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Bifurcations of Maps

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Overview

General introduction

Overview

Part 1

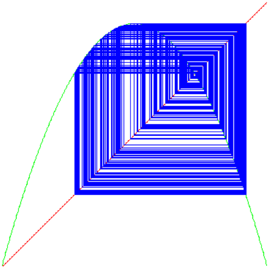
- Setting and stability
- Local codimension 1 bifurcations
- Invariant manifolds and homoclinic bifurcations
- Chaos and Lyapunov exponents

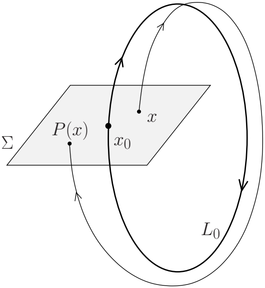
Part 2

- Bifurcations on and of invariant curves
- Codim 2 bifurcations as organizing centers

Maps: Examples

- Models of discrete-time nature; Logistic map, Population biology with off-spring only once per year, Models from economy/game theory with policy adaptation every round.
- Derived from ODE's: Periodically forced ODE's, Poincaré maps or with an Euler-step, bipedal walkers





Setting

Consider a map with parameter α

$$x \mapsto f(x, \alpha) \in \mathbb{R}^n, \qquad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m.$$

Orbit: Sequence of points defined by iterating initial point

$$x_0, x_1 = f(x_0), x_2 = f(f(x_0)), \dots, x_k := f^k(x_0), \dots$$

Fixed point: $f(x_0, \alpha_0)^k - x_0 = 0$.

Minimal period k; fixed point if $k = 1$ or cycle if $k > 1$.

We study dynamics near a fixed point as the parameter α varies and set w.l.o.g. $k = 1$. We will mostly ignore phenomena induced by non-invertibility.

Stability of fixed point (cycle)

Consider evolution of a small perturbation $x_n = x_0 + u_n$:

$$u_{n+1} = f(x_n)^k - x_0 = \underbrace{f(x_0)^k - x_0}_{=0} + Au_n + \underbrace{O(\|u_n\|^2)}_{\textit{ignore}},$$

where $A = f_x(x_0, \alpha_0)^k$. So: Near a fixed point the dynamics is given by the linearized mapping

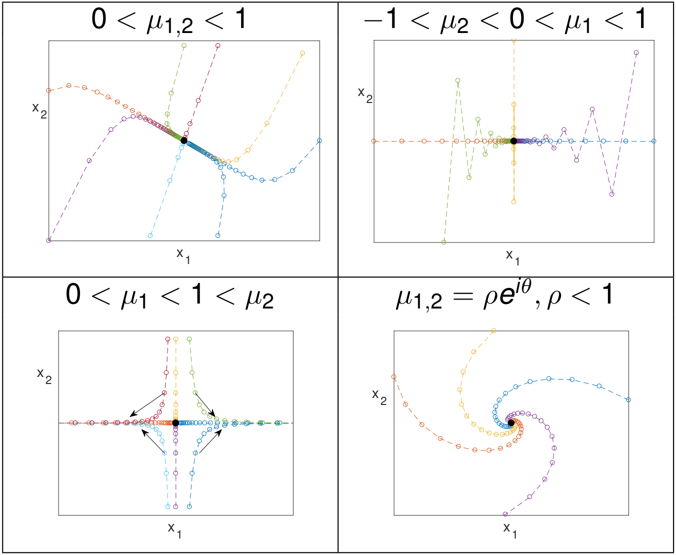
$$u \mapsto Au.$$

The fixed point has **multipliers** (eigenvalues of A)

$$\{\mu_1, \mu_2, \ldots, \mu_n\} = \sigma(A),$$

The fixed point is stable if $\forall i : |\mu_i| < 1$.

Some Linear Phase Portraits



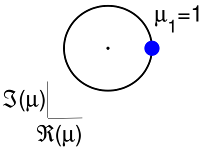
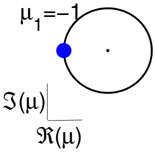
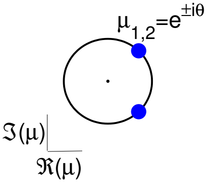
Overview

Occurence of codim 1 bifurcations

Follow a family of fixed points with the defining system

$$F := f(x, \alpha)^k - x.$$

The stability of a fixed point may change as a multiplier crosses the unit circle when a parameter is varied:

Limit Point	Period-Doubling	Neimark-Sacker
		

Limit Point Bifurcation

The fixed point has a simple multiplier $\mu_1 = 1$ and no other multipliers on the unit circle. The simplest example is

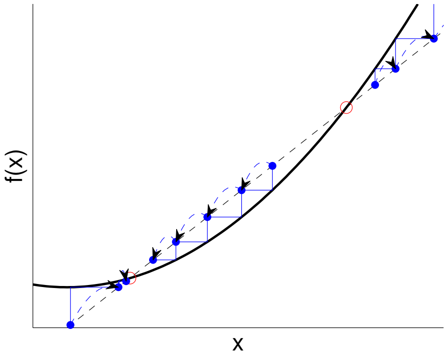
$$\xi \mapsto \alpha + \xi + a\xi^2,$$

where $a \neq 0$.
Other names: saddle-node or fold bifurcation.

The Implicit Function Theorem guarantees the existence of a branch of fixed points $x(\alpha)$ of $f(x,\alpha)^k - x = 0$ as long as $1 \notin \sigma(A)$.

Limit Point Bifurcation

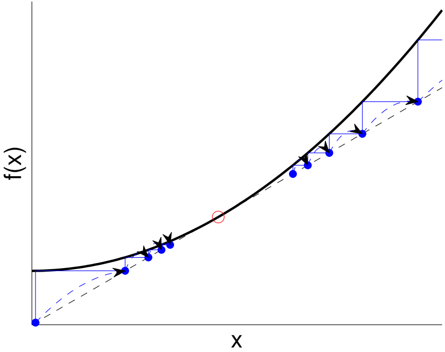
As the parameter crosses the critical value, two fixed points, one stable, one unstable, coalesce and disappear.



$\alpha < \alpha_0$

Limit Point Bifurcation

As the parameter crosses the critical value, two fixed points, one stable, one unstable, coalesce and disappear.



$\alpha = \alpha_0$

Limit Point Bifurcation

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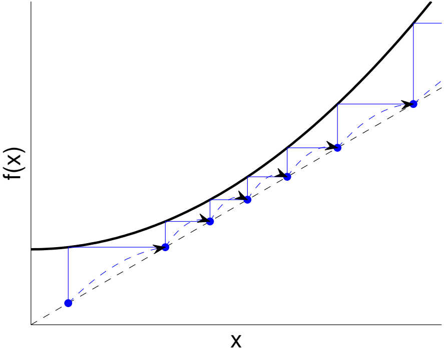


$\alpha > \alpha_0$

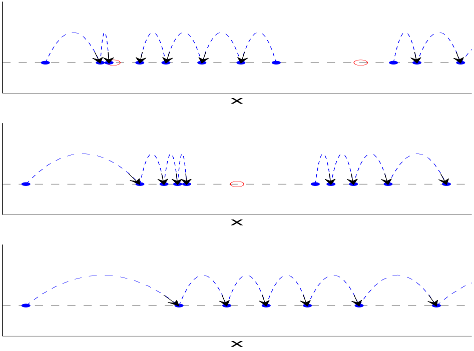
Limit Point Bifurcation

As the parameter crosses the critical value, two fixed points, one stable, one unstable, coalesce and disappear.

$\alpha < \alpha_0$

$\alpha = \alpha_0$

$\alpha > \alpha_0$

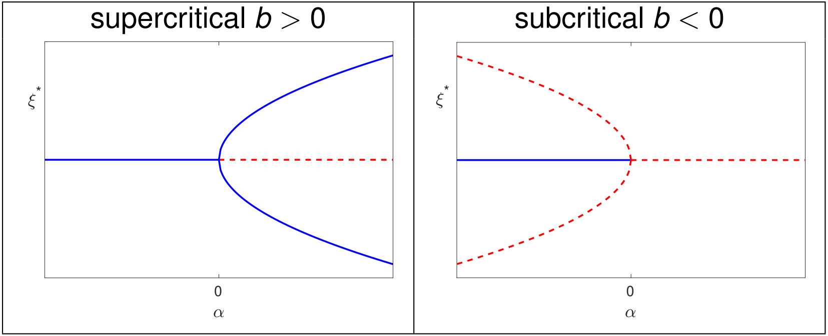


Period-Doubling Bifurcation

The fixed point has a simple multiplier $\lambda_1 = -1$ and no other multipliers on the unit circle. The simplest example is

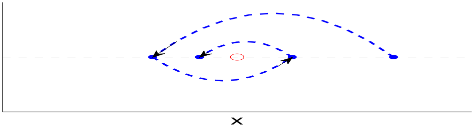
$$\xi \mapsto -\xi(1-\alpha) + b\xi^3,$$

where $b \neq 0$. Other names: flip bifurcation.
Branches of cycles; $\xi^* = 0$ and $\xi^* = \pm\sqrt{\frac{\alpha}{b}}$

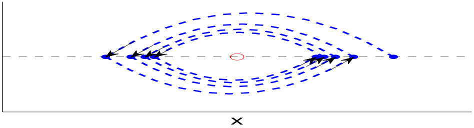


Period-Doubling Bifurcation

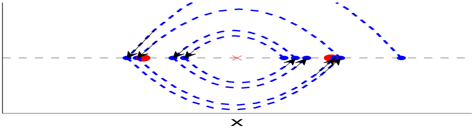
When the parameter crosses the critical value, a cycle of period 2 bifurcates from the fixed point. 2-cycle stable if $b > 0$.



$\alpha < \alpha_0$



$\alpha = \alpha_0$



$\alpha > \alpha_0$

Neimark-Sacker bifurcation

Suppose for a critical value of the parameter $\alpha = \alpha_0$

- the fixed point has critical multipliers $\mu_{1,2} = e^{\pm i\theta_0}$ and no other eigenvalues on the unit circle.
- $e^{iq\theta_0} \neq 1,$ for $q = 1, 2, 3, 4$, i.e. no strong resonances.

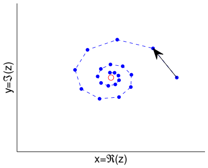
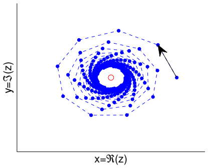
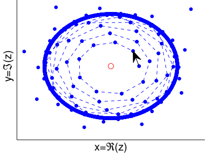
The simplest example is given by

$$z \mapsto z e^{i\theta_0} \left(1 + \alpha + d|z|^2 \right),$$

where $z = x + iy = \rho e^{i\phi}$ is a complex variable and d a complex constant.

Neimark-Sacker bifurcation

If the first Lyapunov coefficient $c := Re(d) \neq 0$, then a unique *closed invariant curve* appears around the fixed point when the parameter crosses the critical value.
Supercritical case $c < 0$: the invariant curve is stable.

$\alpha < \alpha_0$	$\alpha = \alpha_0$	$\alpha > \alpha_0$
		

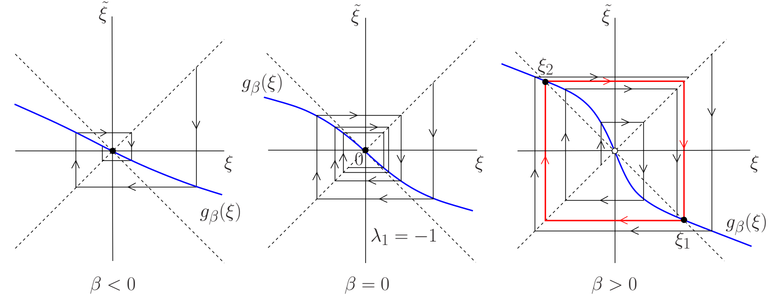
Neimark-Sacker bifurcation

Remarks:

- Subcritical case $c > 0$: an unstable invariant curve disappears as the fixed point becomes unstable when α increases.
- The dynamics on the invariant curve may be a rigid rotation $\phi \mapsto \phi + \theta$ if the rotation number $\theta/(2\pi)$ is (sufficiently) irrational.
- If $\theta/(2\pi)$ is close to rational, the dynamics may be more complicated, see tomorrow.
- Other names used in literature: Hopf (for maps), secondary Hopf, Torus bifurcation

Period-doubling bifurcation revisited

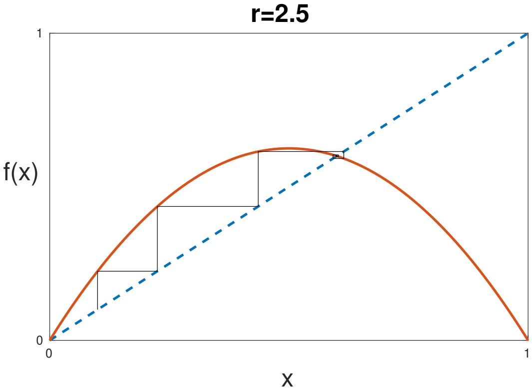
Cobweb: Plot graph and iterate by plotting
$(x, x) \to (x, f(x)) \to (f(x), f(x))$.



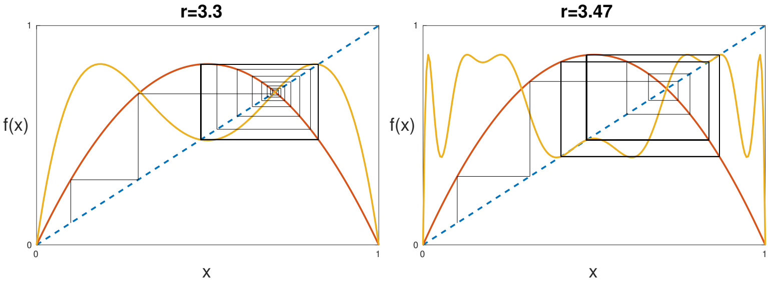
Logistic Map

Mapping the unit-interval $[0, 1]$ to itself with parameter $0 < r < 4$

$$x \mapsto f(x, r) := rx(1 - x)$$



Logistic Map: Period-Doublings



Second and Fourth iterate shown in yellow.
As r increases, see ▸ Movie.

Lyapunov exponents λ

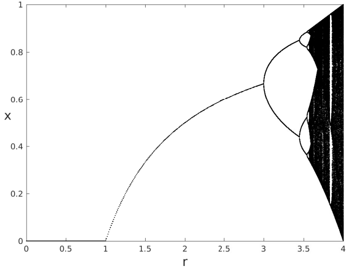
Measure of rate of separation of orbits near $\{x_k\}$
For every iterate consider $f(x_k + \delta\vec{v}) = x_{k+1} + \delta Df(x_k)\vec{v} + ...$
Define growth rate $r_k = \|Df(x_k)\vec{v}\|$

$$\lambda(\vec{v}) := \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} r_k$$

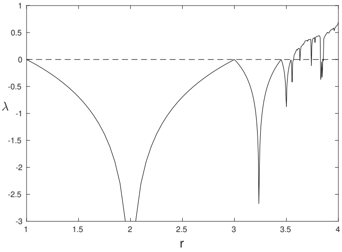
- Choose large N so that average converges.
- There are n exponents for n-dimensional systems.
- $\lambda < 0$ indicates stability, $\lambda = 0$ corresponds to a neutral direction (higher-dimensions), $\lambda > 0$ indicates chaos.

Logistic Map: Positive exponents suggest chaos.

Coordinates of attractor



Lyapunov exponent



r_2	3	r_{16}	3.5644072	
r_4	3.44948974	r_{32}	3.5687594	$r_\infty \approx 3.57$
r_8	3.54409035	r_{64}	3.5696916	

Overview

General introduction

Local Codimension 1 Bifurcations
- Period-Doubling Route to chaos

Invariant Manifolds and Homoclinic tangencies
- Generalized Hénon Map

Bifurcations of Invariant curves
- Phase-locking
- Bifurcations of Invariant Curves

Codim 2 Bifurcations
- 1-Dimensional codim 2 bifurcations
- Strong Resonances 1:1 & 1:2

(Un)Stable Manifolds

Consider a saddle fixed point x_0 with
$\underbrace{|\mu_1| < \cdots < |\mu_i|}_{W^s} < 1 < \underbrace{|\mu_{i+1}| < \cdots < |\mu_n|}_{W^u}$.
A point x in the stable manifold W^s satisfies

$$\lim_{j \to \infty} f(x)^j = x_0.$$

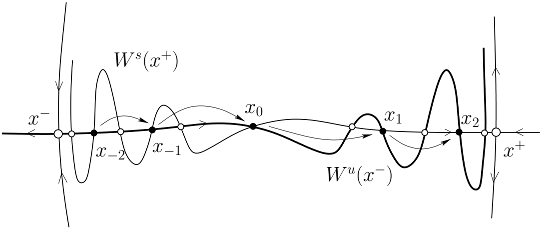
A point x in the unstable manifold W^u (if defined) satisfies

$$\lim_{j \to \infty} f(x)^{-j} = x_0.$$

The (un)stable manifold near the fixed point can be approximated by the (un)stable eigenspace of the linearization.

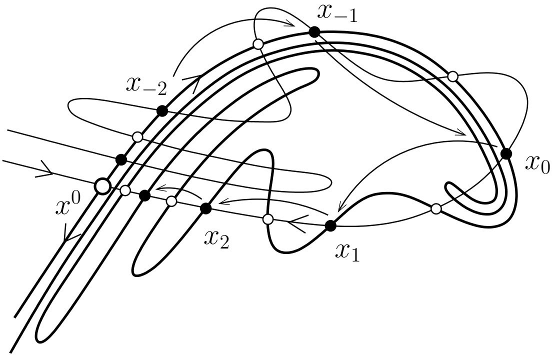
Connecting Orbits

Consider stable and unstable manifolds of saddles $x^{\pm}$.



A heteroclinic orbit $\{x_i\}$ satisfies $\lim_{j \to -\infty} x_j = x^-$ and $\lim_{j \to \infty} x_j = x^+$. For a homoclinic orbit we have $x^- = x^+$.

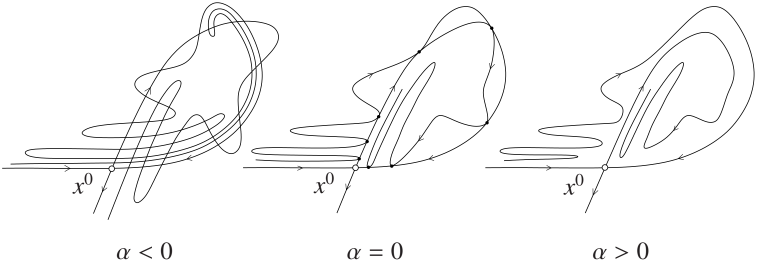
Homoclinic orbits come in pairs



A transversal intersection of manifolds persists for small parameter variations.
A transversal homoclinic orbit allows the construction of Smale's Horseshoe$\Longrightarrow$ Chaotic dynamics

Global bifurcation: Homoclinic tangency

A transversal intersection (dis)appears through a primary tangency of the stable and unstable manifolds.



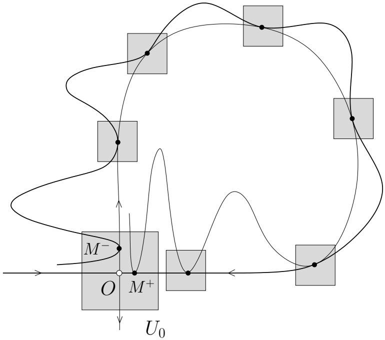
Let's turn to the dynamics near a homoclinic tangency bifurcation curve.

Generalized Hénon Map (GHM): Setting

(A) Map f_0 has a saddle fixed point O with eigenvalues γ, λ, such that $0 < |\lambda| < 1 < |\gamma|$;

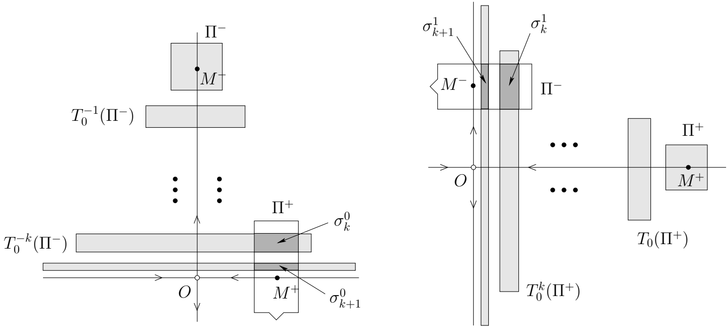
(B) the saddle quantity $\sigma \equiv |\lambda\gamma| = 1$;

(C) the invariant manifolds $W^u(O)$ and $W^s(O)$ have a quadratic tangency at points of a homoclinic orbit Γ.



GHM:Domains of definition

Consider a $(n_0 + k)$-round orbit:
Start at σ_k^0 in Π^+ ending at σ_k^1 in Π^- after k iterations of f_0. Next iterate n_0-times along the homoclinic orbit to come back at Π^+.



(a) (b)

GHM and bifurcations of fixed points

Approximate return map near σ_k^0 defined by

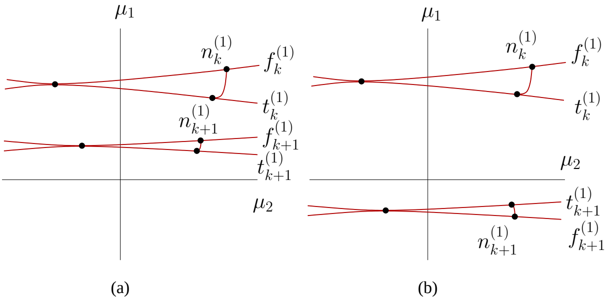
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \alpha_1 - \alpha_2 x - y^2 + Rxy \end{pmatrix},$$

where α is a parameter close to the homoclinic tangency bifurcation curve $\mu_1 = 0$ in the original map f.

- LP bifurcation for $\alpha_1 = \dfrac{(\alpha_2 + 1)^2}{4(R-1)}$
- PD bifurcation for $\alpha_1 = \dfrac{1}{4}(\alpha_2 + 1)^2(3 - R)$
- NS bifurcation for $\alpha_1 = \dfrac{(\alpha_2 - 1)(\alpha_2 - 1 + 2R)}{R^2}$
- Open regions with stable fixed points.

GHM and Newhouse regions

Away from the homoclinic tangency bifurcation curve $\mu_1 = 0$, and for saddle quantity $\sigma < 1$, there is a parameter set with infinitely many stable fixed points with high period.



Overview

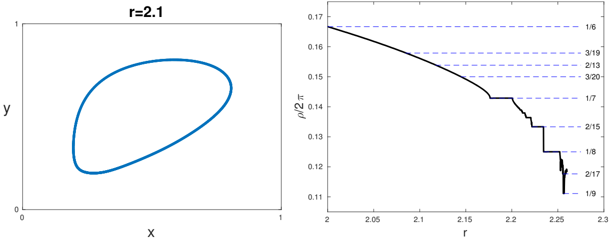
Neimark-Sacker in Delayed Logistic Map

$$F := \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} rx(1-y) \\ x \end{pmatrix}$$

Fixed Point at $x^* = y^* = \frac{r-1}{r}$.

$$DF = \begin{pmatrix} r(1-y) & -rx \\ 1 & 0 \end{pmatrix} \Longrightarrow \det(DF(x^*, y^*)) = rx^* = r - 1$$

So there is a Neimark-Sacker bifurcation at $r = 2$. See ▸ Movie



Model accounting for Phase-Locking

Suppose rotation number $\rho \approx 2\pi p/q$.
Include higher order terms in model to describe period q cycles

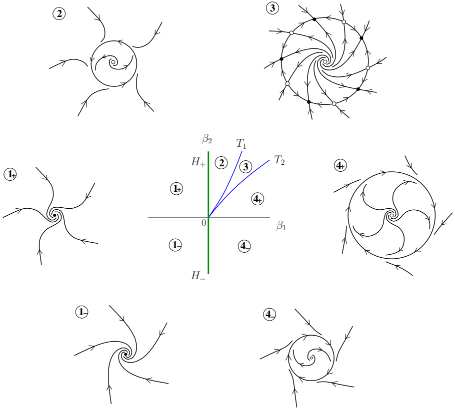
$$z \mapsto z(1+\beta_1)e^{i\theta(\beta)} + \left(\sum_{m=1}^{\lfloor (q-1)/2 \rfloor} A_m(\beta) z|z|^{2m} \right) + B(\beta) z^{q-1} + ...$$

Only really higher order if $q \geq 5$.
Model has the following bifurcations

- Neimark-Sacker bifurcation for $\beta_1 = 0$
- Saddle-Node of period q: $\beta_2 = C_1\beta_1 \pm C_2\beta_1^{(q-2)/2}$ for some constants $C_{1,2}$ depending on A_m, B.

Resonance tongue from NS-bifurcation

As parameters vary through the tongue, a saddle and a node cycle of period q appear on the invariant curve.

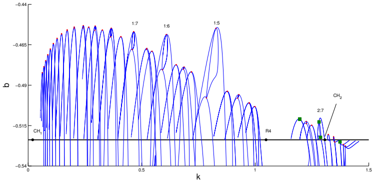


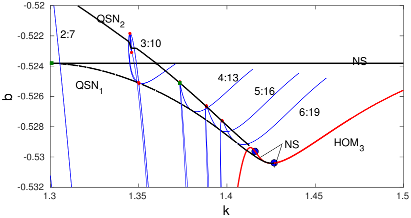
Resonance Tongues in 3D Map

Adaptive Control Map (Frouzakis et al. IJBC 1991)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ bx + k + yz \\ z - \frac{ky}{c+y^2}(bx + k + yz - 1) \end{pmatrix}$$

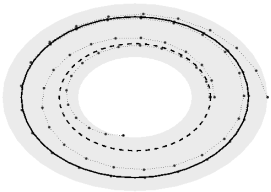
Resonance tongues emerge from the Neimark-Sacker bifurcation at $b = -\frac{c+1}{c+2}$. Fix $c = 0.1$.

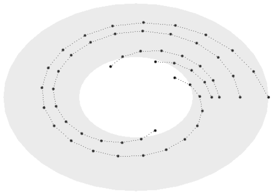




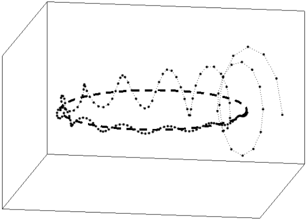
Bifurcations of Invariant Curves 1

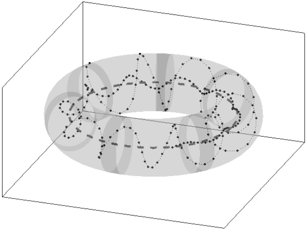
Quasi-periodic Saddle-Node bifurcation



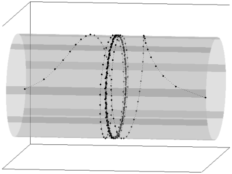


Quasi-periodic Torus bifurcation

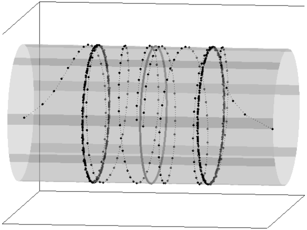


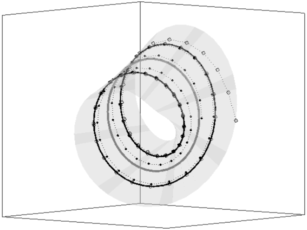


Bifurcations of Invariant Curves 2



Quasi-periodic Doubling bifurcation (two options)





Overview

Why bother about codim 2 bifurcations?

- Numerical continuation yields bifurcation curves depending on two parameters.
- Bifurcation curves divide the parameter plane into several regions with qualitatively different dynamics. Such local and global bifurcations curves come together at codim 2 bifurcation points acting as **organizing centers**.
- For codim 2 bifurcations the normal forms are known, but unfoldings differ depending on critical coefficients.
- The idea is to come to a consistent picture of phase portraits when going around the codim 2 point in the parameter plane.

Recap codim 1 bifurcations

The simplest model systems are so-called normal forms

$$\begin{aligned} \text{LP:} \quad \xi \mapsto & \quad \alpha + x + a\xi^2, \\ \text{PD:} \quad \xi \mapsto & \quad -\xi(1-\alpha) + b\xi^3, \\ \text{NS:} \quad z \mapsto & \quad z e^{i\theta_0}\left(1 + \alpha + d|z|^2\right), \end{aligned}$$

with $a, b, Re(d) \neq 0$ and $q\theta_0 \neq 2\pi$ for $q = 1, 2, 3, 4$.

Codim 2 bifurcations appear through additional multipliers or degeneracies.

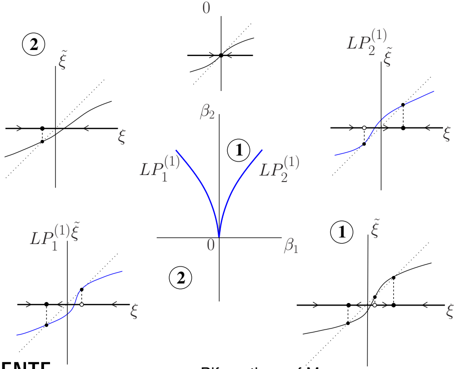
List of local codim 2 bifurcations

Case	degeneracy or additional multipliers	Name
(1)	$\mu_1 = 1, a = 0$	cusp
(2)	$\mu_1 = -1, b = 0$	generalized flip
(3)	$\mu_{1,2} = e^{\pm i\theta_0}, c = Re(d) = 0$	Chenciner
(4)	$\mu_1 = \mu_2 = 1$	1:1 resonance
(5)	$\mu_1 = \mu_2 = -1$	1:2 resonance
(6)	$\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{2\pi}{3}$	1:3 resonance
(7)	$\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{\pi}{2}$	1:4 resonance
(8)	$\mu_1 = 1, \mu_2 = -1$	fold-flip
(9)	$\mu_1 = 1, \mu_{2,3} = e^{\pm i\theta_0}$	fold-NS
(10)	$\mu_1 = -1, \mu_{2,3} = e^{\pm i\theta_0}$	flip-NS
(11)	$\mu_{1,2} = e^{\pm i\theta_1}, \mu_{3,4} = e^{\pm i\theta_2}$	double NS

Cusp: normal form and unfolding

When the quadratic coefficient vanishes along an LP-bifurcation curve, the following normal form characterizes nearby dynamics

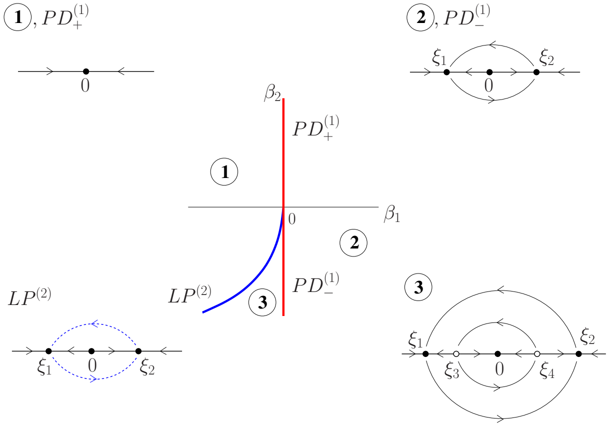
$$\xi \mapsto \beta_1 + \xi + \beta_2 \xi^2 + d\xi^3 + \cdots, \quad d \neq 0$$



Degenerate Period-Doubling

The normal form is given by

$$w \mapsto G(w) = -w(1+\beta_1) + \beta_2 w^3 + e w^5 + \cdots, \quad e \neq 0$$



Strong resonances

If $q\theta_0 = 2\pi, \;\; q = 1, 2, 3, 4$ then a Neimark-Sacker bifurcation becomes a strong resonance.

- The critical normal form for the 1:1 resonance is:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ y + a_1 x^2 + b_1 xy \end{pmatrix} + \cdots$$

- The critical normal form for the 1:2 resonance is:

$$f := \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x + y \\ -y + c_1 x^3 + d_1 x^2 y \end{pmatrix} + \cdots$$

Observation: composition $Df^{-1}(0) \circ f$ is close to the identity.

Vector field Approximation

Theorem (Takens,Neimark): Suppose $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and $D\Phi(0)$ has all eigenvalues on the unit circle. Denote by S the semi-simple part of $D\Phi(0)$. Then there exists a diffeomorphism Ψ and a vector field X such that

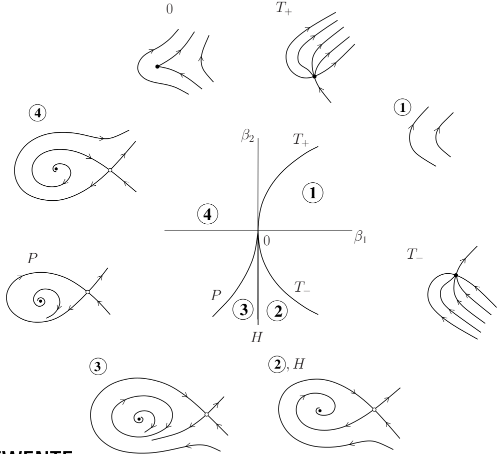
$$\Psi \circ \Phi \circ \Psi^{-1} = \phi_X(t = 1) \circ S$$

in the sense of Taylor series.
Proof: Global Analysis of Dynamical Systems: Festschrift dedicated to Floris Takens for his 60th birthday. Eds. H.W Broer, B. Krauskopf G. Vegter, see Thm 4.6 p20.
Remark:

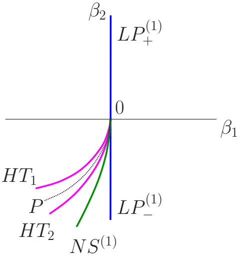
- Φ is the time-1 map of the flow of the vector field X.
- parameters can be included.

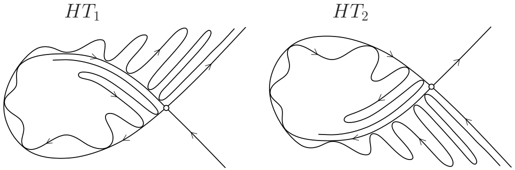
1:1 Resonance Approximately Unfolded

The unfolding of the approximating vector field involves Saddle-node, Hopf and a global homoclinic bifurcation.



1:1 Resonance; Map intricacies

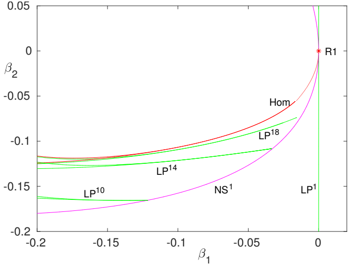




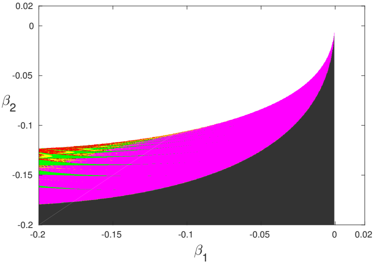
Perturbed 1:1 Resonance Normal Form

In practice, you may observe the following diagrams:

Bifurcation curves



Lyapunov exponents



Note phase-locking (green/yellow), chaos (red), invariant curve (magenta)

1:2 Resonance: normal form

The normal form G (including parameters) is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x+y \\ \beta_1 + (-1+\beta_2)y + c_1 x^3 + d_1 x^2 y \end{pmatrix} + \cdots$$

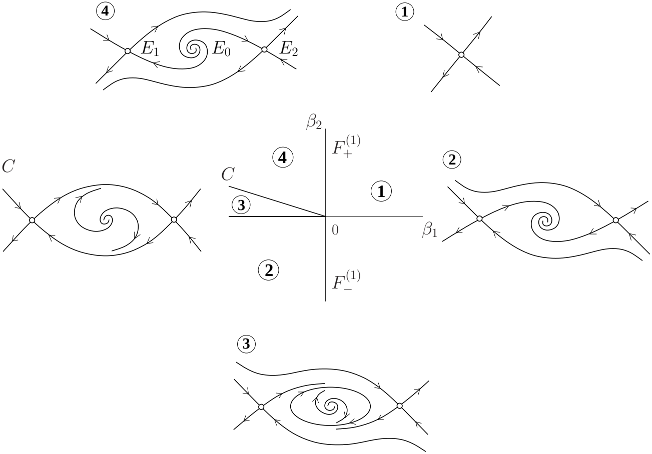
Non-degenerate if $c_1 \neq 0$ and $d_1 + c_1 \neq 0$.
If $c_1 < 0$ a codim 1 branch of Neimark-Sacker bifurcation of double period emanates.

Asymptotic expression of the new branch

$$H^{(2)} : \left(x^2, y, \beta_1, \beta_2\right) = \left(-\frac{1}{c_1}, 0, 1, \left(2 + \frac{d_1}{c_2}\right)\right)\varepsilon$$

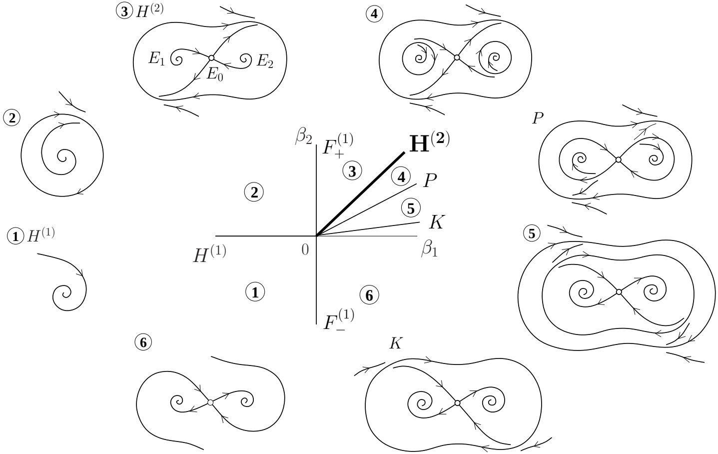
Unfolding $c_1 > 0$

Only NS and PD branches.



Unfolding $c_1 < 0$:

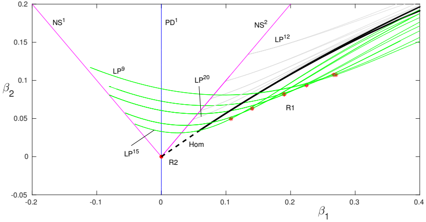
New codim 1 branch $H^{(2)}$ (local bifurcation)



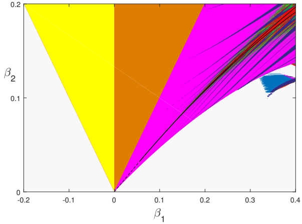
Perturbed 1:2 Resonance Normal Form

In practice, you may observe the following diagrams:

Bifurcation curves



Lyapunov exponents



Note periods 1,2 (yellow/orange), phase-locking (green/dark blue), chaos (red), invariant curve (magenta)