Representations out of polydifferentials and the KZ-system

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Sapporo, August 2009

Goals:

(1)To construct the highest weight representations of Kac-Moody Lie algebras inside algebras of polydifferentials.
(2) To connect this with the Schechtman-Varchenko approach to the KZ system.
(3) To apply this to the associated WZW system

1. Polydifferentials.

Definition: Let M be a manifold, $\mathcal{I}$ a finite set. A *polydifferential* on $M^{\mathcal{I}}$ is a section of $\otimes_{i\in\mathcal{I}}(\wedge^\bullet T^*M)$ (this is an *ordinary* tensor product–no total order on $\mathcal{I}$ is needed, no Koszul rule applies).

On the $\mathfrak{S}_{\mathcal{I}}$-invariant polydifferentials on $M^{\mathcal{I}}$ is defined an exterior derivative up to sign, making it a complex whose cohomology is $(\otimes_{\mathcal{I}} H^\bullet(M))^{\mathfrak{S}_{\mathcal{I}}}$ ($\neq H^\bullet(M^{\mathcal{I}})^{\mathfrak{S}_{\mathcal{I}}}$ unless $H^\bullet(M) = H^{\text{even}}(M)$).

We take for M the Riemann sphere (denoted here $\mathbf{P}$) whose affine coordinate is denoted t. For $X \subset \mathcal{I}$, $dt_X = \prod_{x\in X} dt_x$ is regarded is a (meromorphic) polydifferential of degree $|X|$ on $\mathbf{P}^{\mathcal{I}}$ (no total order on X is needed).

Two modifications here: we allow $\mathcal{I}$ to be (countably) infinite and we work in a relative setting (with base the affine line $\mathbb{C}$, coordinate z, later $\mathbb{C}^n$):

$$\mathbf{P}^{\mathcal{I}}_{\mathbb{C}} : \mathbf{P}^{\mathcal{I}} \times \mathbb{C} \to \mathbb{C}.$$

Let $\mathcal{V}$ be the graded $\mathbb{C}$-vector space of the relative polydifferentials on $\mathbf{P}_{\mathbb{C}}^{\mathcal{I}}$ which is $\mathbb{C}$-spanned by the forms

$$\omega_{I*} = \omega_I(z) := \frac{dt_{i_N} dt_{i_{N-1}} \cdots dt_{i_1}}{(t_{i_N} - t_{i_{N-1}}) \cdots (t_{i_2} - t_{i_1})(t_{i_1} - z)}.$$

where $I = (i_N, i_{N-1}, \ldots, i_1)$ runs over the finite sequences in $\mathcal{I}$. (We write ω_{I*} if we want to leave the variable $z \in \mathbb{C}$ unnamed.) Stipulate $\omega_{\emptyset *} = \omega_* = 1$. Notice that we get zero unless the sequence I is without repetition. Denote by $\widehat{\mathcal{V}}_N$ the space of (possibly infinite) sums of these relative polydifferentials of degree N and put $\widehat{\mathcal{V}} := \oplus_{N=0}^{\infty} \widehat{\mathcal{V}}_N$.

Lemma 1 (Shuffle rule) *The graded vector space $\mathcal{V}$ is closed under product (it is a shuffle algebra): for finite sequences in I and J in $\mathcal{I}$,*

$$\omega_{I*}\omega_{J*} = \sum_{K \text{ a shuffle of } I \text{ and } J} \omega_{K*}$$

Some operators in $\widehat{\mathcal{V}}$.

Let be given complex numbers $(\lambda_i)_{i\in\mathcal{I}}$ and $(c_{i,j})_{i,j\in\mathcal{I},i\neq j}$. For $i\in\mathcal{I}$ define an operator Φ_i in the space of meromorphic relative polydifferentials by

$$\Phi_i := \frac{\lambda_i dt_i}{t_i - z} - \sum_{j\neq i} c_{i,j}\frac{dt_i dt_j}{t_i - t_j}\iota_{\partial/\partial t_j},$$

Here dt_i is the multiplication operator in the space of these polydifferentials and by $\iota_{\partial/\partial t_i}$ its adjoint (which acts in the ith tensor factor by sending dt_i to 1 and 1 to 0). So for a finite subset $X\subset\mathcal{I}$, we have

$$\Phi_i(dt_X) = \Big(\frac{\lambda_i}{t_i - z} - \sum_{x\in X}\frac{c_{i,x}}{t_i - t_x}\Big)dt_i dt_X.$$

(the right hand side is zero when $i\in X$).

Lemma 2 *This operator preserves $\mathcal{V}$ (hence also the completion $\widehat{\mathcal{V}}$), for*

$$\Phi_x(\omega_{I*}) = \sum_{I=I''I'}(\lambda_x - \sum_{i\in I'} c_{x,i})\omega_{I''xI'*},$$

(the right hand side vanishes if $x\in I$).

Proof: Straightforward from shuffle rule.

Residue operators. We regard $\mathrm{Res}_{(t_x=\infty)}$, $x \in \mathcal{I}$, as an operator of $\mathcal{V}$ to itself (so strictly speaking it is the composite of a residue and the pull-back along the projection that suppresses the xth factor).

Observe that for a sequence I in $\mathcal{I}$ without repetition and $x \in \mathcal{I}$, we have

$$\mathrm{Res}_{(t_x=\infty)}\,\omega_{I*} = \begin{cases} -\omega_{J*} & \text{if } I = xJ, \\ 0 & \text{otherwise} \end{cases}$$

A straightforward calculation then yields:

Corollary 3 *Given* $x, y \in \mathcal{I}$, *then* $[\Phi_x, \mathrm{Res}_{(t_y=\infty)}]\omega_{I*} = 0$ *unless* $x = y \notin I$, *in which case it multiplies* ω_{I*} *by the scalar* $\lambda_x - \sum_{i\in I} c_{x,i}$.

2. Representations inside $\widehat{\mathcal{V}}$.

Let $(c_{k,l})_{k,l=1}^{r}$ be a *generalized Cartan matrix*: $c_{k,k} = 2$, and for $k \neq l$, $c_{k,l} \in \mathbb{Z}_{\leq 0}$ with $c_{l,k} = 0 \Leftrightarrow c_{l,k} = 0$

Associated Lie algebra $\tilde{\mathfrak{g}}$ with generators $\tilde{e}_1, \dots, \tilde{e}_r, \tilde{f}_1, \dots, \tilde{f}_r$ subject to $[\tilde{e}_k, \tilde{f}_l] = 0$ for $k \neq l$ and if we put $\tilde{h}_k := [\tilde{e}_k, \tilde{f}_k]$, then

$$[\tilde{h}_k, \tilde{e}_l] = c_{k,l}\tilde{e}_l, \quad [\tilde{h}_k, \tilde{f}_l] = -c_{k,l}\tilde{f}_l, \quad [\tilde{h}_k, \tilde{h}_l] = 0.$$

The *Kac-Moody Lie algebra* $\mathfrak{g}$ is obtained by also imposing the *Serre relations*

$$\mathrm{ad}(\tilde{e}_k)^{1-c_{k,l}}\tilde{e}_l = 0, \quad \mathrm{ad}(\tilde{f}_k)^{1-c_{k,l}}\tilde{f}_l = 0$$

for $k \neq l$.

$\mathfrak{h}$, the Cartan subalgebra: linear span of the h_k's. (In $\mathfrak{g}$ we omit tilde's.)

In what follows we suppose our index set $\mathcal{I}$ endowed with a surjection $\pi : \mathcal{I} \to \{1, \ldots, r\}$ such that each fiber $\mathcal{I}_k := \pi^{-1}(k)$ is countably infinite.
We write $\overline{i}$ for $\pi(i)$.

If $P = (p_N, \ldots, p_1)$ is any sequence in $\{1, \ldots, r\}$, then we put

$$\omega(P) := \sum_{\overline{I}=P} \omega_{I*},$$

where the sum is over all sequences in $\mathcal{I}$ that map under π to P (for $N = 0$, read 1). The right hand side is an element of $\widehat{\mathcal{V}}$ that is invariant under the group $\mathfrak{S}_\pi$ of all permutations of $\mathcal{I}$ that leave π invariant. In fact, these elements give a basis of $\widehat{\mathcal{V}}^{\mathfrak{S}_\pi}$.

Now fix a *dominant weight* relative to the above Lie data, i.e., a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ of r nonnegative integers.

We take for the coefficients defining Φ_i, $c_{i,j} := c_{\bar{i},\bar{j}}$, $\lambda_i := \lambda_{\bar{i}}$:

$$\Phi_i := \frac{\lambda_{\bar{i}} dt_i}{t_x - z} - \sum_{j \neq i} c_{\bar{i},\bar{j}} \frac{dt_i dt_j}{t_i - t_j} \iota_{\partial/\partial t_j},$$

For this choice of coefficients, we put

$$\tilde{f}_k := \sum_{i \in \mathcal{I}_k} \Phi_i \quad \text{(acts in } \widehat{\mathcal{V}}\text{).}$$

Then

$$\tilde{f}_k \omega(P) = \sum_{P = P'' P'} (\lambda_k - c_{k,P'}) \omega(P'' k P')$$

The residue formula suggests to put:

$$\tilde{e}_k \omega(P) := \begin{cases} \omega(P') & \text{if } P = kP', \\ 0 & \text{otherwise} \end{cases}$$

It is then clear that $\tilde{e}_k$ and $\tilde{f}_l$ commute when $k \neq l$ and that $[\tilde{e}_k, \tilde{f}_k]$ multiplies $\omega(P)$ by the scalar $\lambda_k - c_{k,P}$. The residue lemma shows that we have an interpretion $\tilde{e}_k$ as a sum of residues along divisors at infinity:

Lemma 4 *Let* $\omega \in \widehat{\mathcal{V}}^{\mathfrak{S}_\pi}$. *Then the restriction of* $\tilde{e}_k\omega$ *to the hyperplane* $t_i = \infty$, $i \in \mathcal{I}_k$, *as a polydifferential equals* $-\operatorname{Res}_{(t_i=\infty)}\omega$. *In fact, if we identify this residue with a form on* $\mathbf{P}_{\mathbb{C}}^{\mathcal{I}}$ *via its pull-back under the projection* $\mathbf{P}_{\mathbb{C}}^{\mathcal{I}} \to \mathbf{P}_{\mathbb{C}}^{\mathcal{I}-\{i\}}$, *then*

$$\tilde{e}_k(\omega) = -\sum_{i\in\mathcal{I}_k} \operatorname{Res}_{(t_i=\infty)}\omega.$$

In particular, $\tilde{e}_k(\omega) = 0$ *if and only if* ω *is regular along the hyperplanes* $(t_i = \infty)$, $i \in \mathcal{I}_k$.

Proposition 5 *The operators* $\tilde{e}_k, \tilde{f}_k$, $k = 1, \ldots, r$, *define a representation of* $\tilde{\mathfrak{g}}$ *on* $\widehat{\mathcal{V}}^{\mathfrak{S}_\pi}$ *which in addition satisfies the Serre relation* $\mathrm{ad}(\tilde{f}_k)^{1-c_{k,l}}\tilde{f}_l = 0$ *(*$k \neq l$*).*

Denote by $\mathcal{V}(\lambda)$ the smallest subspace of $\widehat{\mathcal{V}}^{\mathfrak{S}_\pi}$ that contains 1 and is invariant under the operators $\tilde{f}_1, \ldots, \tilde{f}_r$.

Theorem 6 *Then $\tilde{\mathfrak{g}}$ acts on $\mathcal{V}(\lambda)$ through the highest weight representation of $\mathfrak{g}$ of weight λ with highest weight vector 1. This highest weight representation is integrable in the sense that each of the e_k and f_k acts in a locally nilpotent fashion.*

Generalization to a tensor product of highest weight representations

Fix dominant weights, $\lambda^{(1)}, \ldots, \lambda^{(n)}$. Work now with n variables $z^1, \ldots, z^n$ instead of one: we consider $\mathbf{P}^{\mathcal{I}}_{\mathbb{C}^n} : \mathbf{P}^{\mathcal{I}} \times \mathbb{C}^n \to \mathbb{C}^n$. For n sequences $I^1, \ldots, I^n$ in $\mathcal{I}$ we have the relative polydifferential

$$\omega_{I^1}(z^1)\omega_{I^2}(z^2)\cdots\omega_{I^n}(z^n).$$

on $\mathbf{P}^{\mathcal{I}}_{\mathbb{C}^n}$. It is zero unless the sequence $I^{(1)} \ldots I^{(n)}$ is without repetition.

$\mathcal{V}(\mathbf{z})$: the graded vector space spanned by these polydifferentials.

$\hat{\mathcal{V}}(\mathbf{z})_N$: the completion of $\mathcal{V}(\mathbf{z})_N$ which allows for infinite sums, $\hat{\mathcal{V}}(\mathbf{z}) := \oplus_N \hat{\mathcal{V}}(\mathbf{z})_N$.

Given n sequences $(P^1, \ldots, P^n)$ in $\{1, \ldots, r\}$, we observe that

$$\prod_{\nu=1}^{n} \omega(P^\nu)(z^\nu) = \sum_{\overline{I^\nu}=P^\nu} \omega_{I^1}(z^1)\omega_{I^2}(z^2)\cdots\omega_{I^n}(z^n),$$

sum is over all n-tuples of sequences $(I^1, \ldots, I^n)$ in $\mathcal{I}$ which map under π to $(P^1, \ldots, P^n)$.

These elements form a $\mathbb{C}$-basis of $\hat{\mathcal{V}}(\mathbf{z})^{\mathfrak{S}_\pi}$ and so the above factorization defines an isomorphism

$$\hat{\mathcal{V}}(\mathbf{z})^{\mathfrak{S}_\pi} \cong \hat{\mathcal{V}}^{\mathfrak{S}_\pi} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \hat{\mathcal{V}}^{\mathfrak{S}_\pi}.$$

The action of $\tilde{f}_k$ operating on the νth factor with dominant weight $\lambda^{(\nu)}$ is denoted $\tilde{f}_k^{(\nu)}$. The sum $\sum_{\nu=1}^{n} \tilde{f}_k^{(\nu)}$ acts as $\tilde{f}_k$ in the tensor representation and hence is simply denoted $\tilde{f}_k$. We do likewise for the other generators of $\tilde{\mathfrak{g}}$.

It is clear that

$$\mathcal{V}(\lambda^*) := \mathcal{V}(\lambda^{(1)}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{V}(\lambda^{(n)})$$

is the smallest subspace of $\widehat{\mathcal{V}}^{\mathfrak{S}_\pi}$ that contains 1 and is invariant under the operators $\tilde{f}_k^{(\nu)}$ and $\tilde{h}_k^{(\nu)}$. It is the tensor product of n highest weight representations.

Notice that the subspace $\mathcal{V}(\lambda^*)_0$ killed by $\mathfrak{h}$ is trivial unless $\sum_{k=1}^r \lambda^{(k)}$ is a sum of positive roots: $\sum_{k=1}^r \lambda^{(k)} = \sum_{k=1}^r m_k \alpha_k$; if $m := \sum_k m_k$, then $\mathcal{V}(\lambda^*)_0 = \mathcal{V}_m(\lambda^*)$ (polydifferentials of degree m).

It follows from our residue formula that:

Theorem 7 *The space of $\mathfrak{g}$-invariants $\mathcal{V}(\lambda^*)^{\mathfrak{g}}$ is the space of polydifferentials in $\mathcal{V}_m(\lambda^*)$ that are regular along every hyperplane at infinity $(t_i = \infty)$, $i \in \mathcal{I}$.*

3. The KZ-connection

Let $\lambda^* = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ and $\mathcal{V}(\lambda^*) = \mathcal{V}(\lambda^{(1)}) \otimes \cdots \otimes \mathcal{V}(\lambda^{(n)})$ (a representation of $\mathfrak{g}$) be as before.

We fix the choice of a symmetric $\mathfrak{g}$-invariant tensor $C \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$. We may regard C as a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}^*$. If $\mathfrak{g}$ is simple, then C is unique up to scalar, and C is nondegenerate, when nonzero.

C will have the form

$$C = C_0 + \sum_{\alpha} C_\alpha,$$

with $C_0 \in \mathfrak{h} \otimes \mathfrak{h}$ and the sum is over all the roots. Here C_0 can be any symmetric tensor invariant under the Weyl group. It then determines C as follows: if α is a positive root and $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ are generators such that $h_\alpha := [e_\alpha, f_\alpha]$ is the corresponding coroot in $\mathfrak{h}$ (this means that

$\alpha(h_\alpha) = 2$), then $C_\alpha = \frac{1}{2}C(\alpha,\alpha)e_\alpha \otimes f_\alpha$ and $C_{-\alpha} = \frac{1}{2}C(\alpha,\alpha)f_\alpha \otimes e_\alpha$ (so $C_{-\alpha}$ is the transpose of C_α).

Comparison with the 1991 paper of Schechtman and Varchenko

They consider the case of a ‘symmetrized generalized Cartan matrix’ and use a presention of $\mathfrak{g}$ derived from that. They work with a fixed weight space and take $\mathcal{I}$ finite and minimal in a sense (the representation is then absent). The interpretation of the KZ connection in terms of polydifferentials that we give below is due to them.

The KZ connection

For $1 \leq \nu < \mu \leq n$, let $C^{(\nu,\mu)}$ be the endomorphism of $\mathcal{V}(\lambda^*)$ obtained by letting C act trough the tensor factors indexed by ν and μ. This operator commutes with the diagonal action of $\mathfrak{g}$ and hence preserves the $\mathfrak{g}$-isotypical summands.

Our base variety will be the subset $U_n \subset \mathbb{C}^n$ defined by $\prod_{\nu<\mu}(z^\nu - z^\mu) \neq 0$.
The KZ connection is the connection on the trivial bundle over U_n with fiber $\mathcal{V}(\lambda^*)$, given by the $\mathsf{End}(\mathcal{V}(\lambda^*))$-valued differential

$$A^C_{KZ} := \sum_{1\leq\nu<\mu\leq n} C^{(\nu,\mu)} \frac{d(z^\nu - z^\mu)}{z^\nu - z^\mu},$$

where $C^{(\nu,\mu)}$ means to have C act on $\mathcal{V}(\lambda^*)$ via its tensor factors indexed by ν and μ. This is clearly a connection with logarithmic singularities on $\mathbb{C}^n$. It is known to be flat.

A Gauss-Manin connection

Central in the discussion is a differential associated to C:

$$\eta^C := \tfrac{1}{2}\textstyle\sum_{i,j\in\mathcal{I},i\neq j} C(\alpha_{\bar{i}},\alpha_{\bar{j}})\frac{d(t_i-t_j)}{t_i-t_j} +$$
$$-\sum_{\nu=1}^{n}\sum_{i\in\mathcal{I}} C(\alpha_{\bar{i}},\lambda^{(\nu)})\frac{d(t_i-z^\nu)}{t_i-z^\nu}$$
$$+\sum_{1\leq\nu<\mu\leq n} C(\lambda^{(\nu)},\lambda^{(\mu)})\frac{d(z^\nu-z^\mu)}{z^\nu-z^\mu}.$$

It is logarithmic and we can formally write it as $-d\log F^C$, where F^C is a product of linear forms with complex exponents. Consider the first order differential operator $d^C := d+\eta^C$. So d^C is the ordinary exterior derivative for the multivalued polydifferentials after they get formally multiplied by the inverse of F^C: $d^C = F^C\, d\,(F^C)^{-1}$. In particular (but still formally), a d^C-closed form is F_C times a d-closed form.

The corresponding connection (of Gauss-Manin type) on the form level requires us to lift each basis vector field $\partial_\nu := \partial/\partial z^\nu$ on U_n to $\mathbf{P}^{\mathcal{I}} \times U_n$ and then covariant derivation with respect to z^ν will be Lie derivation with respect to that lift. In order to ensure that logarithmicity is preserved we take a lift that depends on the argument:

Lemma 8 *Let $\omega := \omega_{I^1}(z^1)\omega_{I^2}(z^2)\cdots\omega_{I^n}(z^n)$ be a basis element of $\mathcal{V}(\mathbf{z})$ and let $\tilde{\partial}_\nu := \partial_\nu + \sum_{i\in I^\nu} \frac{\partial}{\partial t_i}$ (a vector field on $\mathbf{P}^{\mathcal{I}} \times U_n$ that lifts the vector field ∂_ν to U_n). Then the twisted Lie derivative $\mathcal{L}^C_{\tilde{\partial}_\nu} := d^C\iota_{\tilde{\partial}_\nu} + \iota_{\tilde{\partial}_\nu} d^C$ maps ω to $\eta^C(\tilde{\partial}_\nu).\omega$ and the latter lies in $\mathbb{C}[U_n] \otimes_{\mathbb{C}} \mathcal{V}(\mathbf{z})$. This map is $\mathfrak{S}_\pi$-equivariant and defines a connection on $\mathcal{V}(\mathbf{z})$ with logarithmic pole whose form A^C_{GM} lies in*

$$\sum_{\nu<\mu} \frac{d(z^\nu - z^\mu)}{z^\nu - z^\mu} \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}^n}(\mathcal{V}(\mathbf{z})).$$

We refer to this as the Gauss -Manin connection*.*

Theorem 9 *The GM connection on the trivial bundle over U_n with fiber $\mathcal{V}(\lambda^*)$ coincides with the KZ connection: $A^C_{GM} = A^C_{KZ}$.*

4. The WZW system

We assume $\mathfrak{g}$ simple. Let $\tilde{e} \in \mathfrak{g}$ be a longest nonzero iterated commutator of e_k's (spans the highest coroot space). For $\mathbf{z} \in U_n$, we write $e_{\mathbf{z}} := \sum_{k=1}^{r} z^{\nu} \tilde{e}^{(\nu)}$.

Definition. Fix a positive integer ℓ, refered to as the *level*. The *space of conformal blocks of level* ℓ *relative to* $\mathbf{z}$ is the subspace of $\mathcal{V}(\lambda^*)$ killed by $\mathfrak{g}$ and $\tilde{e}_{\mathbf{z}}^{\ell+1}$. These define a subbundle of the trivial bundle $\mathcal{V}(\lambda^*)^{\mathfrak{g}}_{U_n}$, called the *bundle of conformal blocks of level* ℓ*.* We denote this bundle $\mathcal{W}(\lambda^*)_{\ell}$.

There is a natural generator $B \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$, namely the one which takes the value 2 on the highest coroot.
Denote by g_k the number of times e_k occurs and put $g := 1 + \sum_{k=1}^{r} g_k$ (the *dual Coxeter number*).

Proposition 10 *(Beilinson-Feigin) The bundle* $\mathcal{W}(\lambda^*)_{\ell}$ *is flat for the KZ-connection for which* $C = C_{\ell} := \frac{1}{g+\ell} B$*.*

Can characterize $\mathcal{W}(\lambda^*)_\ell$ in terms of a vanishing property.

Assume that $\mathcal{V}(\lambda^*)_0 \neq 0$ (otherwise no issue) so that $\sum_{k=1}^r \lambda^{(k)}$ is a sum of simple roots: $\sum_{k=1}^r \lambda^{(k)} = \sum_{k=1}^r m_k \alpha_k$.

Theorem 11 (vanishing criterion) $\omega \in \mathcal{V}(\lambda^*)$ *defines a section of* $\mathcal{W}(\lambda^*)_\ell$ *if and only if for any subset* $X \subset \mathcal{I}$ *with* $|X \cap \mathcal{I}_k| = (\ell + 1)g_k$, ω *vanishes on the locus* $t_x = \infty$ *for all* $x \in X$ *(i.e., if we blow up this subspace, then* ω *vanishes on it).*

4. Topological interpretation

As before, we assume that $\mathcal{V}(\lambda^*)_0 \neq 0$, so that $\sum_{k=1}^r \lambda^{(k)} = \sum_{k=1}^r m_k \alpha_k$. We put $m := \sum_k m_k$.

For the study of $\mathcal{V}(\lambda^*)_0 = \mathcal{V}_m(\lambda^*)$ no harm is done if we assume (as we will) that $\mathcal{I}$ is finite and $|\mathcal{I}_k| = m_k$ (so that $|\mathcal{I}| = m$).
Now $\mathcal{V}(\lambda^*)_0$ has become a space of relative polydifferentials of *maximal* degree m on $\mathbb{P}^{\mathcal{I}}_{U_n}$.

Denote by $D \subset \mathbf{P}^{\mathcal{I}}_{U_n}$ the sum of the divisors defined by $t_i = \infty$, $t_i = z^\nu$, $t_i = t_j$ ($i \neq j$).
Let $\Omega^\bullet(\log)_{U_n}$ be the $\mathcal{O}_{U^n}$-module of *relative* logarithmic forms on $\mathbb{P}^{\mathcal{I}}_{U_n}$ with polar divisor D that are *anti-invariant* relative to the $\mathfrak{S}_{\mathcal{I}}$-action. Wedging with η_C turns $\Omega^\bullet(\log)_{U_n}$ into a complex ($C \neq 0$ here).

Lemma 12 *For every $\mathbf{z} \in U_n$, $\mathcal{V}(\lambda^*)_0$ maps isomorphically onto $\Omega^m(\log)_{\mathbf{z}}$ with $f(\mathcal{V}(\lambda^*)_2) = f(\mathcal{V}_{m-1}(\lambda^*))$ being mapped onto $\eta^C \wedge \Omega^{m-1}(\log)_{\mathbf{z}}$.*

Corollary 13 (Schechtman-Varchenko) *We get for every $\mathbf{z} \in U_n$ an identification*

$$\mathcal{V}(\lambda^*)^{\mathfrak{g}} \cong H^m(\Omega^\bullet(\text{log})_{\mathbf{z}}, \eta^C_{\mathbf{z}} \wedge).$$

With the help of work of Deligne and Esnault-Schechtman-Viehweg we can obtain a topological interpretation of the right hand side.

On $\mathbf{P}^{\mathcal{I}}_{U_n} - D$ we have a local system $\mathbb{L}^C$ of rank one with connection form η^C.

There is a natural blow up $\mathcal{X} \to \mathbf{P}^{\mathcal{I}}_{U_n}$, minimal for the property that the full transform of D is a normal crossing divisor. Then η^C has on every irreducible component a residue. Let $\tilde{D} \subset \mathcal{X}$ be a union of such irreducible components with the property that if an irreducible component has *integral* residue, then it is in $\tilde{D}$ if and only if that residue is nonpositive (so this $\tilde{D}$ is not unique). We then have an inclusion

$$j : \mathbf{P}^{\mathcal{I}}_{U_n} - D \subset \mathcal{X} - \tilde{D}.$$

The following sharpens a theorem of Schechtman-Varchenko.

Theorem 14 *For every* $\mathbf{z} \in U_n$*, we have a natural identification of* $\mathcal{V}(\lambda^*)^{\mathfrak{g}} \cong H^m(\Omega^\bullet(\log)_{\mathbf{z}}, \eta_{\mathbf{z}}^C \wedge)$ *with* $\mathbb{H}^m(\mathbf{P}_{\mathbf{z}}^{\mathcal{I}} - D_{\mathbf{z}}; R^\bullet j_! \mathbb{L}_{\mathbf{z}}^C)^{sign}$*. This identifies the Gauss-Manin given by* A_{GM}^C *with the usual one.*

It is conjectured that the bundle of conformal blocks has a flat inner product. We conjecture more:

Conjecture 15 *For every* $\mathbf{z} \in U_n$*, the flat isomorphism* $\mathcal{V}(\lambda^*) \cong \mathbb{H}^m(\mathbf{P}_{\mathbf{z}}^{\mathcal{I}} - D_{\mathbf{z}}; R^\bullet j_! \mathbb{L}_{\mathbf{z}}^C)^{sign}$ *identifies* $\mathcal{W}(\lambda^*)_\ell$ *with the bidegree* $(m, 0)$*-part of the image of*

$$H_c^m(\mathbf{P}_{\mathbf{z}}^{\mathcal{I}} - D_{\mathbf{z}}; \mathbb{L}_{\mathbf{z}}^{C_\ell})^{sign} \to \mathbb{H}^m(\mathbf{P}_{\mathbf{z}}^{\mathcal{I}} - D_{\mathbf{z}}; R^\bullet j_! \mathbb{L}_{\mathbf{z}}^C)^{sign}$$

So this bidegree $(m, 0)$-part should be flat and the inner product then should come from Hodge theory (use that $\mathbb{L}_{\mathbf{z}}^{C_\ell}$ has a flat metric).

The conjecture holds for the case $\mathfrak{g} = \mathfrak{sl}(2)$; this essentially follows from work of *T.R. Ramadas* (Ann. of Math. 2009) (we gave a more direct proof (J. Geom. and Physics 2009) by deriving this from the equivalent formulation: if $\omega \in \Omega^{\bullet}(\log)_{\mathbf{z}}$ satisfies the vanishing conditions of Thms 7 and 11, then $F_{\mathbf{z}}^{C_\ell}\omega_{\mathbf{z}}$ is square integrable).