Progress on fold-flip and other codim-2 bifurcations of fixed points

Yuri A. Kuznetsov

joint work with *Hil Meijer* and *Lennaert van Veen*

Utrecht University, NL

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1. Introduction

References:

- *Kuznetsov, Yu.A., Meijer, H.G.E., and van Veen, L.* The fold-flip bifurcation. Preprint 1270, Department of Mathematics, Universiteit Utrecht, The Netherlands (2003) [to appear in *IJBC* **14** (2004)]
- *Kuznetsov, Yu.A. and Meijer, H.G.E.* Numerical normal forms for codim 2 bifurcations of fixed points with at most two critical eigenvalues. Preprint 1290, Department of Mathematics, Universiteit Utrecht, The Netherlands (2003)
- *Kuznetsov, Yu.A.* Elements of Applied Bifurcation Theory, 3rd edition. Springer-Verlag, New York, 2004 [to appear]

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1.1. Codim 1 bifurcations of fixed points

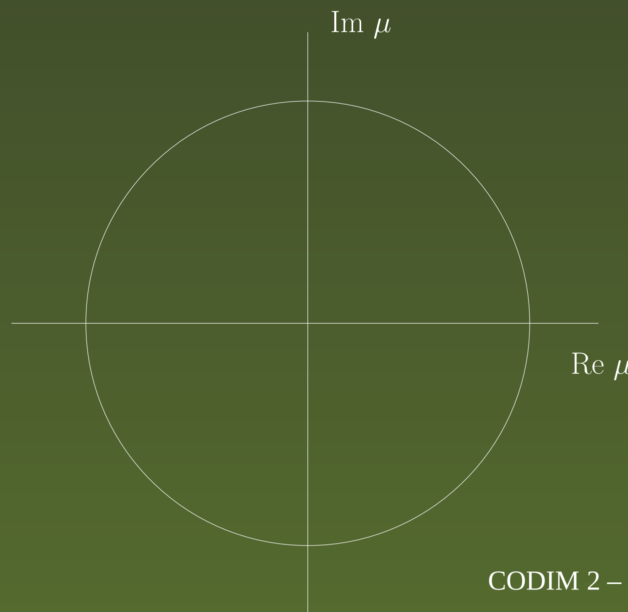
Consider a smooth family of maps

$$x \mapsto f(x, \alpha), \;\; x \in \mathbb{R}^n, \alpha \in \mathbb{R}^m.$$

Fixed points satisfy $f(x^0, \alpha^0) - x^0 = 0$ and have multipliers

$$\{\mu_1, \mu_2, \ldots, \mu_n\} = \sigma(A),$$

where $A = f_x(x^0, \alpha^0)$.



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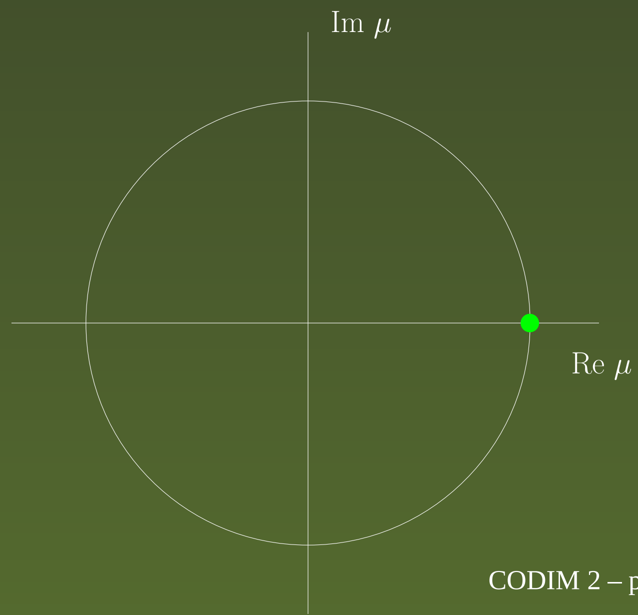
- Fold: $\mu_1 = 1$;



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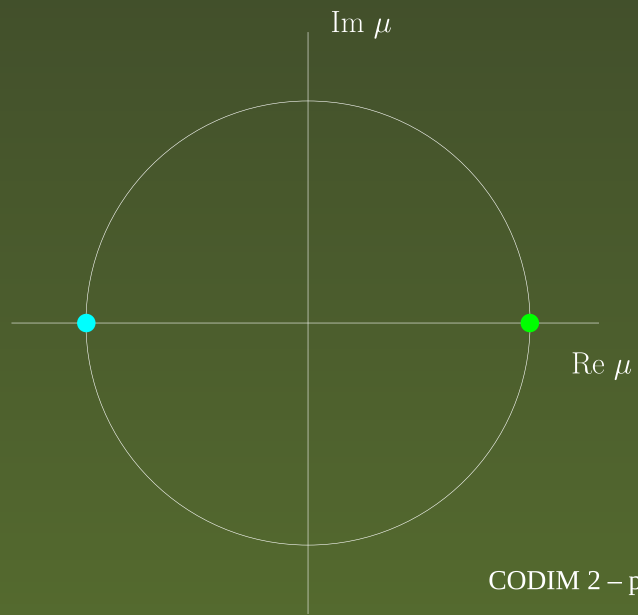

1.1. Codim 1 bifurcations of fixed points

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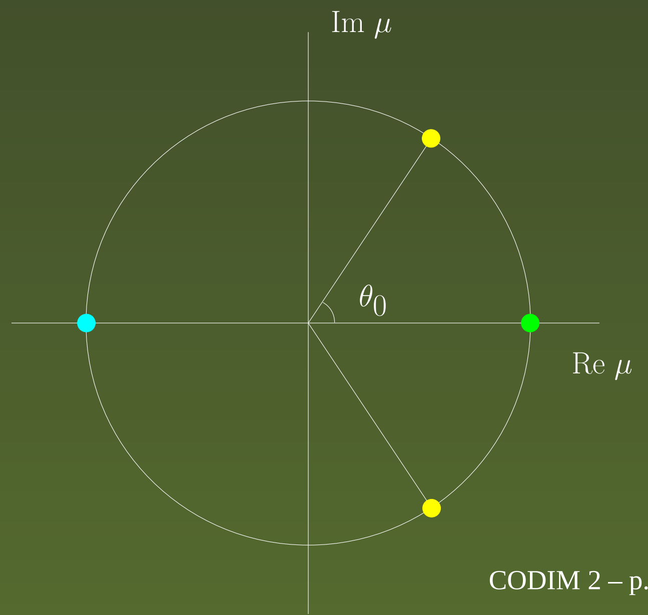
$$\{\mu_1, \mu_2, \ldots, \mu_n\} = \sigma(A),$$

where $A = f_x(x^0, \alpha^0)$.

- Fold: $\mu_1 = 1$;
- Flip: $\mu_1 = -1$;
- Neimark-Sacker: $\mu_{1,2} = e^{\pm i\theta_0}$.



Fold

Using $x = x^0 + y$, write $y \mapsto f(x^0 + y, \alpha^0) - x^0$ as

$$y \mapsto F(y) = Ay + \frac{1}{2}B(y,y) + O(\|y\|^3).$$

A smooth normal form on the critical center manifold:

$$w \mapsto w + \frac{1}{2}bw^2 + O(w^3), \quad b = \langle p, B(q,q) \rangle,$$

where

$$Aq = q, \quad A^T p = p, \quad \langle q, q \rangle = \langle p, q \rangle = 1.$$

The topological normal form:

$$w \mapsto \beta + w + \nu w^2, \quad \beta \in \mathbb{R}^1, \quad \nu = \text{sgn}\, b.$$

Flip

Using $x = x^0 + y$, write $y \mapsto f(x^0 + y, \alpha^0) - x^0$ as

$$y \mapsto F(y) = Ay + \frac{1}{2}B(y,y) + \frac{1}{6}C(y,y,y) + O(\|y\|^4).$$

A smooth normal form on the critical center manifold:

$$w \mapsto -w + \frac{1}{6}cw^3 + O(w^4), \quad c = \langle p, C(q,q,q) + 3B(q,(I_n - A)^{-1}B(q,q))\rangle,$$

where

$$Aq = -q, \quad A^T p = -p, \quad \langle q,q\rangle = \langle p,q\rangle = 1.$$

The topological normal form:

$$w \mapsto -(1+\beta)w + \nu w^3, \quad \beta \in \mathbb{R}^1, \quad \nu = \text{sgn}\, c.$$

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Neimark-Sacker

Using $x = x^0 + y$, write $y \mapsto f(x^0 + y, \alpha^0) - x^0$ as

$$y \mapsto F(y) = Ay + \frac{1}{2}B(y,y) + \frac{1}{6}C(y,y,y) + O(\|y\|^4).$$

Provided $e^{ik\theta_0} \neq 1, k = 1,2,3,4$, a smooth normal form on the critical center manifold:

$$w \mapsto e^{i\theta_0}w + \frac{1}{2}c_1 w|w|^2 + O(|w|^4),\ w \in \mathbb{C},$$

where

$$c_1 = \langle p, C(q,q,\bar{q}) + 2B(q,(I_n - A)^{-1}B(q,\bar{q})) + B(\bar{q},(e^{2i\theta_0}I_n - A)^{-1}B(q,q))\rangle$$

and

$$Aq = e^{i\theta_0}q,\ \ A^T p = e^{-i\theta_0}p,\ \ \langle p,q\rangle = 1.$$

1.2. List of codim 2 bifurcations

(1) $\mu_1 = 1, b = 0$ (cusp)

(2) $\mu_1 = -1, c = 0$ (generalized flip)

(3) $\mu_{1,2} = e^{\pm i\theta_0}, d = \text{Re}\,[e^{-i\theta_0} c_1] = 0$ (Chenciner bifurcation)

(4) $\mu_1 = \mu_2 = 1$ (1:1 resonance)

(5) $\mu_1 = \mu_2 = -1$ (1:2 resonance)

(6) $\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{2\pi}{3}$ (1:3 resonance)

(7) $\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{\pi}{2}$ (1:4 resonance)

(8) $\mu_1 = 1, \mu_2 = -1$ (fold-flip)

(9) $\mu_1 = 1, \mu_{2,3} = e^{\pm i\theta_0}$ (“fold-Hopf for maps")

(10) $\mu_1 = -1, \mu_{2,3} = e^{\pm i\theta_0}$ (“flip-Hopf for maps")

(11) $\mu_{1,2} = e^{\pm i\theta_1}, \mu_{3,4} = e^{\pm i\theta_2}$ (“Hopf-Hopf for maps")

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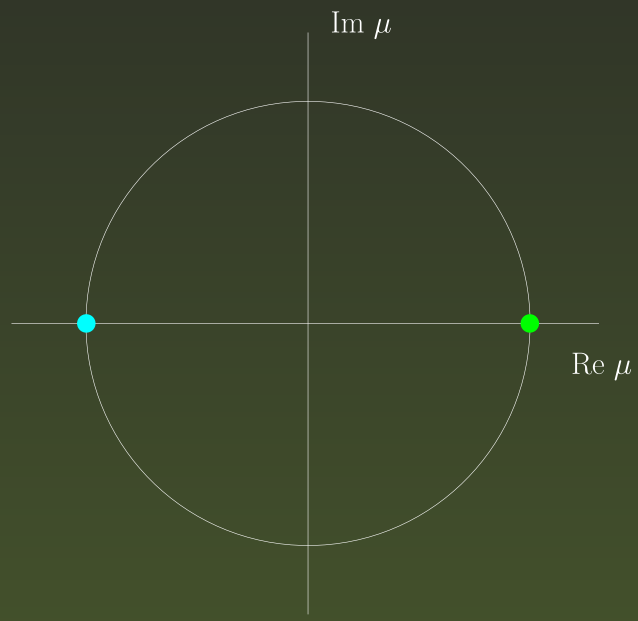
2. Fold-flip bifurcation on the plane

$$\xi \mapsto F(\xi, \alpha), \;\; \xi \in \mathbb{R}^2, \alpha \in \mathbb{R}^2$$

$$F(0,0) = 0, \quad \{\mu_1, \mu_2\} = \sigma(F_\xi(0,0))$$

$$\mu_1 = 1, \quad \mu_2 = -1$$



Gheiner, J. Codimension-two reflection and nonhyperbolic invariant lines, *Nonlinearity* **7** (1994), 109-184.

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2.1. Critical normal form

Proposition 1: [Gheiner, 1994]

Suppose a smooth map $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ *has the form*

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 + \displaystyle\sum_{i+j=2,3} \frac{1}{i!j!} g_{ij} \xi_1^i \xi_2^j \\ -\xi_2 + \displaystyle\sum_{i+j=2,3} \frac{1}{i!j!} h_{ij} \xi_1^i \xi_2^j \end{pmatrix} + O(\|\xi\|^4)$$

and $h_{11} \neq 0$. *Then* F_0 *is smoothly equivalent near the origin to*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + \frac{1}{2}a(0)x_1^2 + \frac{1}{2}b(0)x_2^2 + \frac{1}{6}c(0)x_1^3 + \frac{1}{2}d(0)x_1x_2^2 \\ -x_2 + x_1x_2 \end{pmatrix} + O(\|x\|^4),$$

where

$$
\begin{aligned}
a(0) &= \frac{g_{20}}{h_{11}}, \\
b(0) &= g_{02}h_{11}, \\
c(0) &= \frac{1}{h_{11}^2}\left(g_{30} + \frac{3}{2}g_{11}h_{20}\right), \\
d(0) &= \frac{3g_{02}(h_{02}h_{20} + 2h_{21} - 2g_{11}h_{20}) - g_{20}(3h_{02}^2 + 2h_{03})}{6h_{11}} \\
&\quad - g_{11}^2 + g_{12} + \frac{1}{2}g_{11}h_{02} - h_{02}^2 - \frac{2}{3}h_{03}.
\end{aligned}
$$

2.2. Parameter-dependent normal form

Proposition 2: *Consider a two-parameter family of planar maps*

$$\xi \mapsto F(\xi, \alpha), \;\; \xi \in \mathbb{R}^2, \alpha \in \mathbb{R}^2,$$

where $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *is smooth and such that*

1. $F_0 : \mathbb{R}^2 \to \mathbb{R}^2, \; \xi \mapsto F_0(\xi) = F(\xi, 0)$ *satisfies* **Proposition 1**;
2. *The map* $T : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$, *defined by*

$$\begin{pmatrix} \xi \\ \alpha \end{pmatrix} \mapsto T(\xi, \alpha) = \begin{pmatrix} F(\xi, \alpha) - \xi \\ \det F_\xi(\xi, \alpha) + 1 \\ \mathrm{Tr}\, F_\xi(\xi, \alpha) \end{pmatrix}$$

is regular at $(\xi, \alpha) = (0, 0)$.

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Then F is smoothly equivalent near the origin to a family

$$x \mapsto N(x,\mu) + O(\|x\|^4), \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \ \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \in \mathbb{R}^2,$$

with

$$N(x,\mu) = \begin{pmatrix} \mu_1 + (1+\mu_2)x_1 + \frac{1}{2}a(\mu)x_1^2 + \frac{1}{2}b(\mu)x_2^2 + \frac{1}{6}c(\mu)x_1^3 + \frac{1}{2}d(\mu)x_1x_2^2 \\ -x_2 + x_1x_2 \end{pmatrix},$$

where all coefficients are smooth functions of μ, and their values at $\mu = 0$ are given in **Proposition 1**. *Moreover, $N(Rx,\mu) = RN(x,\mu)$ for*

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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2.3. Local bifurcations of $x \mapsto N(x, \mu)$

1. There is a curve

$$t_{fold} : (x_1, x_2, \mu_1) = \left(-\frac{\mu_2}{a_0} + O(\mu_2^2), 0, \frac{\mu_2^2}{2a_0} + O(\mu_2^3) \right),$$

   on which a nondegenerate fold bifurcation of N occurs if $a_0 \neq 0$.

2. There is a curve $t_{flip} : (x_1, x_2, \mu_1) = (0, 0, 0)$ on which a nondegenerate flip bifurcation of N occurs if $b_0 \neq 0$.

3. If $b_0 > 0$ and $\mu_1 < 0$, there is a curve

$$t_{NS} : (x_1, x_2, \mu_2) = \left(0, \sqrt{-\frac{2\mu_1}{b_0}} + O(\mu_1^{3/2}), \frac{(d_0 + 2b_0)\mu_1}{b_0} + O(\mu_1^2) \right),$$

   on which a nondegenerate Neimark-Sacker bifurcation of the second iterate of N occurs, provided

$$c_{NS} = b_0 c_0 - a_0^2 b_0 - 3a_0 b_0 - a_0 d_0 \neq 0.$$

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2.4. Approximating vector field

Proposition 3:

$$RN(x,\mu) = \varphi^1(x,\mu) + O(\|\mu\|^2) + O(\|x\|^2\|\mu\|) + O(\|x\|^4),$$

where φ^t is the flow generated by the system

$$\dot{x} = X(x,\mu), \;\; x \in \mathbb{R}^2, \; \mu \in \mathbb{R}^2,$$

and the vector field X is given by

$$X(x,\mu) = \begin{pmatrix} \mu_1 + \left(-\frac{1}{2}a_0\mu_1 + \mu_2\right)x_1 + \frac{1}{2}a_0 x_1^2 + \frac{1}{2}b_0 x_2^2 + d_1 x_1^3 + d_2 x_1 x_2^2 \\ \frac{1}{2}\mu_1 x_2 - x_1 x_2 + d_3 x_1 x_2^2 + d_4 x_2^3 \end{pmatrix}$$

with

$$d_1 = \frac{1}{6}\left(c_0 - \frac{3}{2}a_0^2\right), \; d_2 = \frac{1}{2}\left(d_0 + \frac{1}{2}b_0(2 - a_0)\right), \; d_3 = \frac{1}{4}(a_0 - 2), \; d_4 = \frac{1}{4}b_0.$$

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2.5. Heteroclinic connection and limit cycles

Proposition 4:

If $a_0, b_0 > 0$ and $\mu_1 < 0$ then the vector field $X(x, \mu)$ has two saddles, which are always connected by a heteroclinic orbit along the x_1-axis. There exists another heteroclinic orbit for

$$t_J : \mu_2 = \frac{\mu_1}{3 + a_0} \left((a_0 + 2) \frac{d_0 + 2b_0}{b} + \frac{c_0 - a_0 - a_0^2}{a_0} \right) + o(\mu_1).$$

The slopes of t_J and the Neimark-Sacker bifurcation curve t_{NS} coincide if and only if $c_{NS} = 0$.

Moreover, near the origin, the approximating vector field $X(x, \mu)$ has at most one limit cycle in the upper half-plane.

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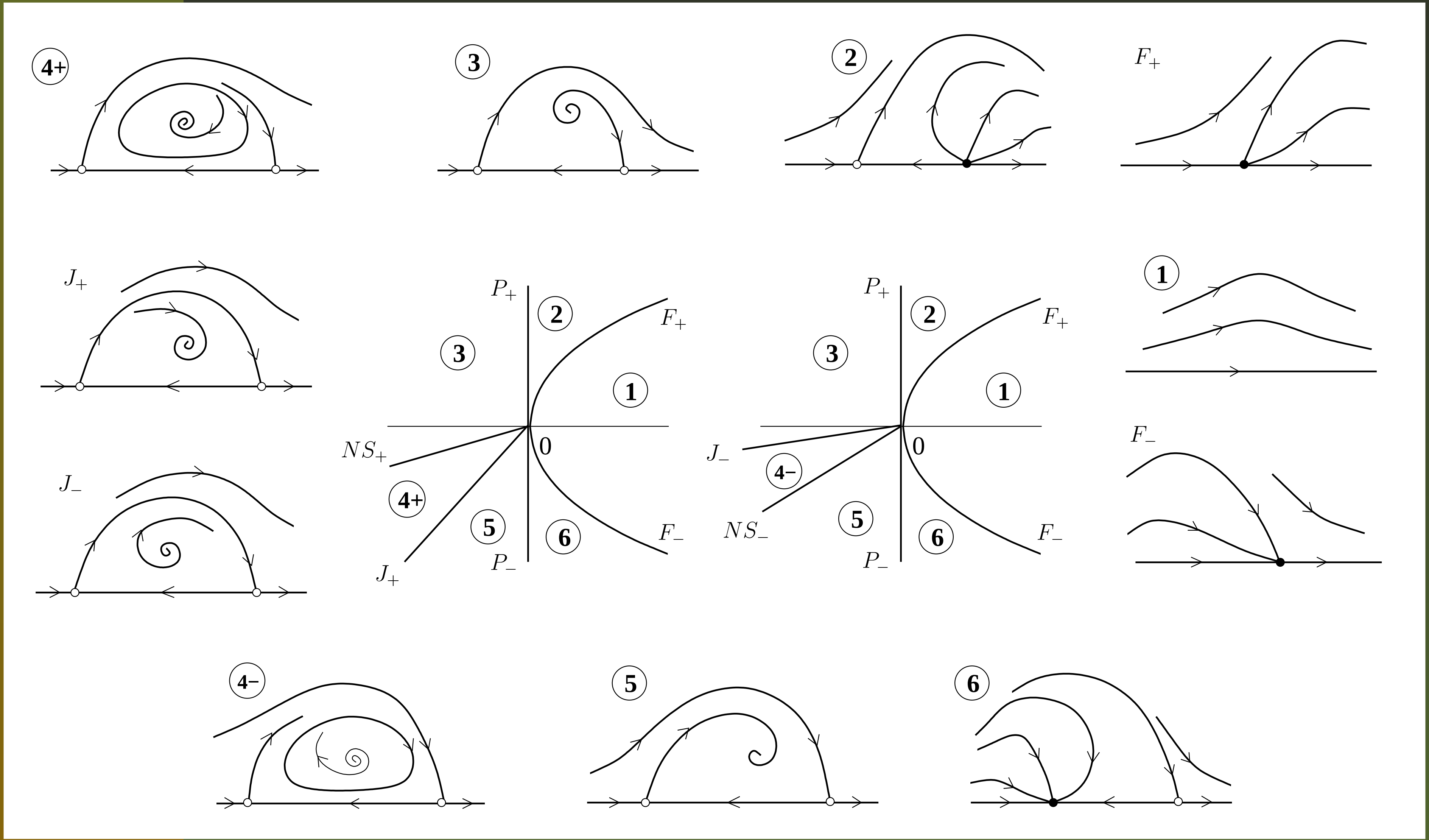
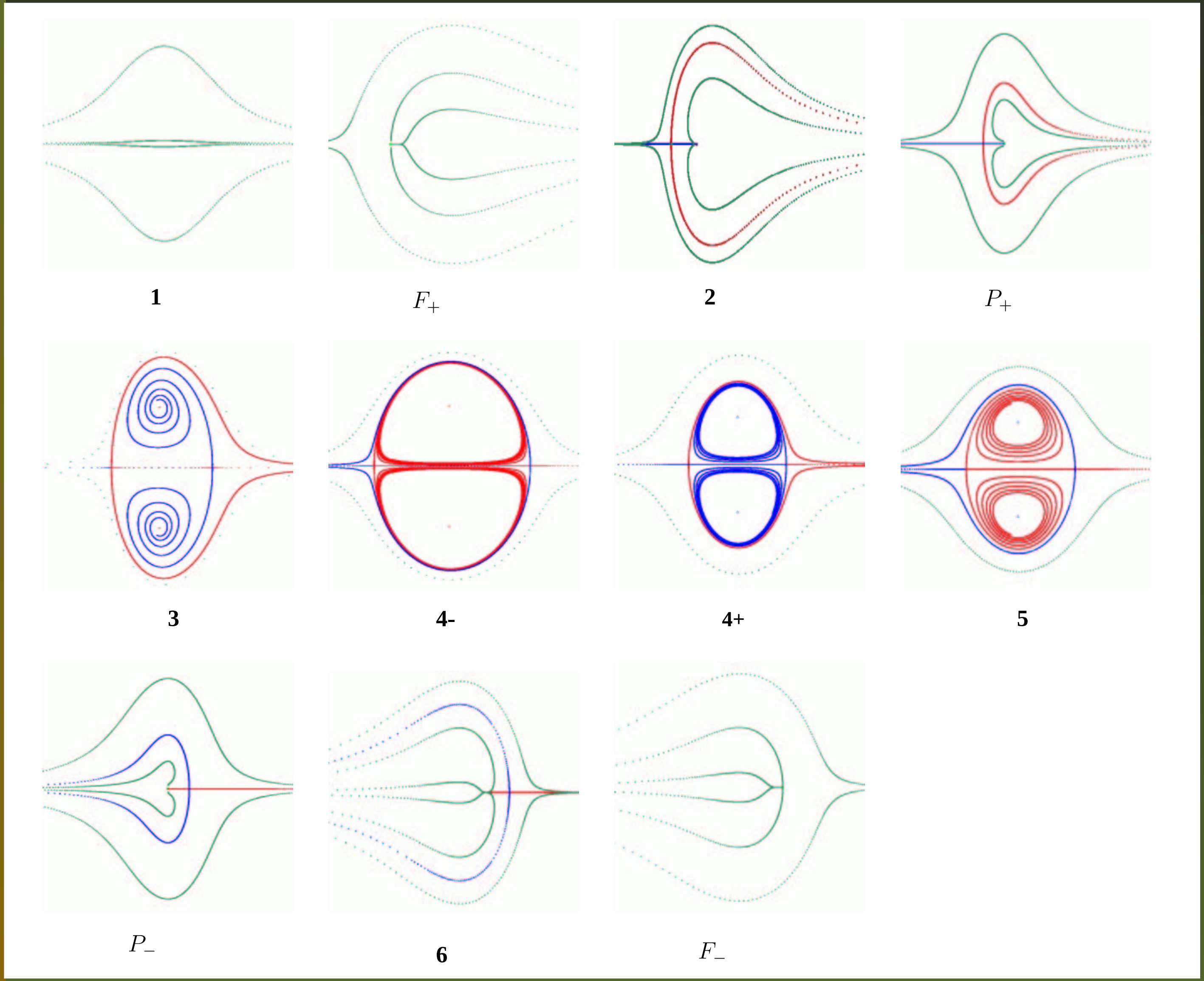
2.6. Bifurcation diagram $(a_0 > 0, b_0 > 0)$



2.6. Maps $(a_0 > 0, b_0 > 0)$



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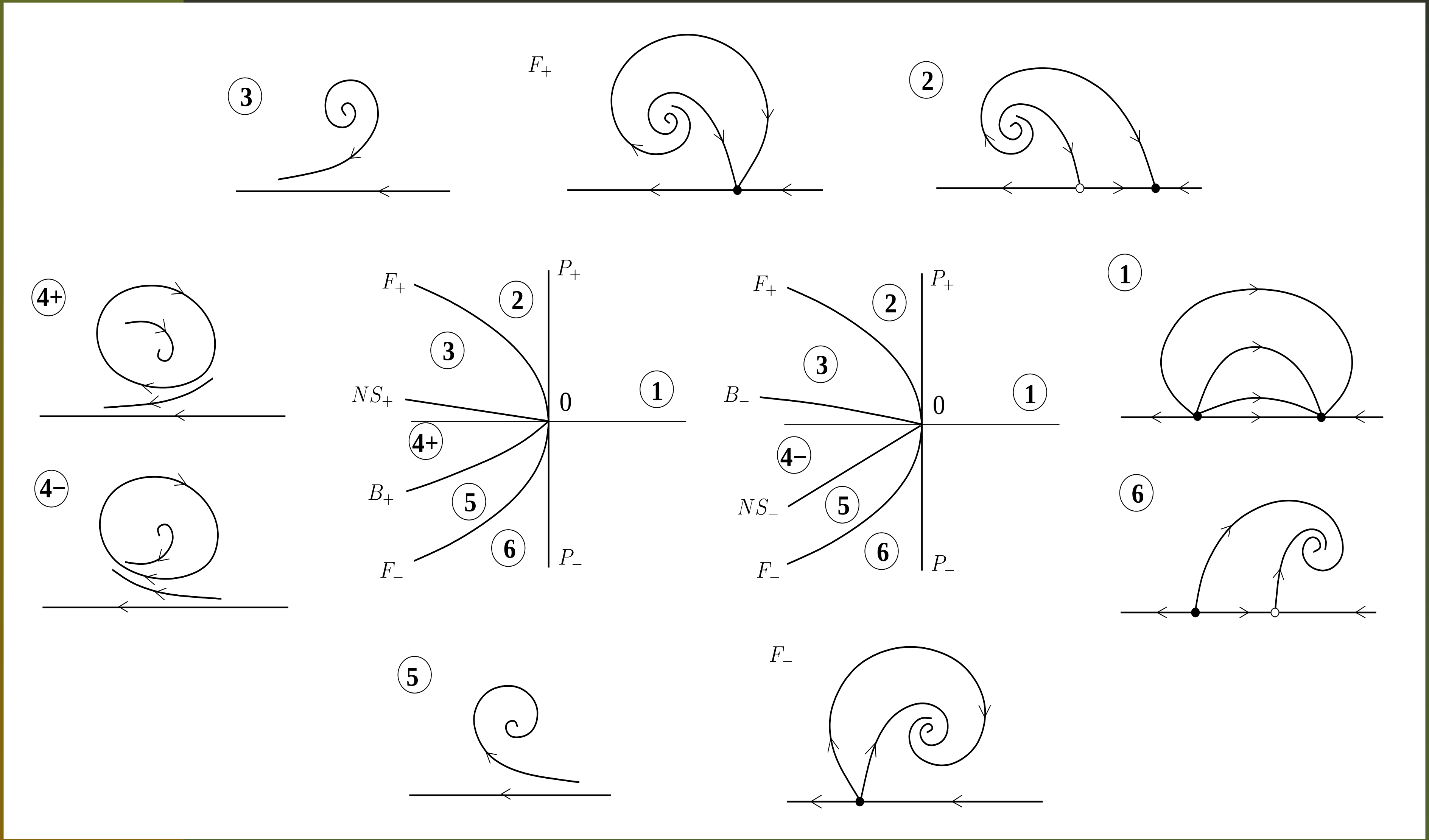
2.6. Bifurcation diagram $(a_0 < 0, b_0 > 0)$



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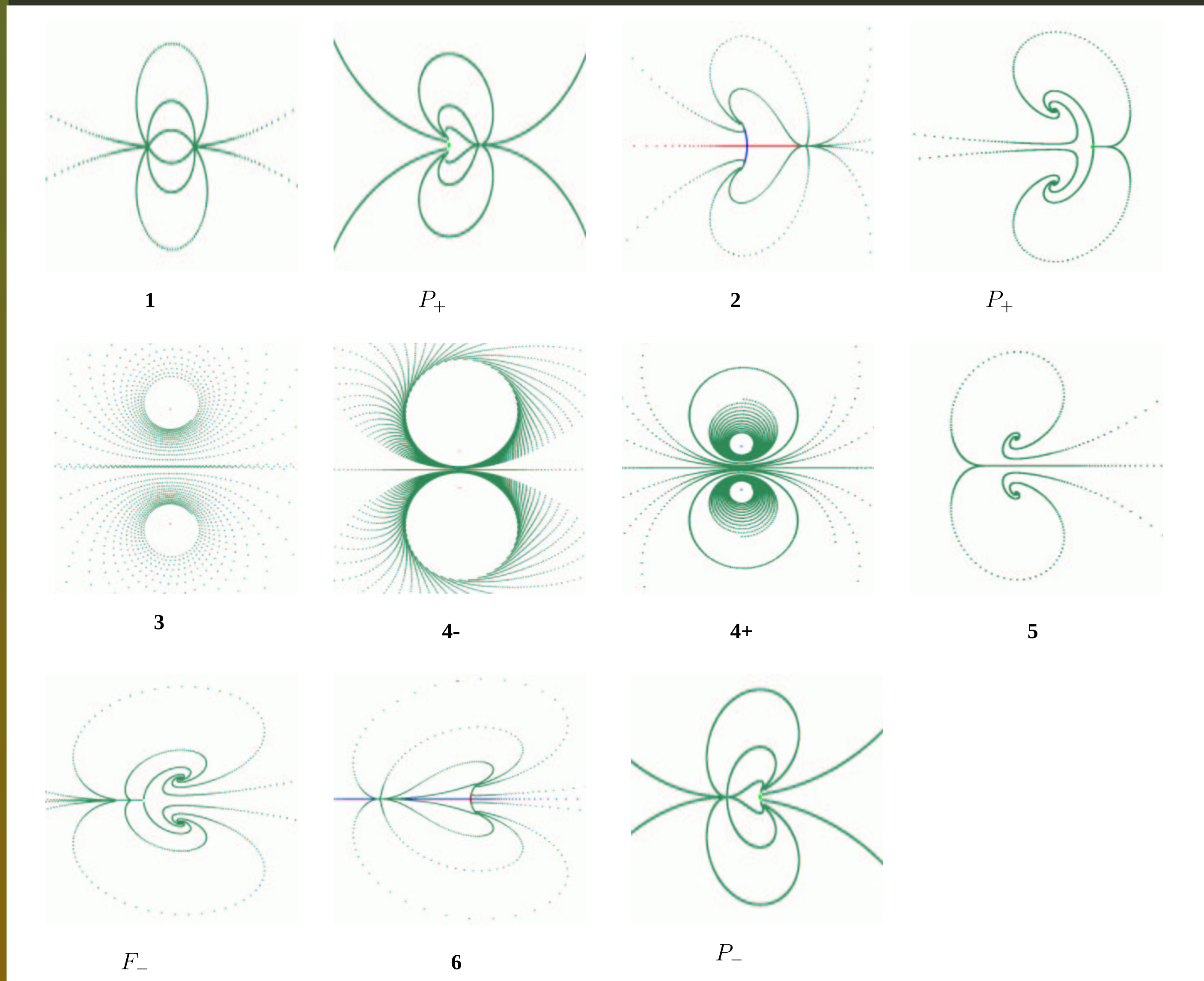

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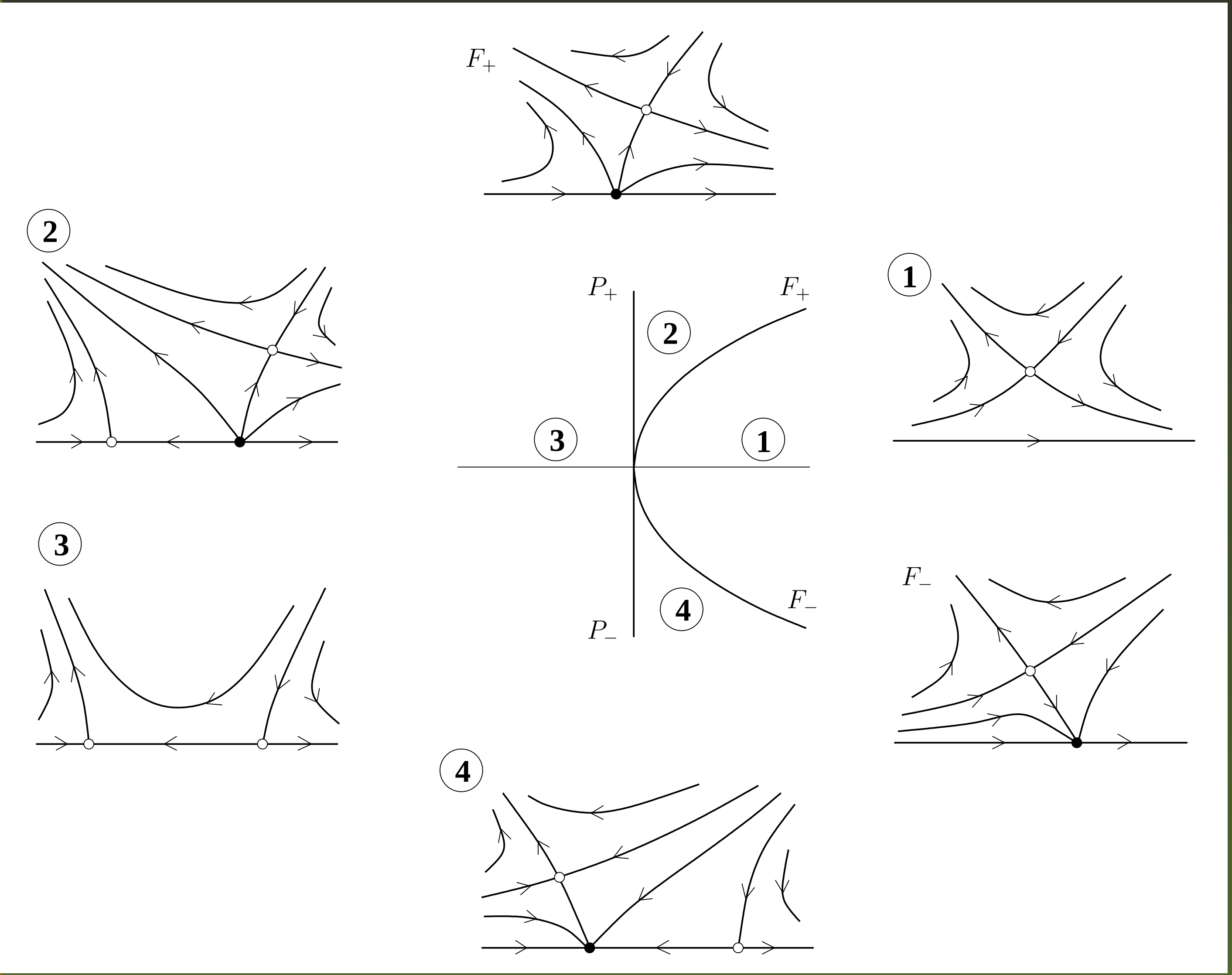

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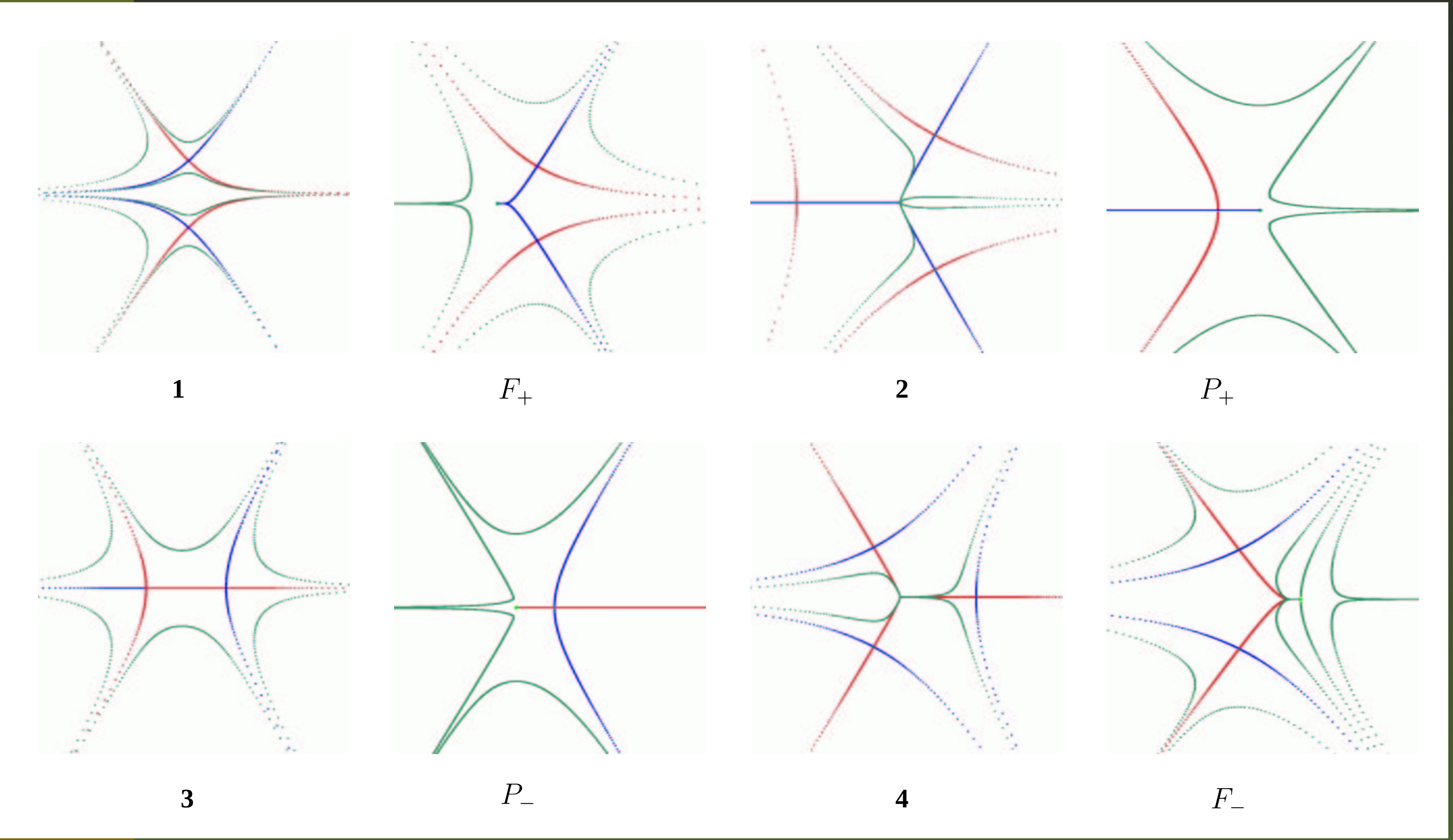

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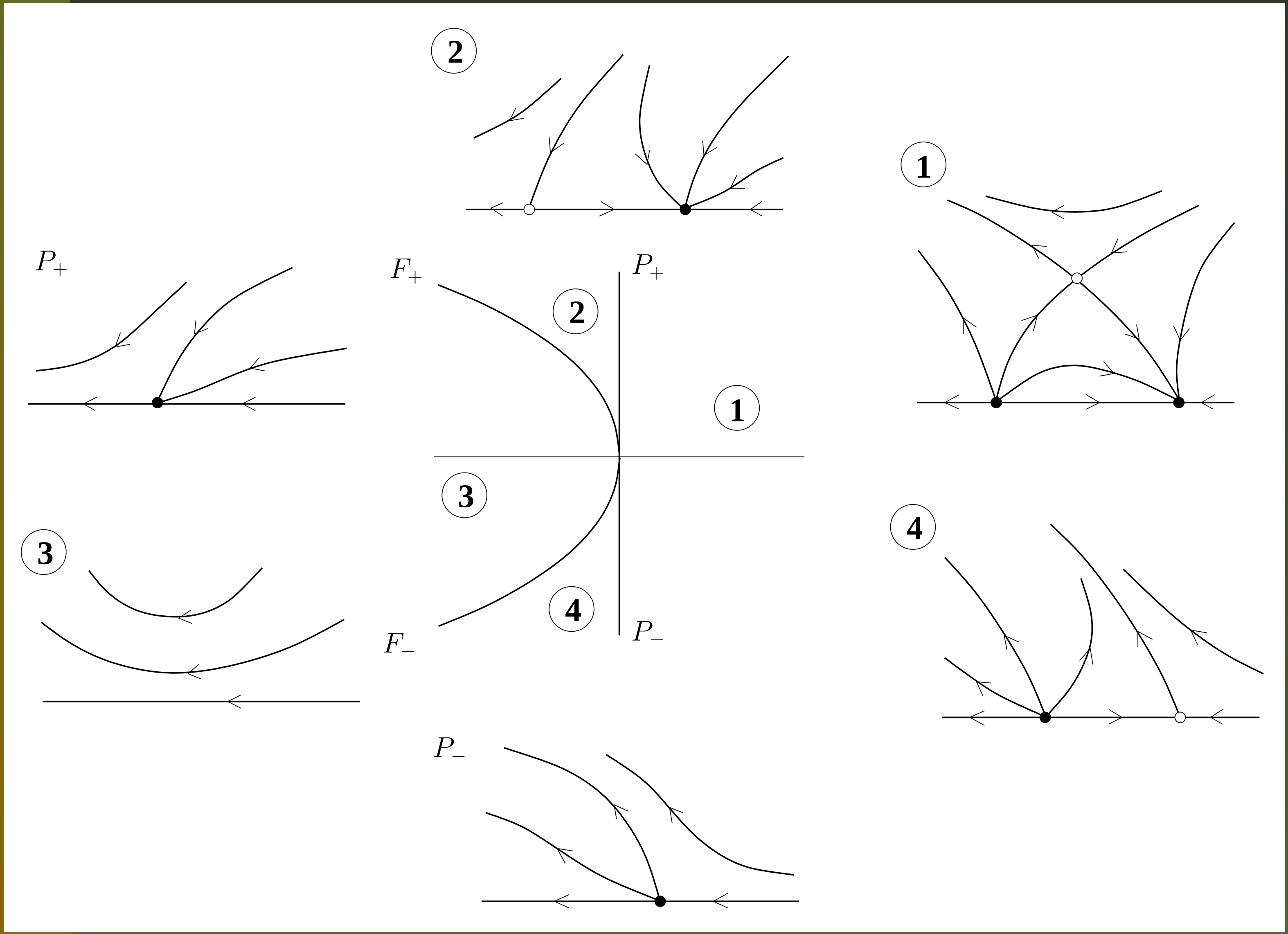

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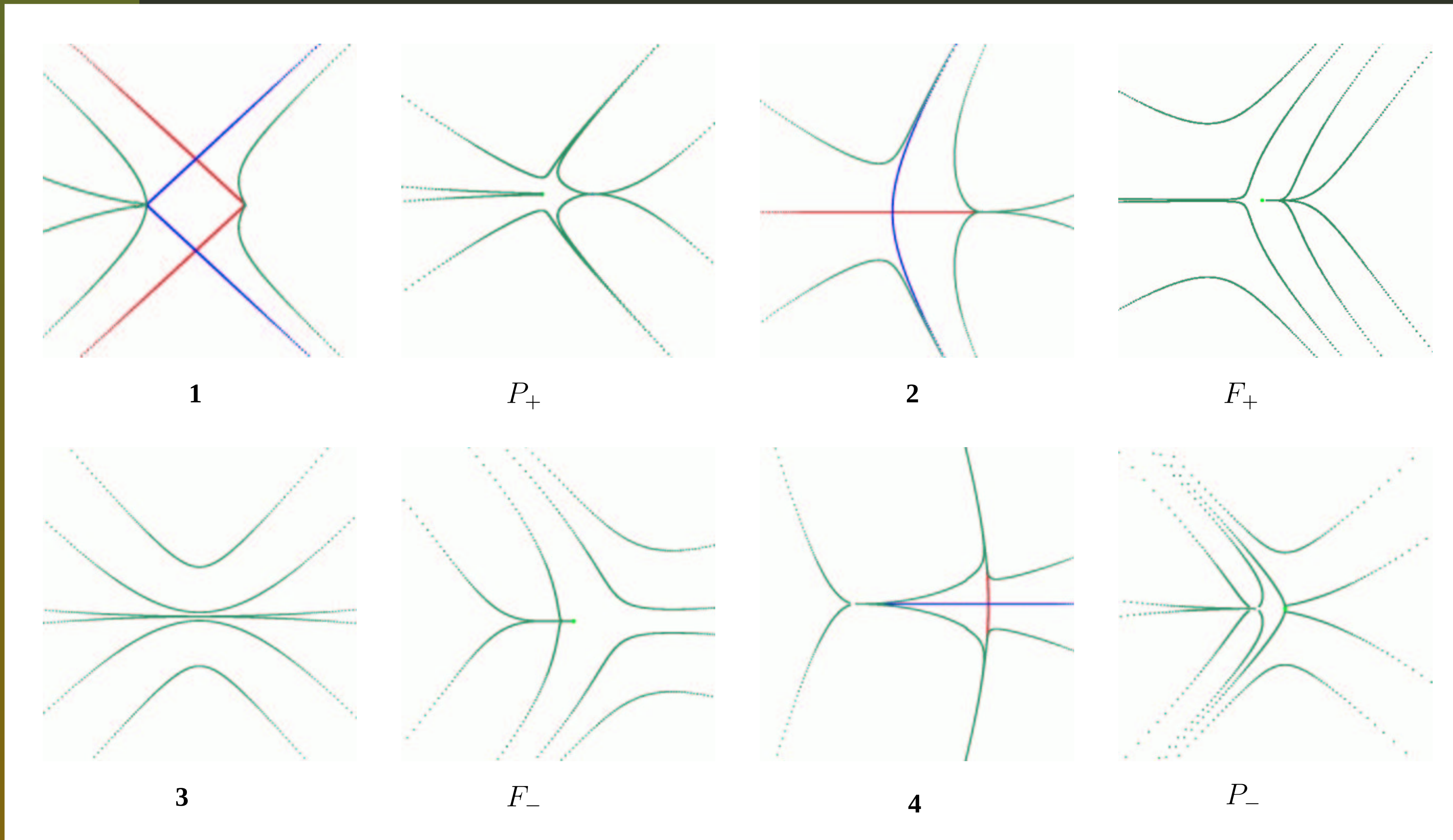
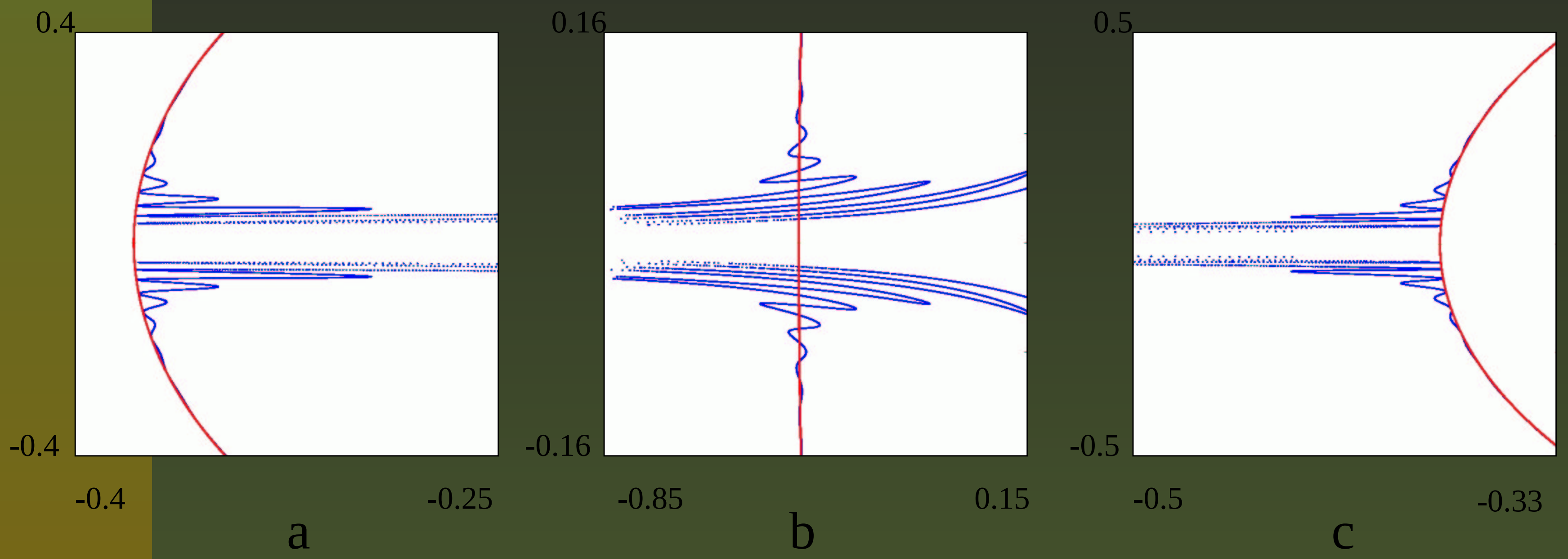


2.7. Heteroclinic structures



$$a = b = d = 1, \;\; c = 0.5, \;\; \mu_1 = -0.2.$$

(a) $\mu_2 = -0.35346$ (b) $\mu_2 = -0.35347$ (c) $\mu_2 = -0.35349$

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3. Critical normal form coefficients for codim 2 bifurcations

Write the critical map as

$$\tilde{x} = F(x), \;\; x \in \mathbb{R}^n,$$

and restrict it to its n_0-dimensional center manifold W^c:

$$x = H(w), \;\; H : \mathbb{R}^{n_0} \to \mathbb{R}^n,$$

The restricted map becomes

$$\tilde{w} = G(w), \;\; G : \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}.$$

The invariancy of the center manifold, $\tilde{x} = H(\tilde{w})$, yields the homological equation:

$$f(H(w)) = H(G(w)).$$

Let

$$F(x) = Ax + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + \frac{1}{24}D(x,x,x,x) + \frac{1}{120}E(x,x,x,x,x) + \cdots$$

and expand the functions G, H into Taylor series with unknown coefficients,

$$G(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} g_\nu w^\nu, \quad H(w) = \sum_{|\nu|\geq 1} \frac{1}{\nu!} h_\nu w^\nu,$$

assuming that the restricted map is put into the normal form up to a certain order.

Collecting the coefficients of the w^ν-terms in the homological equation gives a linear system for h_ν :

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$$L_\nu h_\nu = R_\nu .$$

When R_ν depends on the unknown coefficient g_ν of the normal form, L_ν is singular and the Fredholm solvability condition

$$\langle p, R_\nu \rangle = 0$$

gives the expression for g_ν. Here p is any vector satisfying $\bar{L}_\nu^T p = 0$. If the null-space of L_ν is spanned by q, $h_\nu = L_\nu^{INV} R_\nu$ satisfying $\langle p, h_\nu \rangle = 0$ can be found from the nonsingular bordered system

$$\begin{pmatrix} L_\nu & q \\ \bar{p}^T & 0 \end{pmatrix} \begin{pmatrix} h_\nu \\ s \end{pmatrix} = \begin{pmatrix} R_\nu \\ 0 \end{pmatrix} .$$

3.1. Cusp: $Aq = q, A^T p = p, \langle p, q \rangle = 1, \;\; b = \langle p, B(q,q) \rangle = 0$

The critical normal form

$$w \mapsto G(w) = w + \frac{1}{6} e w^3 + \cdots$$

on the center manifold

$$H(w) = wq + \frac{w^2}{2} h_2 + \frac{w^3}{6} h_3 + \cdots$$

Collecing quadratic and cubic terms in the homological equation, we get

$$w^2 : \;\; (A - I_n) h_2 = -B(q,q) \;\; \Longrightarrow \;\; h_2 = -(A - I_n)^{INV} B(q,q)$$

$$w^3 : \;\; (A - I_n) h_3 = eq - C(q,q,q) - 3B(q,h_2) \;\; \Longrightarrow$$

$$e = \langle p, C(q,q,q) + 3B(q,h_2) \rangle$$

3.2. Generalized flip: $Aq = -q,\ A^T p = -p,\ \langle p, q \rangle = 1,\quad c = 0$

$$w \mapsto G(w) = -w + \frac{1}{120} g w^5 + \cdots$$

$$H(w) = wq + \frac{w^2}{2} h_2 + \frac{w^3}{6} h_3 + \frac{w^4}{24} h_4 + \frac{w^5}{120} h_5 + \cdots$$

where

$$h_2 = -(A - I_n)^{-1} B(q, q)$$

$$h_3 = -(A + I_n)^{INV} [C(q, q, q) + 3B(q, h_2)]$$

$$h_4 = -(A - I_n)^{-1} [4B(q, h_3) + 3B(h_2, h_2) + 6C(q, q, h_2) + D(q, q, q, q)]$$

$$\begin{aligned} g &= \langle p, 5B(q, h_4) + 10B(h_2, h_3) + 10C(q, q, h_3) + 15C(q, h_2, h_2) \\ &\quad + 10D(q, q, q, h_2) + E(q, q, q, q, q) \rangle \end{aligned}$$

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Example I: Periodically forced SEIR model

$$\begin{cases} \dot{S} &= \mu - \mu S - \beta S I \\ \dot{E} &= \beta S I - (\mu + \alpha) E \\ \dot{I} &= \alpha E - (\mu + \gamma) I \end{cases}$$

where $\beta = \beta_0(1 + \delta \cos(2\pi t))$ and

$$\mu = 0.02, \alpha = 35.842, \gamma = 100.$$

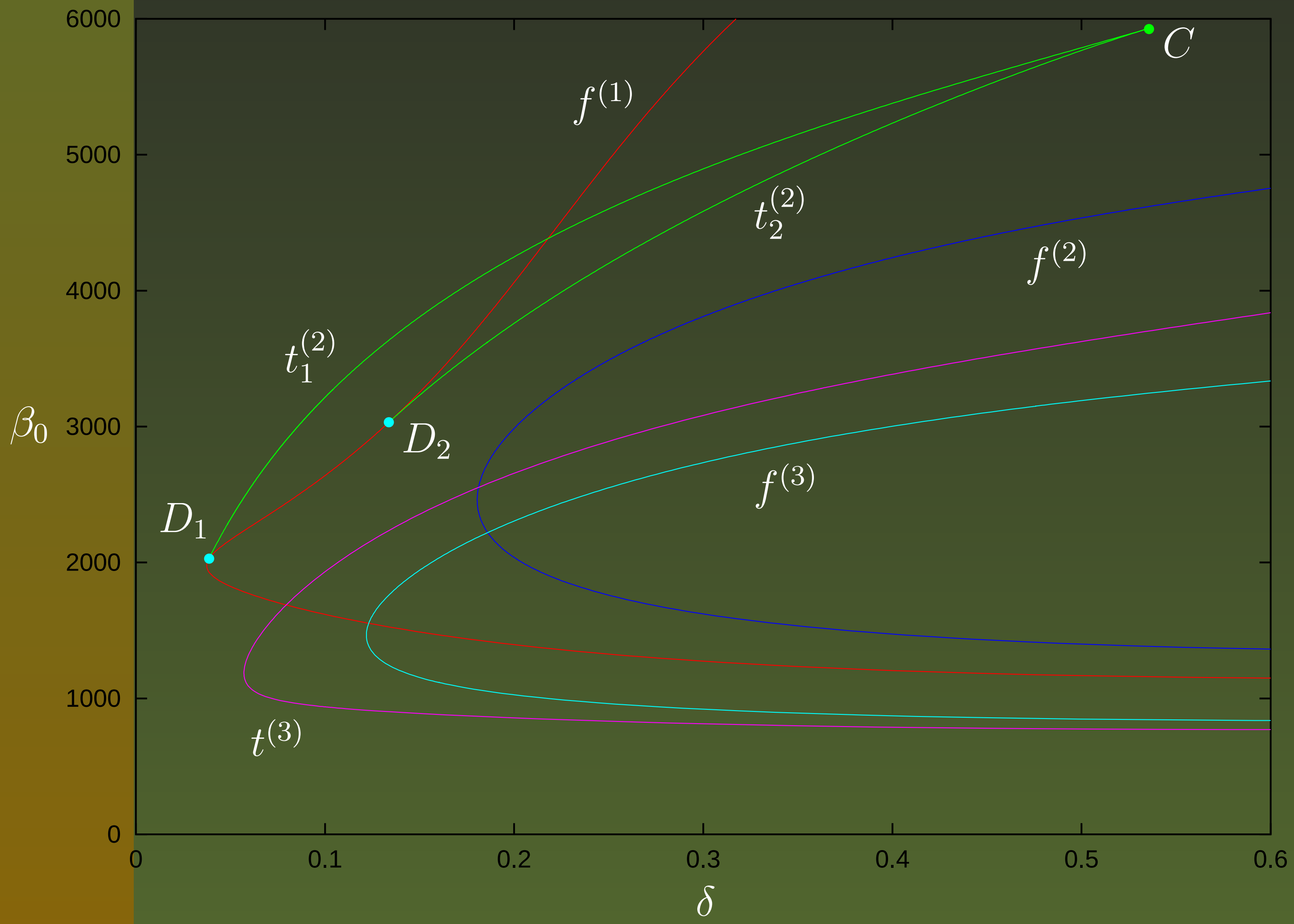
Cusp: $C : (\delta, \beta_0) \approx (0.5327, 5928)$ with $e = -0.224\ldots$

Generalized flip:

$$D_1 : (\delta, \beta_0) \approx (0.03815, 2015) \text{ with } g = 0.764\ldots$$

$$D_2 : (\delta, \beta_0) \approx (0.1328, 3019) \text{ with } g = -0.0313\ldots$$

Example I: Bifurcation diagram



3.3. Fold-flip: $Aq_k = \pm q_k, A^T p_k = \pm p_k, \langle p_k, q_k \rangle = 1, k = 1, 2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{1}{2} a_0 x^2 + \frac{1}{2} b_0 y^2 + \frac{1}{6} c_0 x^3 + \frac{1}{2} d_0 x y^2 \\ -y + xy \end{pmatrix} + \cdots$$

$$a_1 = \langle p_1, B(q_1, q_1) \rangle, e_1 = \langle p_2, B(q_1, q_2) \rangle \neq 0, b_1 = \langle p_1, B(q_2, q_2) \rangle$$

$$h_{20} = (A - I_n)^{INV}[a_1 q_1 - B(q_1, q_1)],\ h_{11} = (A + I_n)^{INV}[e_1 q_2 - B(q_1, q_2)]$$

$$h_{02} = (A - I_n)^{INV}[b_1 q_1 - B(q_2, q_2)]$$

$$c_1 = \langle q_1, C(q_1, q_1, q_1) + 3B(q_1, h_{20}) \rangle, c_2 = \langle q_1, C(q_1, q_2, q_2) + B(q_1, h_{02}) + 2B(q_2, h_{11}) \rangle$$

$$c_3 = \langle p_2, C(q_1, q_1, q_2) + B(q_2, h_{20}) + 2B(q_1, h_{11}) \rangle, c_4 = \langle p_2, C(q_2, q_2, q_2) + 3B(q_2, h_{02}) \rangle$$

$$a_0 = \frac{a_1}{e_1}, b_0 = b_1 e_1, c_0 = \frac{c_1}{e_1^2}, d_0 = c_2 + \frac{1}{e_1} \left(b_1 c_3 - \frac{1}{3}(2e_1 + a_1) c_4 \right)$$

Example II: The extended Lorenz-84 model

$$\begin{cases} \dot{X} &= -Y^2 - Z^2 - \alpha X + \alpha F - \gamma U^2 \\ \dot{Y} &= XY - \beta XZ - Y + G \\ \dot{Z} &= \beta XY + XZ - Z \\ \dot{U} &= -\delta U + \gamma UX + T \end{cases}$$

There is a fold-flip bifurcation of a limit cycle for

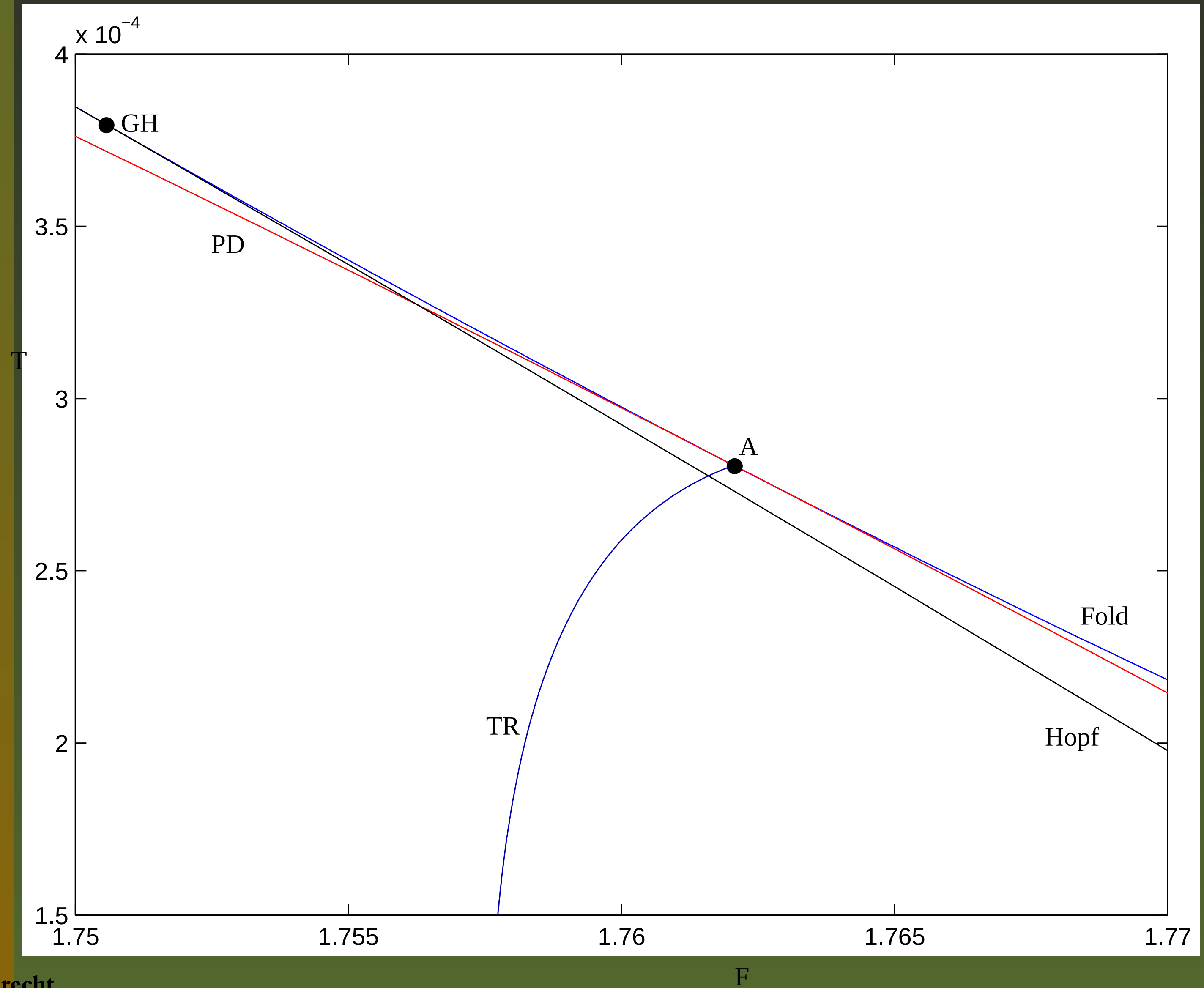
$$\alpha = 0.25, \ \ \beta = 1, \ \ \gamma = 0.987, \ \ \delta = 1.04, \ \ G = 0.2,$$

$$F = 1.76205328796\ldots, \ \ T = 0.000280597685\ldots$$

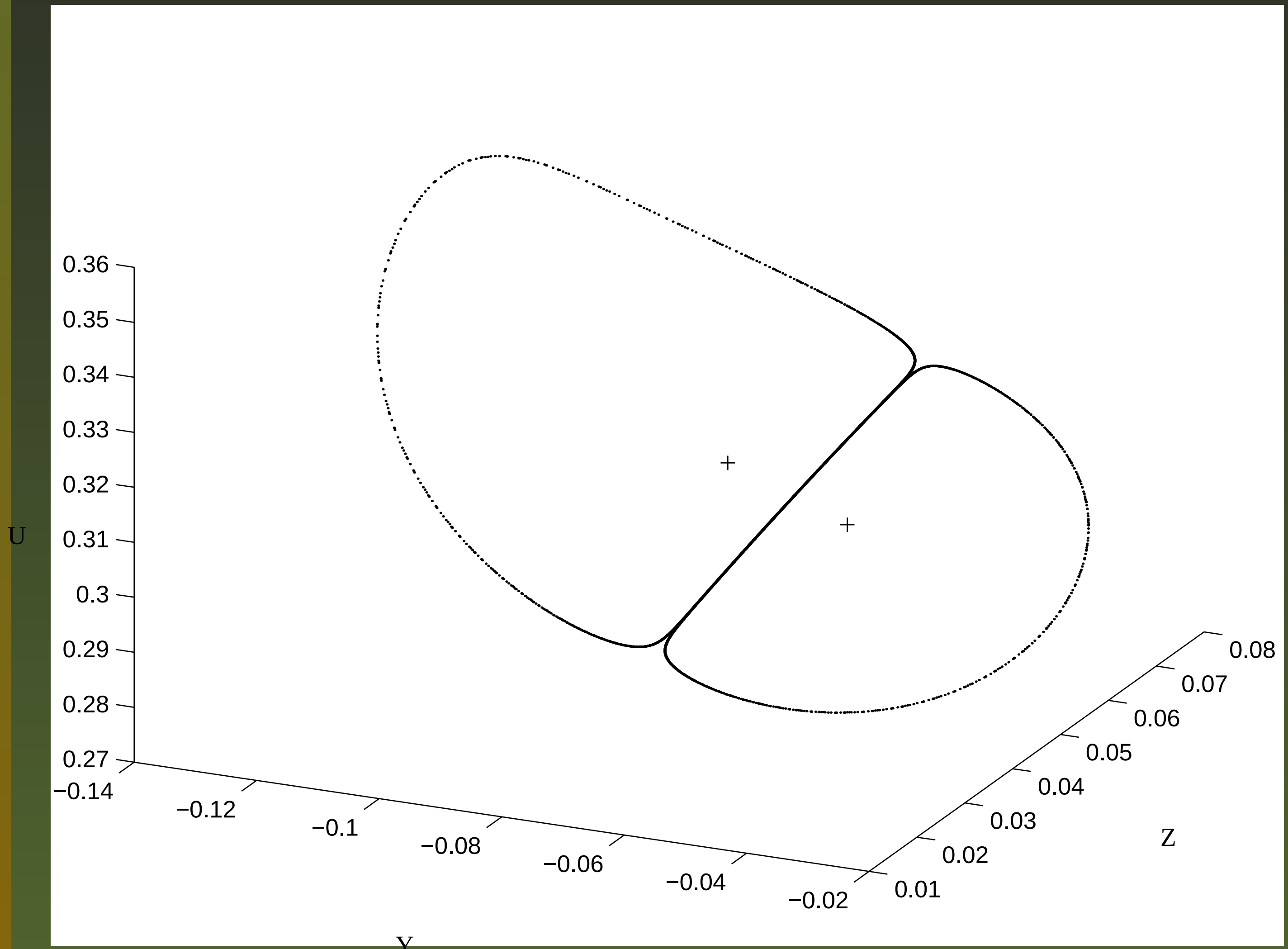
with the critical normal form coefficients

$$a_0 = 0.002047\ldots, b_0 = 4.4010\ldots, c_0 = -0.02336\ldots, d_0 = 232.682\ldots$$

Example II: Bifurcation curves near the fold-flip point A



Example II: Invariant torus



3.4. Chenciner: $Aq = e^{i\theta_0}q,\ A^T p = e^{-i\theta_0}p,\ \langle p, q\rangle = 1,\ \mathrm{Re}[e^{-i\theta_0}c_1] = 0$

$$w \mapsto G(w) = e^{i\theta_0}w + \frac{1}{2}c_1 w|w|^2 + \frac{1}{12}c_2 w|w|^4 + \cdots,\ e^{ik\theta_0} \neq 1, k = 1, 2, \ldots, 5,$$

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{j+k\geq 2} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k,$$

where

$$h_{20} = -(A - e^{2i\theta_0}I_n)^{-1}B(q, q),\quad h_{11} = -(A - I_n)^{-1}B(q, \bar{q})$$

$$c_1 = \langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11})\rangle$$

$$\begin{aligned} h_{21} &= (A - e^{i\theta_0}I_n)^{INV}[c_1 q - C(q, q, \bar{q}) - B(\bar{q}, h_{20}) - 2B(q, h_{11})], \\ h_{30} &= -(A - e^{3i\theta_0}I_n)^{-1}[C(q, q, q) + 3B(q, h_{20})] \end{aligned}$$

$$h_{31} = -(A - e^{2i\theta_0} I_n)^{-1} \{[D(q,q,q,\bar{q}) + 3C(q,q,h_{11}) + 3C(q,\bar{q},h_{20})$$
$$+ 3B(q,h_{21}) + 3B(h_{11},h_{20}) + B(\bar{q},h_{30})] - 3c_1 h_{20} e^{i\theta_0}\}$$
$$h_{22} = -(A - I_n)^{-1} [D(q,q,\bar{q},\bar{q}) + C(q,q,h_{02}) + C(\bar{q},\bar{q},h_{20}) + 4C(q,\bar{q},h_{11})$$
$$+ B(h_{20},h_{02}) + 2B(h_{11},h_{11}) + 2B(q,h_{12}) + 2B(\bar{q},h_{21})]$$

$$c_2 = \langle p, E(q,q,q,\bar{q},\bar{q}) + D(q,q,q,\bar{h}_{20}) + 6D(q,q,\bar{q},h_{11}) + 3D(q,\bar{q},\bar{q},h_{20})$$
$$+3C(q,h_{20},\bar{h}_{20}) + 6C(q,h_{11},h_{11}) + 3C(q,q,\bar{h}_{21}) + 6C(q,\bar{q},h_{21})$$
$$+6C(\bar{q},h_{11},h_{20}) + C(\bar{q},\bar{q},h_{30}) + 3B(h_{20},\bar{h}_{21}) + 6B(h_{11},h_{21})$$
$$+3B(q,h_{22}) + B(h_{02},h_{30}) + 2B(\bar{q},h_{31})\rangle$$

The bifurcation is nodegenerate if $\frac{1}{2}\left(\mathrm{Im}[e^{-i\theta_0}c_1]\right)^2 + \mathrm{Re}[e^{-i\theta_0}c_2] \neq 0$

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3.5. Resonance 1:1

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} w_1 + w_2 \\ w_2 + \frac{1}{2} a w_1^2 + b w_1 w_2 \end{pmatrix} + \cdots$$

$$H(w_1, w_2) = w_1 q_0 + w_2 q_1 + \frac{1}{2} h_{20} w_1^2 + h_{11} w_1 w_2 + \frac{1}{2} h_{02} w_2^2 + \cdots ,$$

where $A q_0 = q_0, A q_1 = q_1 + q_0, A^T p_0 = p_0, A^T p_1 = p_1 + p_0$ with

$$\langle p_0, q_1 \rangle = \langle p_1, q_0 \rangle = 1, \;\; \langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 0$$

$$\begin{aligned} w_1^2 &: \quad (A - I_n) h_{20} = -B(q_0, q_0) + a q_1 \\ w_1 w_2 &: \quad (A - I_n) h_{11} = -B(q_0, q_1) + h_{20} + b q_1 \\ w_2^2 &: \quad (A - I_n) h_{02} = -B(q_0, q_0) + 2 h_{11} + h_{20} \end{aligned}$$

$$a = \langle p_0, B(q_0, q_0) \rangle, \;\; b = \langle p_0, B(q_0, q_1) \rangle + \langle p_1, B(q_0, q_0) \rangle$$

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3.6. Resonance 1:2

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} -w_1 + w_2 \\ -w_2 + \frac{1}{6} c w_1^3 + \frac{1}{2} d w_1^2 w_2 \end{pmatrix} + \cdots$$

$$H(w_1, w_2) = w_1 q_0 + w_2 q_1 + \sum_{2 \le j+k \le 3} \frac{1}{j!k!} h_{jk} w_1^j w_2^k + \cdots ,$$

where $Aq_0 = -q_0, Aq_1 = -q_1 + q_0, A^T p_0 = -p_0, A^T p_1 = -p_1 + p_0,$

$$\langle p_0, q_1 \rangle = \langle p_1, q_0 \rangle = 1, \;\; \langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 0$$

$$h_{20} = -(A - I_n)^{-1} B(q_0, q_0), \;\; h_{11} = -(A - I_n)^{-1} [B(q_0, q_1) + h_{20}]$$

$$c = \langle p_0, C(q_0, q_0, q_0) + 3B(q_0, h_{20}) \rangle$$

$$d = \langle p_0, 2B(q_0, h_{11}) + B(q_1, h_{20}) + C(q_0, q_0, q_1) \rangle + \langle p_1, 3B(q_0, h_{20}) + C(q_0, q_0, q_0) \rangle$$

3.7. Resonance 1:3 $Aq = e^{i\theta_0}q,\ A^T p = e^{-i\theta_0}p,\ \langle p, q\rangle = 1,\ \theta_0 = \frac{2\pi}{3}$

$$w \mapsto e^{i\theta_0}w + \frac{1}{2}b\bar{w}^2 + \frac{1}{2}cw|w|^2 + \cdots$$

$$H(w_1, w_2) = wq + \bar{w}\bar{q} + \sum_{j+k\geq 2} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k,$$

where

$$b = \langle p, B(\bar{q}, \bar{q})\rangle$$

$$h_{20} = (A - e^{2i\theta_0}I_n)^{INV}[\bar{b}\bar{q} - B(q,q)],\quad h_{11} = -(A - I_n)^{-1}B(q,\bar{q})$$

$$c = \langle p, C(q,q,\bar{q}) + 2B(q, h_{11}) + B(\bar{q}, h_{20})\rangle$$

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3.8. Resonance 1:4 $Aq = e^{i\theta_0} q,\ A^T p = e^{-i\theta_0} p,\ \langle p, q \rangle = 1,\ \theta_0 = \frac{\pi}{2}$

$$w \mapsto e^{i\theta_0} w + \frac{1}{2} c w |w|^2 + \frac{1}{6} d \bar{w}^3 + \cdots$$

$$H(w_1, w_2) = wq + \bar{w}\bar{q} + \sum_{j+k \geq 2} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k,$$

where

$$h_{20} = -(A + I_n)^{-1} B(q, q), \quad h_{11} = -(A - I_n)^{-1} B(q, \bar{q})$$

$$c = \langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) \rangle$$

$$d = \langle p, C(\bar{q}, \bar{q}, \bar{q}) + 3B(\bar{q}, \bar{h}_{20}) \rangle$$

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4. Open problems

- $\mu_1 = 1, \mu_{2,3} = e^{\pm i\theta_0}$ [R. Vitolo, 2003]
- $\mu_1 = -1, \mu_{2,3} = e^{\pm i\theta_0}$ [J. Los, 1989]
- $\mu_{1,2} = e^{\pm i\theta_1}, \mu_{3,4} = e^{\pm i\theta_2}$ [G. Iooss & J. Los, 1988]
- Periodic normal forms for codim 2 bifurcations of limit cycles using BVP techniques
- Implementation into the standard software:
  1. Automatic differentiation of Poincaré maps
  2. Directional derivatives

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