Application of semigroup theory to reaction-diffusion equations

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Outline

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on a bounded domain $\Omega \subset \mathbb{R}^m (m \leq 3)$ with Dirichlet or Neumann conditions on its piecewise smooth boundary $\partial\Omega$ where $u = (u_1, .., u_n)^T, D = diag(d_1, .., d_n)$ is a diagonal matrix, $C = (c_{ij})$ and $f = (f_1, .., f_n)^T$.

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$C = (c_{ij})$ and $f = (f_1, .., f_n)^T$.

Semigroup theory is a way to see evolution equations of the form: $\frac{d}{dt}u(t) = R(u(t))$ where R is an operator, as ODEs on a Banach function space.

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Is called a strongly continous semigroup or C^0 semigroup.

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Moreover if:

- $\|T(t)\| \leq 1, \forall t \geq 0$, T is called semigroup of contractions

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It is defined on its domain $D(C) \subseteq X$, the set where the limit exists.
It is proven that $D(C)$ is dense in X and C is a closed operator.

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- $t \mapsto T(t)x$ is continuous on $[0, \infty), \forall x \in X$
- If C is the infinitesimal generator of $T(t)$ then : $T(t)x \in D(C)$ and $\frac{d}{dt}(T(t)x) = CT(t)x = T(t)Cx$ $\forall x \in D(C), t \in \mathbb{R}_+$

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- C is closed and densely defined
- $(0,\infty) \subset \rho(C)$, the resolvent set of C, and $\|R(\lambda)\| = \left\|(\lambda I - C)^{-1}\right\| \leq \lambda^{-1}, \forall \lambda > 0$

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Here $H^k(\Omega)$ is the Sobolev space of (equivalence classes of) functions $u : \Omega \to \mathbb{R}$ that have weak derivatives up to and including order k in $L^2(\Omega)$ with the norm

$$|u|_k^\Omega = \left[\int_\Omega \sum_{|\alpha| \le k} |D^\alpha u|^2 dx \right]^{\frac{1}{2}}$$

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- $|T(\xi)x - x| \to 0$ as $|\xi| \to 0$ in any closed subsector of $\Delta_\theta, \forall x \in X$

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$\Rightarrow$ A generates a bounded analytic semigroup of angle δ.

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- $t \to T(t)$ is norm continous on $(0, \infty)$
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- if T is analytic $\Rightarrow$ S is analytic
- if T is compact $\Rightarrow$ S is compact

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$C_0^2(\overline{\Omega}) = \left\{ u \in C^2(\overline{\Omega}) | u = 0 \text{ on } \partial\Omega \right\}$ in $H^2(\Omega)$
The space $L^2(\Omega)$ is a Hilbert space and A is dissipative because for $u \in C_0^2(\overline{\Omega})$ we have: $< \Delta u, u > \leq 0 \Rightarrow < Au, u > \leq 0$ for $u \in D_A$.

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Therefore A generates a contraction semigroup on $L^2(\Omega)$.
Moreover the semigroup generated by A is also analytic and compact.

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The domain D_A is a Banach space with the same norm of $H^2(\Omega)$. We define now $\widetilde{A}$ as the restriction of A to the subspace $D_{A^2} = \{u \in D_A | Au \in D_A\}$ and $\widetilde{T}(t)$ the restriction of $T(t)$ to the subspace D_A of $L^2(\Omega)$.

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In the case of Neumann boundary conditions the result is valid as well and A defined as before still generates a contraction semigroup.

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Since f is locally Lipschitz and $\|T(t)\| \leq Me^{\omega t}$, Picard iteration shows that $C + f$ generates a non-linear C^0 semigroup $F(t)$. Since the integral equation above is not defined $\forall t \in \mathbb{R}_+$, this semigroup is only defined on an interval $[0, \alpha)$.

Reaction-diffusion equations

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We consider now an n-component reaction-diffusion system:
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on a bounded domain $\Omega \subset \mathbb{R}^m (m \leq 3)$ with Dirichlet or Neumann conditions on its smooth boundary $\partial\Omega$ where d_i and c_{ij} are real numbers, $d_i > 0$, $f_i : \mathbb{R}^n \to \mathbb{R}$ are smooth functions in $u_1, \cdots, u_n$ with $f_i(0) = 0$.

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We write this system as $\dot{u} = D\Delta u + Cu + f(u)$ where $u = (u_1, .., u_n)^T$, $D = diag(d_1, .., d_n)$ is a diagonal matrix, $C = (c_{ij})$ and $f = (f_1, .., f_n)^T$.

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Principle of linearized stability

The principle of linearized stability in the finite dimensional case says that, if 0 is an equilibrium of the system of differential equations $\dot{u} = f(u)$ and all the eigenvalues of the Jacobian matrix Df have real part less than zero, then the zero solution is stable. We see now how this result is also valid for evolution equations under some assumptions.

Principle of linearized stability

Consider the equation: $\dot{u}(t) = A(u(t)) + f(u(t))$,$u(0) = u_0$, $t > 0$ where A is a sectorial operator on X and $f : X \to X$ is smooth and suppose 0 is a solution. We have $u(t) = F(t)u_0$, where $F(t)$ is the non linear semigroup associated with the equation above.

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Definition: The zero solution of the equation above is called stable in X if $\forall \epsilon > 0$,$\exists \delta > 0$ such that: $u_0 \in X$,$\|u(0)\| \leq \delta \Rightarrow$ the solution is defined $\forall t > 0$, $\|u(t)\| \leq \epsilon$, $\forall t \geq 0$.

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The zero solution is called asymptotically stable is it is stable and moreover $\exists \delta_0 > 0$ such that if $\|u(0)\| \leq \delta_0$ then $\lim_{t\to\infty} \|u(t)\| = 0$

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Theorem (Principle of linear stability):
Suppose $s(A) < 0$ and $F : X \to X$ is smooth in a neighborhood of 0.
Then $\forall \omega \in [0, -s(A)]$ there exists positive constants $M = M(\omega)$,$r = r(\omega)$ such that if $u_0 \in X$,$u_0 \geq r \Rightarrow$ we have that the solution is defined $\forall t > 0$ and $\|u(t)\| \leq Me^{-\omega t} \|u_0\|$,$t \geq 0$
Therefore the zero solution is asymptotically stable.

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$$\frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + f_1(u, v)$$

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with $u(t, 0) = u(t, \pi) = 0$ and f_1 and f_2 are smooth functions. This is a particular case covered by the previous theory so the system defines a nonlinear local semigroup on $H^2([0, \pi])$.

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The Jacobian matrix is: $Df = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

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The eigenvalues are found by:
$\lambda^2 - (\mathrm{Tr}\, M)\lambda + \mathrm{Det}\, M = 0 = \lambda^2 - (m_{11} + m_{22})\lambda + m_{11} m_{22} - m_{21} m_{12}$

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The spectrum of this operator consists of eigenvalues satisfying for all integer k the equation:

$$\begin{vmatrix} m_{11} - \lambda - k^2 d_1 & m_{12} \\ m_{21} & m_{22} - \lambda - k^2 d_2 \end{vmatrix} = 0$$

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So we get: $\lambda^2 + \lambda \left[k^2(d_1 + d_2 - (m_{11} + m_{22})\right] + h(k^2) = 0$
where $h(k^2) = k^4 d_1 d_2 - k^2(m_{11} d_2 + m_{22} d_1) + (m_{11} m_{22} - m_{21} m_{12})$.

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and to have $h(k^2) < 0$ the following must be satisfied:

- $m_{11}d_2 + m_{22}d_1 > 0$

and the minimum of $h(k^2)$ must be below 0, this gives:

- $\frac{(m_{11}d_2 + m_{22}d_1)^2}{4d_1 d_2} > \text{Det } M$

Turing instability

Finally we have that, to have diffusion-driven instability the following conditions must be satisfied:

- $\operatorname{Tr} M = m_{11} + m_{22} < 0$
- $\operatorname{Det} M = m_{11}m_{22} - m_{21}m_{12} > 0$
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In case these conditions are satisfied we have that the spatially homogeneous stable state becomes unstable if there is integer k in a range $k_1 < k < k_2$ where k_1 and k_2 are given by:

$$k_{1,2}^2 = \frac{(m_{11}d_2 + m_{22}d_1)}{2d_1d_2} \pm \frac{\sqrt{(m_{11}d_2 + m_{22}d_1)^2 - 4d_1d_2 DetM}}{2d_1d_2}$$