# THE HODGE POLYNOMIAL OF $\overline{\mathcal{M}}_{3,1}$

E. GETZLER AND E. LOOIJENGA

ABSTRACT. We give formulas for the Dolbeault numbers of $\overline{\mathcal{M}}_{3,1}$, using the first author's calculations of the weights of the cohomology of $\mathcal{M}_{2,2}$ and $\mathcal{M}_{2,3}$ and the second author's calculation of the weights of the cohomology of $\mathcal{M}_{3,1}$.

Recall that the Serre polynomial of a quasi-projective variety $X$, or more generally, of a Deligne-Mumford stack whose coarse moduli space is a quasi-projective variety, is the polynomial

$$\mathsf{e}(X) = \sum_{i,j} u^i v^j \sum_{k=0}^{\infty} (-1)^k h^{i,j}(H_c^k(X,\mathbb{C})).$$

If a finite group $\Gamma$ acts on $X$, the equivariant Serre polynomial $\mathsf{e}_\Gamma(X)$ is defined in the same way, and takes values in $R(\Gamma)$. For background to all of this, see [1].

If $X$ is smooth and projective, or more generally, if $X$ is a smooth Deligne-Mumford stack whose coarse moduli space is projective, then the Serre polynomial of $X$ equals its Hodge polynomial:

$$\mathsf{e}(X) = \sum_{i,j} (-u)^i (-v)^j h^{i,j}(X).$$

In particular, Poincaré duality implies the functional equation

$$\mathsf{e}(X)(u,v) = q^{\dim(X)}\, \mathsf{e}(X)(u^{-1}, v^{-1}),$$

where throughout this paper, $q$ denotes $uv$. All of this is also true equivariantly.

If $2g-2+n>0$, let $\mathcal{M}_{g,n}$ be the moduli stack of smooth projective curves of genus $g$ together with $n$ distinct marked points, and let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack of stable curves of (arithmetic) genus $g$ with $n$ distinct marked smooth points [4]; these stacks are smooth, of dimension $3g-3+n$, and the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$ is projective. Furthermore, there is a flat map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ which represents $\overline{\mathcal{M}}_{g,n+1}$ as the universal curve over $\overline{\mathcal{M}}_{g,n}$. In particular, $\overline{\mathcal{M}}_{g,1}$ is the universal stable curve of genus $g$. In this paper, we calculate $\mathsf{e}(\overline{\mathcal{M}}_{3,1})$. In particular, we show that $h^{i,j}(\overline{\mathcal{M}}_{3,1}) = 0$ unless $i=j$.

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There is a standard identification of $R(\mathbb{S}_n)$ with the space of symmetric functions (in an infinite number of variables $\{x_i \mid i \geq 0\}$) of degree $n$, which we denote by $\Lambda_n$. If $X$ is a quasi-projective variety with action of $\mathbb{S}_n$, let $\mathsf{e}_n(X)$ be its Serre polynomial, thought of as an element of $\Lambda_n[u,v]$.

There is a formula permitting the calculation of the $\mathbb{S}_n$-equivariant Hodge polynomial of $\overline{\mathcal{M}}_{g,n}$ in terms of the $\mathbb{S}_m$-equivariant Serre polynomials of the moduli stacks $\mathcal{M}_{h,m}$, where $h \leq g$ and $2h+m \leq 2g+n$.

Let $\hat{\Lambda}$ be the completion of the ring of symmetric functions:

$$\hat{\Lambda} = \prod_{n=0}^{\infty} \Lambda_n .$$

We may identify $\hat{\Lambda}$ with the algebra of power series in the variables

$$h_n = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \dots x_{i_n} .$$

Introduce an auxilliary variable $\lambda$, and let $\hat{\Lambda}[\![\lambda]\!]$ be the ring of power series in $\lambda$ with coefficients in the ring of symmetric functions. We may identify $\hat{\Lambda} \otimes \mathbb{Q}$ with the algebra of power series in the variables

$$p_n = \sum_{i=0}^{\infty} x_i^n .$$

The ring $\hat{\Lambda}[\![\lambda]\!]$ is a (special) lambda-ring: the Adams operations on $\hat{\Lambda}[\![\lambda]\!]$ are determined by the formulas $\psi_k(p_n) = p_{kn}$ and $\psi_k(\lambda) = \lambda^k$. Let $\mathsf{Exp}$ be the operation

$$\mathsf{Exp}(f) = \sum_{k=0}^{\infty} \sigma_k(f) = \exp\left( \sum_{k=0}^{\infty} \frac{\psi_k(f)}{k} \right) : \lambda\hat{\Lambda}[\![\lambda]\!] \to 1 + \lambda\hat{\Lambda}[\![\lambda]\!],$$

and let $\mathsf{Log}$ be its inverse, given by the formula

$$\mathsf{Log}(f) = \sum_{k=0}^{\infty} \frac{\mu(k)}{k} \psi_k(f) : 1 + \lambda\hat{\Lambda}[\![\lambda]\!] \to \lambda\hat{\Lambda}[\![\lambda]\!].$$

Let $\Delta$ be the differential operator on $\hat{\Lambda}$ (and by extension of coefficients, on $\hat{\Lambda}[\![\lambda]\!]$) give by the formula

$$\Delta = \sum_{k=1}^{\infty} \lambda^{2k} \left( \frac{k}{2} \frac{\partial^2}{\partial p_k^2} + \frac{\partial}{\partial p_{2k}} \right) .$$

We may define two elements of $\hat{\Lambda}[\![\lambda]\!]$, by the formulas

$$\Psi = \sum_{2g-2+n>0, n>0} \lambda^{2g-2+n} \, \mathsf{e}_{\mathbb{S}_n}(\overline{\mathcal{M}}_{g,n}) \quad \text{and}$$

$$\Phi = \sum_{2g-2+n>0, n>0} \lambda^{2g-2+n} \, \mathsf{e}_{\mathbb{S}_n}(\overline{\mathcal{M}}_{g,n}) .$$

The following is Theorem (8.13) of [3].

**Theorem 1.** $\Psi = \mathsf{Log}\big(\exp(\Delta)\,\mathsf{Exp}(\Phi)\big)$

We see that to use this formula to calculate $\mathsf{e}(\overline{\mathcal{M}}_{3,1})$, we need the Serre polynomial of $\mathcal{M}_{3,1}$, together with the following equivariant Serre polynomials:

| $\mathcal{M}_{g,n}$ | $\mathsf{e}_{\mathbb{S}_n}(\mathcal{M}_{g,n})$ |
|---|---|
| $\mathcal{M}_{0,3}$ | $s_3$ |
| $\mathcal{M}_{0,4}$ | $qs_4 - s_{2^2}$ |
| $\mathcal{M}_{0,5}$ | $q^2s_5 - qs_{32} + s_{31^2}$ |
| $\mathcal{M}_{0,6}$ | $q^3s_6 - q^2s_{42} + q(s_{41^2} + s_{321}) - (s_{41^2} + s_{3^2} + s_{2^21^2})$ |
| $\mathcal{M}_{0,7}$ | $q^4s_7 - q^3s_{52} + q^2(s_{3^21} + s_{51^2} + s_{421})$ |
| | $- q(s_{51^2} + s_{43} + s_{421} + s_{2^31} + s_{41^3} + s_{3^21} + s_{321^2})$ |
| | $+ s_{321^2} + s_{52} + s_{31^4} + s_{32^2} + s_{421}$ |
| $\mathcal{M}_{1,1}$ | $qs_1$ |
| $\mathcal{M}_{1,2}$ | $q^2s_2$ |
| $\mathcal{M}_{1,3}$ | $q^3s_3 - s_{1^3}$ |
| $\mathcal{M}_{1,4}$ | $q^4s_4 - q^2s_4 - qs_{21^2} + s_{31}$ |
| $\mathcal{M}_{1,5}$ | $q^5s_5 - q^3(s_5 + s_{41}) + q^2(s_{32} - s_{31^2})$ |
| | $+ q(s_{41} + s_{32} + s_{31^2}) - (s_5 + s_{32} + s_{2^21} + s_{1^5})$ |
| $\mathcal{M}_{2,1}$ | $q^4s_1 + q^3s_1$ |
| $\mathcal{M}_{2,2}$ | $q^5s_2 + q^4(s_2 + s_{1^2}) - s_2$ |
| $\mathcal{M}_{2,3}$ | $q^6s_3 + q^5(s_3 + s_{21}) - q^4s_3 - q(s_3 + s_{21})$ |

Here, if $\mu = (\mu_1 \geq \cdots \geq \mu_\ell)$ is a partition of $n$, $s_\mu$ denotes the associated Schur polynomial, given by the Jacobi-Trudi formula $s_\mu = \det(h_{\mu_i+j-i})$. The above Serre polynomials are calculated in [1] (for genus 0 and 1) and [2] (for genus 2).

Applying Theorem 1 (and taking advantage of J. Stembridge's symmetric function package `SF` [6] for `maple`), we see that

$$\mathsf{e}(\overline{\mathcal{M}}_{3,1}) = \mathsf{e}(\mathcal{M}_{3,1}) + 3q^6 + 15q^5 + 29q^4 + 29q^3 + 16q^2 + 4q.$$

**Theorem 2.** *The Hodge polynomial* $\mathsf{e}(\overline{\mathcal{M}}_{3,1})$ *of* $\overline{\mathcal{M}}_{3,1}$ *equals*

$$q^7 + 5q^6 + 16q^5 + 29q^4 + 29q^3 + 16q^2 + 5q + 1.$$

*Proof.* We must show that $\mathsf{e}(\mathcal{M}_{3,1}) = q^7 + 2q^6 + q^5 + q + 1$. This almost, but not quite, follows from [5]. There, it was shown that the polynomial

$$\sum_{i,j,k} u^iv^jt^kh^{i,j}(H^k_c(\mathcal{M}_{3,1},\mathbb{C})) = q^7t^{14} + q^62t^{12} + q^5t^{10} + q(t^8 + t^7) + 2t^6,$$

which would imply that $\mathsf{e}(\mathcal{M}_{3,1}) = q^7 + 2q^6 + q^5 + 2$.

Unfortunately, there was a small error in that calculation; namely, two terms were omitted from the calculation (4.8) of the Poincaré polynomial of $W\backslash T'$ in case $R$ is of type $E_7$, associated to the Dynkin subdiagrams $D_6 \times A_1$

and $D_4 \times A_1^2$ of $E_7$. Once they are taken into account (the first contributes $t^7$, the second $qt^8$), we see that

$$\sum_{i,j,k} u^i v^j t^k h^{i,j}(H^k(\mathcal{M}_{3,1}, \mathbb{C})) = q^7 t^{14} + q^6 2t^{12} + q^5 t^{10} + qt^8 + t^7 + (aqt + b)(t^7 + t^6),$$

where $a$ and $b$ equal 0 or 1. This proves the theorem. □

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Northwestern University, 2033 Sheridan Rd., Evanston, IL 60208-2730, USA
*E-mail address*: `getzler@math.nwu.edu`

Utrecht University, P.O. Box 80010, 3508 TA Utrecht, The Netherlands
*E-mail address*: `looijeng@math.uu.nl`