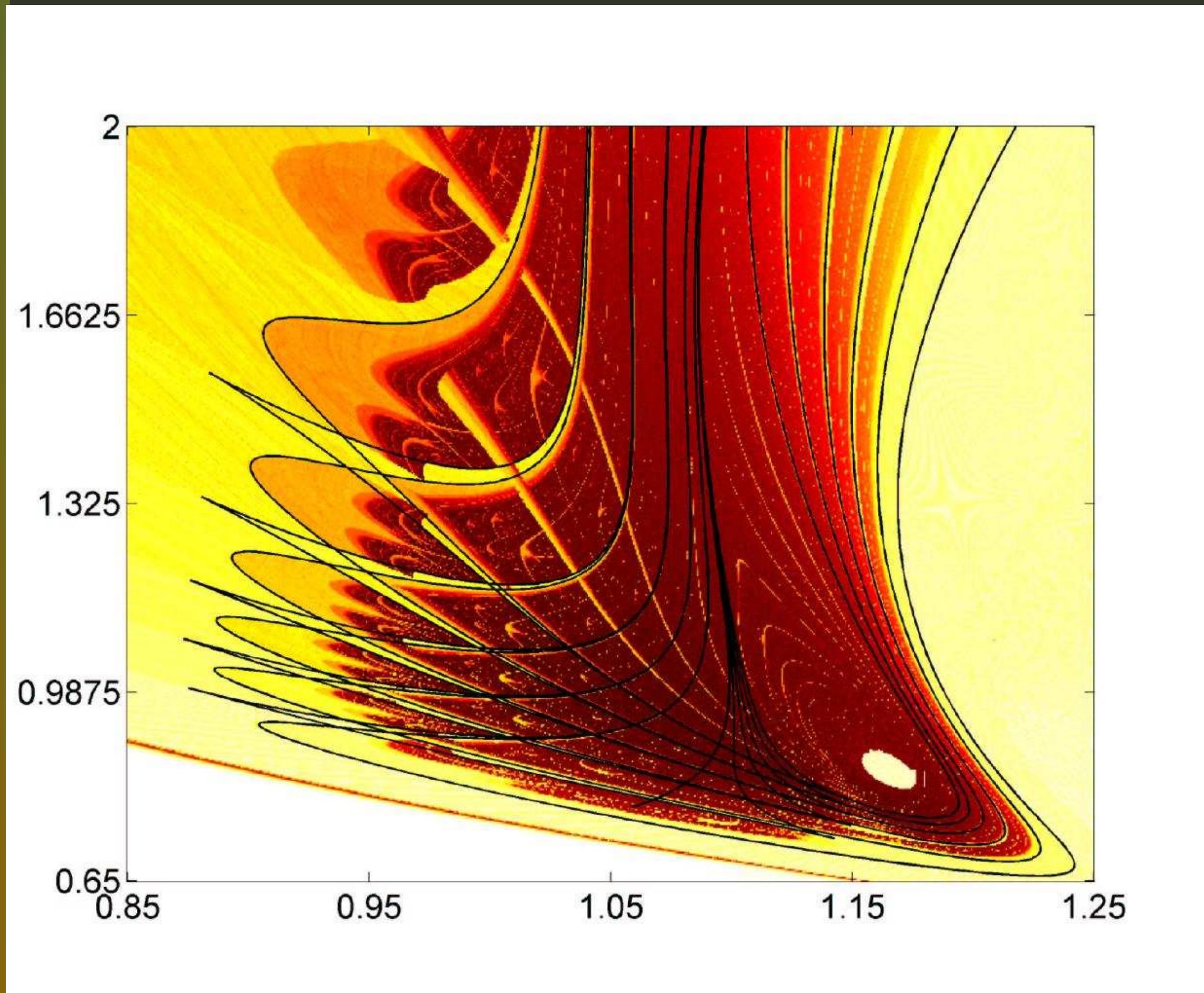
Homoclinic tangency curve



Detailed bifurcation diagram



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Open questions

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- Should all this be integrated in AUTO ?