Foliations in moduli spaces of abelian varieties and dimension of leaves

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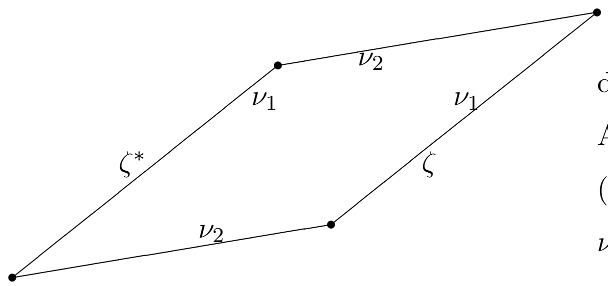
Introduction

In this talk I consider moduli spaces of polarized abelian varieties in characteristic p. Also we consider deformation spaces of p-divisible groups. In these spaces we describe **foliations**, as constructed in [25].

What are the dimensions of leaves in these foliations, see (2.8)? We **compute the dimensions** of the central leaves, both in the unpolarized case and in the polarized case. These computations use either a result on "minimal p-divisible groups", see [29], plus a result by Wedhorn, see [32], or a construction basically due to Ching-Li Chai which gives a generalization of Serre-Tate coordinates, see [2].

The dimension of a leaf in the *unpolarized case*, $\zeta = \mathcal{N}(X)$, see (8.3):



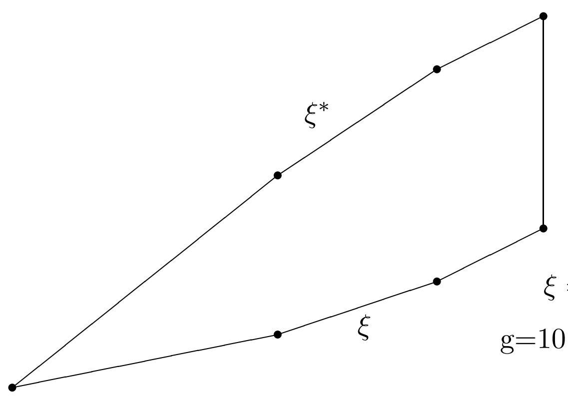
$\dim(\mathcal{C}_X(D)) = \sum_{0<i<h} (\zeta^*(i) - \zeta(i))$.

An example:

$(m_1, n_1) = (4, 1), \quad (m_2, n_2) = (1, 5),$

$\nu_1 = 4/5, \quad \nu_2 = 1/6, \quad h = 11.$

The dimension of a leaf in the *polarized case*, $\xi = \mathcal{N}(A)$, see (9.3):



$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) =$

$= \sum_{0<i\leq g} (\xi^*(i) - \xi(i)).$

An Example:

$\xi = (4, 1) + (2, 1) + 2{\cdot}(1, 1) + (1, 2) + (1, 4),$

g=10; slopes: $\{5 \times \frac{4}{5}, 3 \times \frac{2}{3}, 4 \times \frac{1}{2}, 3 \times \frac{1}{3}, 5 \times \frac{1}{5}\}$.

These results find their natural place in joint work with Ching-Li Chai, which we expect finally to appear in [3]. I thank Chai for the beautiful things I learned from him.

Historical remark. In summer 2000 I gave a talk in Oberwolfach on foliations in moduli spaces of abelian varieties. After my talk, in the evening of Friday 4-VIII-2000 Bjorn Poonen asked me several questions, especially related to the problem I raised to determine the dimensions of central leaves. Our discussion resulted in Problem 21 in [5]. His expectations coincided with computations I had made of these dimensions for small values of g. Then I jumped to the conclusion what those dimensions for general g could be; that is what was proved later, and reported on here, see (8.3), (9.3). I thank Bjorn Poonen for his interesting questions; our discussion was valuable for me.

1 Notation

(1.1) We fix a prime number p. All base schemes and base fields will be in characteristic p. We write K for a field, and we write k and Ω for algebraically closed fields of characteristic p.

We write $\mathcal{A}_g \otimes \mathbb{F}_p$ for the moduli space of polarized abelian varieties of dimension g in characteristic p. Let X_0 be a p-divisible group over K; we write $\mathrm{Def}(X_0)$ for the (infinitesimal, local) deformation space.

For an abelian variety A, or an abelian scheme, and a prime number p we write $A[p^\infty] = \cup_i A[p^i] = X$ for its p-divisible group.

For a p-divisible group in characteristic p one can define its Newton Polygon; this is basically given by the "p-adic values of Frobenius"; for a precise definition see [15], or [4], and see Section 4; we write $\mathcal{N}(X)$ for the Newton polygon of X. A basic theorem tells us that the isogeny class of a p-divisible group over an algebraically closed field is "the same" as its Newton polygon:

(1.2) Theorem (Dieudonné and Manin), see [15], "Classification theorem " on page 35,

$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon}\}.$$

This means: for every p-divisible group X over a field we define its Newton polygon $\mathcal{N}(X)$; over en algebraically closed field, every Newton polygon comes from a p-divisible group and

$$X \quad \sim_k \quad Y \quad \Longleftrightarrow \quad \mathcal{N}(X) = \mathcal{N}(Y).$$

(1.3) NP - EO - Fol. Here is a short survey of strata and foliations, to be explained below. For an abelian variety A, with a polarization (sometimes supposed to be principal) we can study the following objects:

NP $A \mapsto A[p^\infty] \mapsto A[p^\infty]/\sim$ over an algebraically closed field: the isogeny class of its p-divisible group; by the Dieudonné - Manin theorem we can identify this isogeny class of p-divisible groups with the Newton polygon of A. We obtain the Newton polygon strata.

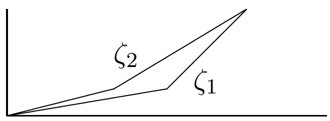
EO $(A,\lambda) \mapsto (A,\lambda)[p] \mapsto (A,\lambda)[p]/\cong$ over an algebraically closed field: we obtain EO-strata; see [23]. Important feature (Kraft, Oort): the number of geometric isomorphism classes of group schemes of a given rank annihilated by p is *finite*.

Fol $\quad (A,\lambda) \mapsto (A,\lambda)[p^\infty] \mapsto (A,\lambda)[p^\infty]/\cong$ over an algebraically closed field: we obtain a foliation of an open Newton polygon stratum; see [25]. Note that for $f < g-1$ the number of leaves is *infinite*.

Note: $X \cong Y \Rightarrow \mathcal{N}(X) = \mathcal{N}(Y)$; conclusion: every central leaf in **Fol** is contained in exactly one Newton polygon stratum in **NP**.
Note: $X \cong Y \Rightarrow X[p] = Y[p]$; conclusion: every central leaf in **Fol** is contained in exactly EO-stratum in **EO**.
However, a NP-stratum may contain many EO-strata, an EO-stratum may intersect several NP-strata. If the p-rank is smaller than $g-1$ a NP-stratum contains infinitely many central leaves. Whether an EO-stratum equals a central leaf is studied and answered in Section 4.

(1.4) For Newton polygons we introduce a partial ordering.

We write $\zeta_1 \succ \zeta_2$ if ζ_1 is "below" ζ_2,
i.e. if no point of ζ_1 is strictly above ζ_2.



If S is a base scheme, and $\mathcal{X} \to S$ is a p-divisible group over S we write

$$\mathcal{W}_\zeta(S) = \{s \in S \mid \mathcal{N}(\mathcal{X}_s) \prec \zeta\} \subset S$$

and

$$\mathcal{W}^0_\zeta(S) = \{s \in S \mid \mathcal{N}(\mathcal{X}_s) = \zeta\} \subset S.$$

See (10.4).

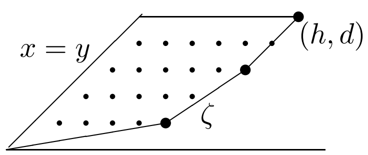
(1.5) **The dimension of Newton polygon strata, unpolarized case.** We fix integers $h \geq d \geq 0$, and we write $c := h - d$. We consider Newton Polygons ending at (h,d). For such a NP ζ we write:

$$\diamond(\xi) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y < d, \quad y < x, \quad (x,y) \ \text{ on or above } \ \zeta\},$$

and we write

$$\boxed{\dim(\zeta) := \#(\diamond(\zeta)).}$$

Example:



$\zeta = 2 \times (1,0) + (2,1) + (1,5) =$

$= 6 \times \frac{1}{6} + 3 \times \frac{2}{3} + 2 \times \frac{1}{1}; \quad h = 11.$

Here $\dim(\zeta) = \#(\diamond(\zeta)) = 22$.

(1.6) **Theorem** see [22], Th. 3.2 and [24], Th. 2.10. *Let X_0 be a p-divisible group over a field K; let $\zeta \succ \mathcal{N}(X_0)$. Then:*

$$\dim(\mathcal{W}_\zeta(\mathrm{Def}(X_0))) \quad = \quad \dim(\zeta).$$

For polarized abelian varieties and for quasi-polarized p-divisible groups one can try to determine the dimension of the Newton polygon strata. For an arbitrary polarization see (11.1). For *principal polarizations*:

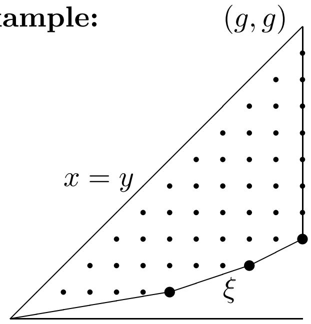
(1.7) The dimension of Newton polygon strata, polarized case. We fix an integer g. For every *symmetric* NP ξ of height $2g$ we define:

$$\triangle(\xi) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leq g, \quad (x,y) \text{ on or above } \xi\},$$

and we write

$$\boxed{\mathrm{sdim}(\xi) := \#(\triangle(\xi)).}$$

Example:



$$\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)) = \#(\triangle(\xi))$$

$$\xi = (5,1) + (2,1) + 2{\cdot}(1,1) + (1,2) + (1,5),$$

$$\text{g=11; slopes: } \{6 \times \tfrac{5}{6}, 3 \times \tfrac{2}{3}, 4 \times \tfrac{1}{2}, 3 \times \tfrac{1}{3}, 6 \times \tfrac{1}{6}\}.$$

This case: $\dim(\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)) = \mathrm{sdim}(\xi) = 48$.

(1.8) Theorem, see [22], Th. 3.4 and [24], Th. 4.1. *Let ξ be a symmetric Newton polygon. Then:*

$$\dim\left(\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)\right) = \mathrm{sdim}(\xi).$$

2 Foliations, see [25]

(2.1) Let X be a p-divisible group and let $\mathcal{Y} \to S$ be a p-divisible group over a base S. We write

$$\mathcal{C}_X(S) = \{s \in S \mid \exists\Omega, \exists \mathcal{Y}_s \otimes \Omega \cong X \otimes \Omega\}.$$

Choose a quasi-polarized p-divisible group (X,λ). Let $(\mathcal{Y},\mu) \to S$ be a quasi-polarized p-divisible group over a base S. We write

$$\mathcal{C}_{(X,\lambda)}(S) = \{s \in S \mid \exists\Omega, \exists(\mathcal{Y},\mu)_s \otimes \Omega \cong (X,\lambda) \otimes \Omega\}.$$

(2.2) Theorem, see [25], Th. 2.3.

$$\mathcal{C}_X(S) \subset \mathcal{W}^0_{\mathcal{N}(X)}(S)$$

is a closed set.

A proof can be given using (10.5), (10.6) and (10.7).

(2.3) Note that we gave a "point-wise" definition of $\mathcal{C}_X(S)$; we can consider $\mathcal{C}_X(S) \subset S$ as a closed set, or as a subscheme with induced reduced structure; however is this last definition "invariant under base change"?. It would be much better to have a "functorial definition" and a nature-given scheme structure on $\mathcal{C}_X(S)$.

Note that the proof of this theorem is quite involved. One of the ingredients is the notion of "completely slope divisible p-divisible groups" introduced by T. Zink, and theorems on p-divisible groups over a normal base, see [33] and [31].

Considering the situation in the moduli space with enough level structure in order to obtain a fine moduli scheme, we see that $\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p)$ is regular.

We write $\mathcal{C}_x$ for the irreducible component of $\mathcal{C}_{(A,\lambda)[p^\infty]}\mathcal{A}$ passing through $[(A,\lambda)] = x$; this is called the *central leaf* passing thought x.
Remark (Chai & FO). In fact, for $\mathcal{N}(A) \neq \sigma$, i.e. A is not supersingular, it is known that $\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A})$ is *geometrically irreducible;* see (11.3).

(2.4) Isogeny correspondences, unpolarized case. Let $\psi : X \to Y$ be an isogeny between p-divisible groups. Then there exist finite surjective morphisms

$$\mathcal{C}_X(\mathrm{Def}(X)) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_Y(\mathrm{Def}(Y)).$$

(2.5) Corollary. *The dimension of* $\mathcal{C}_X(\mathrm{Def}(X))$ *only depends on the isogeny class of* X.

(2.6) Isogeny correspondences, polarized case. Let $\psi : A \to B$ be an isogeny, and let λ respectively μ be a polarization on A, respective on B, and suppose there exists an integer $n \in \mathbb{Z}_{>0}$ such that $\psi^*(\mu) = n{\cdot}\lambda$. Then there exist finite surjective morphisms

$$\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_{(B,\mu)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p).$$

See [25], 3.16.

(2.7) Corollary. The dimension of $\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)$ only depends on the isogeny class of (X,λ).
Remark / Notation. In fact, this dimension only depends on the isogeny class of X. We write

$$c(\xi) := \dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)\right), \quad X = A[p^\infty], \quad \xi := \mathcal{N}(X);$$

this is well-defined: all irreducible components have the same dimension.

A proof of all previous results on isogeny correspondences, and the independence of the dimension of the leaf in an isogeny class can be given using (10.5), (10.6) and (10.7); see [25], 2.7 and 3.13.

(2.8) Here is the central question of this talk. Consider a p-divisible group X_0 over a field K, and let $D = \mathrm{Def}(X_0)$. Consider $g \in \mathbb{Z}_{>0}$ and let $\mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ be the moduli space of principally polarized abelian varieties; choose $[(A,\lambda)] = x \in \mathcal{A}$ and $(A,\lambda)[p^\infty] = (X,\lambda)$. Determine

$$\text{unpolarized case:} \quad \dim(\mathcal{C}_{X_0}(D)) = ?; \qquad \text{polarized case:} \quad \dim(\mathcal{C}_{(X,\lambda)}(\mathcal{A})) = ?.$$

(2.9) Remark. Isogeny correspondences in characteristic p in general blow up and down in a rather wild pattern. The dimension of Newton polygon strata and of EO-strata in general depends very much on the degree of the polarization in consideration. However for p-rank strata the dimension in the whole of $\mathcal{A}_g \otimes \mathbb{F}_p$ solely depends on the p-rank, see [19], Theorem 4.1. It seems a miracle that the dimension on central leaves does not depend on the degree of a polarization.

In [25] we also find the definition of *isogeny leaves*, and we see that any irreducible component of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ up to a finite morphism is isomorphic with the product of a central leaf and an isogeny leaf, see [25], 5.3. Note that all central leaves with the same Newton polygon have the same dimension, see (2.5) and (2.7). However for Newton polygon strata, and hence also for isogeny leaves, the dimension in general depends very much on the degree of the polarization; for more information see (11.1).

Here we do not consider isogeny leaves, and a central leaf just will be called a leaf.

3 Cayley - Hamilton, see [22]

We like to compute the dimension of a Newton polygon stratum. In (1.6) and (1.8) we have seen "easy" formulas to compute these dimensions (in the unpolarized, or in the principally polarized case). However, up to now there seems to be no really easy proof that these are indeed the correct formulas. In [14] the dimension of the supersingular locus $W_\sigma \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is computed: every irreducible component of W_σ has dimension equal to $[g^2/4]$. Using purity, see [11], and if one would know beforehand that Newton polygon strata in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ are nested as predicted by the Newton polygon graph, we have a proof of (1.8). However *proofs work the other way around.*

(3.1) For a group scheme G over a perfect field K we write $a(G) := \dim_K(\mathrm{Hom}(\alpha_p, G))$. For a local-local p-divisible group X the fact $a(X) = 1$ implies that this Dieudonné module $\mathbb{D}(X)$ is generated by one element over the Dieudonné ring. It turns out that Newton polygon strata around a points where $a = 1$ are smooth (in the local deformation space in the unpolarized case, and in the principally polarized case). In this case the local dimension of the deformation space is computed in[22]. More precisely:

(3.2) Theorem(CH - unpolarized). *Let X_0 be a p-divisible group over a perfect field K. Suppose $a(X_0) = 1$. Let $\gamma = \mathcal{N}(X_0)$, and let $\gamma \prec \beta$. Let ρ be the ordinary Newton polygon with same dimension and height as X_0. Let $D = \mathrm{Def}(X_0)$. Define R_β by:*

$$\mathcal{W}_\beta(D) = \mathrm{Spf}(R_\beta);\ \text{then} \quad R_\beta \quad \cong \quad \frac{K[[z_{x,y} \mid (x,y) \in \Diamond(\rho)]]}{(z_{x,y} \mid (x,y) \notin \Diamond(\beta))} \quad \cong \quad K[[z_{x,y} \mid (x,y) \in \Diamond(\beta)]].$$

(3.3) Theorem (CH - polarized). *Let (A_0, λ) be a principally polarized abelian variety over a perfect field K. Suppose $a(A_0) = 1$. Let $\zeta = \mathcal{N}(X_0)$, and let $\zeta \prec \xi$. Let ρ be the ordinary Newton polygon. Let $D = \mathrm{Def}(A_0, \lambda_0)$. Define R_ξ by:*

$$\mathcal{W}_\xi(D) = \mathrm{Spf}(R_\xi);\ \text{then} \quad R_\xi \quad \cong \quad \frac{K[[z_{x,y} \mid (x,y) \in \triangle(\rho)]]}{(z_{x,y} \mid (x,y) \notin \triangle(\beta))} \quad \cong \quad K[[z_{x,y} \mid (x,y) \in \triangle(\beta)]].$$

Suppose $q \in D$, and $Y = \mathcal{X}_q$ with $\mathcal{N}(Y) = \beta$. We consider the central leaf $\mathcal{C}_Y(D) \subset D = \mathrm{Def}(X_0)$. Can we describe this locus in the coordinates $z_{x,y}$ given?

(3.4) Question. *Under the identification given in* (3.2) *it might be that $\mathcal{C}_Y(D)$ can be described by:*

$$\mathcal{C}_Y(D) \quad \stackrel{?}{=} \quad \mathrm{Spf}(K[[z_{x,y} \mid (x,y) \in \triangle(\beta,\beta^*)]]) \subset \mathcal{W}_\beta(D) = \mathrm{Spf}(K[[z_{x,y} \mid (x,y) \in \triangle(\beta)]]).$$

Write $q \in D = \mathrm{Def}(A_0, \lambda_0)$ with $(B, \mu) = (A_q, \lambda_q)$ and $\mathcal{N}(B) = \xi$.

(3.5) Question *Under the indentification given in* (3.3) *it might be that*

$$\mathcal{C}_{(B,\mu)}(D) \quad \stackrel{?}{=} \quad \mathrm{Spf}(K[[z_{x,y} \mid (x,y) \in \Diamond(\xi,\xi^*)]]) \subset \mathcal{W}_\xi(D) = \mathrm{Spf}(K[[z_{x,y} \mid (x,y) \in \triangle(\xi)]]).$$

4 **Minimal p-divisible groups,** see [29] and [30]

(4.1) Suppose $m, n \in \mathbb{Z}_{\geq 0}$ are coprime integers. There exists a p-divisible group $G_{m,n}$ over $\mathbb{F}_p$; for the construction see [15], page 35, or see [23], 15.5. This p-divisible group has the following properties:

$$\dim(G_{m,n}) = m, \textit{ its Serre dual has } \dim(G^t_{m,n}) = n, \textit{ it is simple over } k,$$
$$\textit{and every } k\textit{-simple } p\textit{-divisible group } X \textit{ with } \dim(X) = m \textit{ and } \dim(X^t) = n$$
$$\textit{is } k\textit{-isogenous with } G_{m,n} \otimes k.$$

We attach to this isogeny class the Newton polygon consisting of $h = m + n$ slopes equal to $\nu = m/(m+n)$. We write $\mathcal{N}((G_{m,n}) = (m, n) = h{\cdot}(m/(m+n))$.

This explains the notation used in (1.2): if $X \sim_k \sum_i G_{m_i,n_i}$ then $\mathcal{N}(X) = \sum_i (m_i, n_i)$; the Newton polygon of X is obtained by ordering the slopes $m_i/(m_i + n_i)$ with multiplicity $m_i + n_i$ in non-decreasing order.

(4.2) In the isogeny class of $G_{m,n}$ we single out one p-divisible group $H_{m,n}$ specifically; for a description see [11], 5.3 - 5.7; the p-divisible group $H_{m,n}$ is defined over $\mathbb{F}_p$, it is isogenous with $G_{m,n}$, and

$$\textit{the endomorphism ring } \mathrm{End}(H_{m,n} \otimes k) \textit{ is the maximal order}$$
$$\textit{in the endomorphism algebra } \mathrm{End}(H_{m,n} \otimes k);$$

these conditions determine $H_{m,n} \otimes \overline{\mathbb{F}_p}$ up to isomorphism. This p-divisible group $H_{m,n}$ is called *minimal.*

If $\zeta = \sum_i (m_i, n_i)$ we write $H(\zeta) := \sum_i H_{m_i,n_i}$, the minimal p-divisible group with Newton polygon equal to ζ. We write $G(\zeta) = H(\zeta)[p]$, the minimal BT_1 group scheme attached to ζ.

In case $\mu \in \mathbb{Z}_{>0}$ we write

$$H_{d,c} = (H_{m,n})^{\mu}, \quad \text{where} \quad d := \mu m, \quad c := \mu n.$$

In [29] and [30] we study the following question:

Starting from a p-divisible group X we obtain a BT_1 group scheme:

$$[p] : \{X \mid \text{a } \ p-\text{divisible group}\}/\cong_k \quad \longrightarrow \{G \mid \text{a } \ \mathrm{BT}_1\}/\cong_k; \quad X \mapsto G := X[p].$$

This map is known to be surjective. Does $G = X[p]$ determine the isomorphism class of X? This seems a strange question, and in general the answer is “NO”. It is the main theorem of [29] that the fiber of this map over G up to $\cong_k$ is precisely one p-divisible group X if G is minimal:

(4.3) Theorem. *If $G = G(\zeta)$ is minimal over k, and X and Y are p-divisible groups with $X[p] \cong G \cong Y[p]$, then $X \cong Y$; hence $X \cong H(\zeta) \cong Y$.*

However things are different if G is not minimal: it is one of the main results of [30] that for a non-minimal BT_1 group scheme G there are infinitely many isomorphism classes X with $X[p] \cong G$.

Note the following important corollaries.
Notation: $\mathcal{S}_G(D) = \{s \in S \mid \mathcal{X}_s \otimes \Omega \cong G \otimes \Omega\}$.

(4.4) Corollary. *Suppose X is a p-divisible group and $G = X[p]$; let $D = \mathrm{Def}(X)$. Study the inclusion $\mathcal{C}_X(D) \subset \mathcal{S}_G(D)$. Then:*

$$X \quad \text{is minimal} \quad \Rightarrow \quad \mathcal{C}_X(D) = \mathcal{S}_G(D).$$

Notation: $\mathcal{S}_{(A_0,\lambda)[p]}(\mathcal{A}) = \{s \in S \mid \ (A,\lambda)_s[p] \otimes \Omega \cong (A_0,\lambda)[p] \otimes \Omega\}$. In fact, we should introduce a level structure, so that $\mathcal{A}$ is a fine moduli scheme, with universal family (A,λ), etc.

(4.5) Corollary. *Let (A_0,λ) be a polarized abelian variety. Then*

$$A_0[p] \quad \text{is minimal} \quad \Rightarrow \quad \mathcal{C}_{(A_0,\lambda)[p^\infty]}(\mathcal{A}) = \mathcal{S}_{(A,\lambda)[p]}(\mathcal{A}).$$

5 A theorem by Torsten Wedhorn on the dimension of EO-strata, see [32]

Let X be a p-divisible group, $D = \mathrm{Def}(X)$. Let $G := X[p]$. Consider the EO-stratum $S_G(D)$.

(5.1) Theorem (Wedhorn).

$$\dim(S_G(D)) = \dim(D) - \dim\left(\underline{\mathrm{Aut}}(G)\right).$$

See [32], 6.10.

Let(X,λ) be a principally quasi-polarized p-divisible group over a field of characteristic $\mathrm{char}(k) = p > 2$. Write $(G,\lambda) = (X,\lambda)[p]$.

(5.2) Theorem (Wedhorn) $\boxed{p > 2}$.

$$\dim(S_{(G,\lambda)}(\mathrm{Def}(X,\lambda))) = \dim(\mathrm{Def}(X,\lambda)) - \dim\left(\underline{\mathrm{Aut}}((G,\lambda))\right).$$

See [32], 2.8 and 6.10.

As we will see, we can avoid using this theorem.

6 Computation of the dimension of automorphism schemes

Consider minimal BT_1 group schemes like $H_{d,c}[p]$ as considered in Section 4. Consider homomorphism group schemes between such, automorphism group schemes and their dimensions.

(6.1) Proposition. *Suppose $a,b,d,c \in \mathbb{Z}_{\geq 0}$; assume that $a/(a+b) \geq d/(d+c)$. Then:*

$$\dim\left(\underline{\mathrm{Hom}}(H_{a,b}[p], H_{d,c}[p])\right) = bd = \dim\left(\underline{\mathrm{Hom}}(H_{d,c}[p], H_{a,b}[p])\right);$$

$$\dim\left(\underline{\mathrm{Aut}}(H_{d,c}[p])\right) = dc.$$

For methods of proof, see [16], Sections 4 and 5, [18], [30], 2.4.

In fact, much more is true in case of minimal p-divisible groups. For $I, J \in \mathbb{Z}_{>0}$ we have

$$\underline{\mathrm{Hom}}(H_{a,b}[p^I], H_{d,c}[p^J]) = \underline{\mathrm{Hom}}(H_{a,b}[p], H_{d,c}[p]).$$

(6.2) Proposition. *Suppose $d, c \in \mathbb{Z}_{\geq 0}$ with $d > c$. Let λ be a principal quasi-polarization on $H_{d,c} \times H_{c,d}$. Then:*

$$\dim(\underline{\mathrm{Aut}}((H_{d,c} \times H_{c,d}, \lambda)[p])) = c(c+1) + dc.$$

Moreover:

$$\dim(\underline{\mathrm{Aut}}(((H_{1,1})^r, \lambda)[p])) = \frac{1}{2} \cdot r(r+1)$$

for a principal quasi-polarization λ.

7 Serre-Tate coordinates, see [2], see [1], §7

(7.1) Serre-Tate coordinates: the ordinary case. Let (A, λ) be an *ordinary* principally polarized abelian variety; write $(X, \lambda) = (A, \lambda)[p^\infty]$. Deformations of (A, λ) are described by extensions of $(X, \lambda)_{\mathrm{et}}$ by $(X, \lambda)_{\mathrm{loc}}$. This shows that $\mathrm{Def}(X, \lambda)$ has the structure of a formal group. Let $n \in \mathbb{Z}_{\geq 3}$ be not divisible by p and let $[(A, \lambda, \gamma)] = a \in \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$. Using the Serre-Tate theorem, see (10.1), we see that we have an isomorphism:

$$(\mathcal{A}_{g,1,n} \otimes \mathbb{F}_p)^{/a} \quad \cong \quad (\mathbb{G}_m[p^\infty])^{g(g+1)/2},$$

canonically up to $\mathbb{Z}_p$-linear transformations: *the Serre-Tate canonical coordinates*; see [13]; see [17], Introduction.

One can try to formulate an analogous result around a non-ordinary point. Generalizations of Serre-Tate coordinates run into several difficulties. In an arbitrary deformation there is no reason that the slope filtration on the p-divisible group remains constant (as it does in the ordinary case). Even supposing that the slope filtration remains constant, or more strongly, supposing that the slope subfactors remain constant does not give the desired generalization. However it turns out that if we suppose that *under deformation the geometric isomorphism type of the p-divisible group remains geometrically constant*, we arrive at a satisfactory generalization of Serre-Tate coordinates. This is precisely the tool provided by the theory of *central leaves* as in [25]. We follow ideas basically due to Ching-Li Chai. We extract from [2], and from [1], §7 the information we need here.

Let Z be a p-divisible group, and $D = \mathrm{Def}(Z)$. Suppose that $Z = X_1 \times \cdots \times X_u$, where the summands are isoclinic of slopes $\nu_1 > \cdots > \nu_u$. Write $Z_{i,j} = X_i \times X_j$.

(7.2) Proposition.

$$\dim\left(\mathcal{C}_Z(D)\right) \quad = \sum_{1 \leq i < j \leq u} \dim\left(\mathcal{C}_{Z_{i,j}}(\mathrm{Def}(Z_{(i,j)}))\right).$$

Note that the "group-like structure" on the leaf $\mathcal{C}_Z(D)$ can be described using the theory of "cascades" as in [17], 0.4.

Let (Z, λ) be a principally quasi-polarized p-divisible group, and $D = \mathrm{Def}(Z, \lambda)$. Suppose that $Z = X_1 \times \cdots \times X_u$, where the summands are iscolinic of slopes $\nu_1 > \cdots > \nu_u$. Then the heights of X_i and X_{u+1-i} are equal and $\nu_i = 1 - \nu_{u+1-i}$. We have the following pairs of summands:

$X_i + X_j$, with $1 \leq i < j < u + 1 - i$ and $X_{u+1-j} + X_{u+1-i}$, and

$X_i + X_{u+1-i}$ for $1 \le i \le t/2$.
In this ways all pairs are described. Note that
$Z_{i,j} := X_i + X_j + X_{u+1-j} + X_{u+1-i}$ for $1 \le i < j < u+1-i$, and
$S_i := X_i + X_{u+1-i}$ for $1 \le i \le u/2$, and $S_{(u+1)/2}$ if u is odd
are principally quasi-polarized p-divisible groups (write the induced polarization again by λ on each of them).

(7.3) Proposition.
$$\dim\left(\mathcal{C}_{(Z,\lambda)}(\mathrm{Def}(Z,\lambda))\right) =$$
$$= \sum_{1\le i<j<u+1-i} \dim\left(\mathcal{C}_{Z_{(i,j)}}(\mathrm{Def}(Z_{(i,j)})\right) + \sum_{1\le i\le u/2} \dim\left(\mathcal{C}_{(S_i,\lambda)}(\mathrm{Def}(S_i,\lambda))\right).$$

Note that
$$\{(i,j) \mid 1 \le i < j < u+1-i\} \xrightarrow{\sim} \{(I,J) \mid 1 \le I < J \quad \text{and} \quad u+1-I < J \le u\}$$
is a bijection under the map $(i,j) \mapsto (I = u+1-j, J = u+1-i)$; indeed $i < j$ implies $I < J$ and $j < u+1-i$ gives $j = u+1-I < J = u+1-i$. In this case λ gives an isomorphism $X_i \times X_j \xrightarrow{\sim} X_J \times X_I$.

(7.4) Example. The group structure on a leaf can be easily understood in the case of *two slopes*. This was the starting point for Chai to describe the relevant generalization of Serre-Tate coordinates from the ordinary case to the arbitrary case.

(7.5) Theorem (Chai). *Let X be isoclinic of slope ν_X, height h_X and Y isoclinic of slope ν_Y and height h_Y. Suppose $\nu_Y > \nu_X$. Write $Z = Y \times X$. The central leaf $C = \mathcal{C}_Z(\mathrm{Def}(Z))$ has the structure of a p-divisible group, isoclinic of slope $\nu_Y - \nu_X$, of height $h_X{\cdot}h_Y$, and*
$$\dim\left(\mathcal{C}_Z(\mathrm{Def}(Z))\right) \quad = \quad (\nu_Y - \nu_X){\cdot}h_X{\cdot}h_Y.$$
Suppose moreover there exists a principal quasi-polarization λ on Z; this implies $h_X = h_Y$ and $\nu_X = 1 - \nu_Y$. *The central leaf $\mathcal{C}_{(Z,\lambda)}(\mathrm{Def}(Z,\lambda))$ has the structure of a p-divisible group, isoclinic of slope $\nu_Y - \nu_X$, of height $h_X{\cdot}(h_X+1)/2$, and*
$$\dim\left(\mathcal{C}_{(Z,\lambda)}(\mathrm{Def}(Z,\lambda)\right) \quad = \quad \frac{1}{2}(\nu_Y - \nu_X){\cdot}h_X{\cdot}(h_X+1).$$
See [1], 7.5.2.

(7.6) Remark. Let Z be an *isoclinic* p-divisible group. Then $\dim\left(\mathcal{C}_Z(\mathrm{Def}(Z))\right) = 0$. This can be seen as following from a generalization of the previous theorem: take $\nu_Y = \nu_X$. This fact was already know: see [11], 2.17.

8 The dimension of central leaves, the unpolarized case

(8.1) Notation. Let ζ be a Newton polygon, and $(x,y) \in \mathbb{Q} \times \mathbb{Q}$. We write:
$(x,y) \prec \zeta$ if (x,y) is on or above ζ,
$(x,y) \precneqq \zeta$ if (x,y) is strictly above ζ,
$(x,y) \succ \zeta$ if (x,y) is on or below ζ,

$(x,y) \gneq \zeta$ if (x,y) is strictly below ζ.

Let ζ be a Newton polygon. Suppose that the slopes of ζ are $1 \geq \beta_1 \geq \cdots \geq \beta_h \geq 0$; this polygon has slopes $\beta_h, \cdots, \beta_1$ (non-decreasing order), and it is lower convex. We write ζ^* for the polygon starting at $(0,0)$ constructed using the slopes $\beta_1, \cdots, \beta_h$ (non-increasing order); note that ζ^* is upper convex, and that the beginning and end point of ζ and of ζ^* coincide.

We write:

$$\Diamond(\zeta;\zeta^*) := \{(x,y) \in \mathbb{Z} \mid (x,y) \prec \zeta, \quad (x,y) \gneq \zeta^*\}; \qquad \boxed{\mathrm{cdu}(\zeta) := \#\left(\Diamond(\zeta;\zeta^*)\right)};$$

"cdu" = <u>d</u>imension of <u>c</u>entral leaf, <u>u</u>npolarized case.

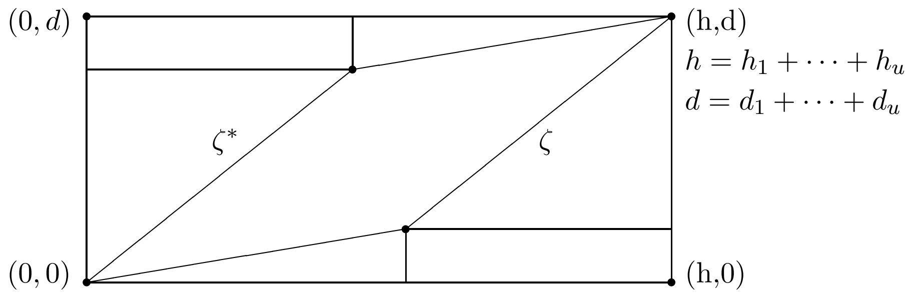
We suppose $\zeta = \sum_{1 \leq i \leq u} \mu_i{\cdot}(m_i, n_i)$, written in such a way that $\gcd(m_i, n_i) = 1$ for all i, and $\mu_i \in \mathbb{Z}_{>0}$, and $i < j \Rightarrow (m_i/(m_i + n_i)) > (m_j/(m_j + n_j))$. Write $d_i = \mu_i{\cdot}m_i$ and $c_i = \mu_i{\cdot}n_i$ and $h_i = \mu_i{\cdot}(m_i + n_i)$; write $\nu_i = m_i/(m_i + n_i) = d_i/(d_i + c_i)$ for $1 \leq i \leq u$. Note that the slope $\nu_i = \mathrm{slope}(G_{m_i,n_i}) = m_i/(m_i + n_i) = d_i/h_i$: this Newton polygon is the "Frobenius-slopes" Newton polygon of $\sum (G_{m_i,n_i})^{\mu_i}$. Note that the slope ν_i appears h_i times; these slopes with these multiplicities give the set $\{\beta_j \mid 1 \leq j \leq h := h_1 + \cdots + h_u\}$ of all slopes of ζ.

(8.2) Combinatorial Lemma, the unpolarized case. *The following numbers are equal*

$$\#\left(\Diamond(\zeta;\zeta^*)\right) =: \mathrm{cdu}(\zeta) \quad = \quad \sum_{i=1}^{i=h} \ (\zeta^*(i) - \zeta(i)) =$$

$$= \sum_{1 \leq i < j \leq u} (d_i c_j - d_j c_i) = \sum_{1 \leq i < j \leq u} (d_i h_j - d_j h_i) = \sum_{1 \leq i < j \leq u} h_j{\cdot}h_i{\cdot}(\nu_i - \nu_j).$$

Example:

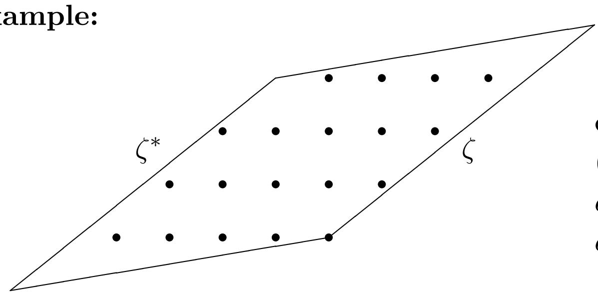


A **proof** for this lemma is not difficult. The equality $\mathrm{cdu}(\zeta) = \sum (\zeta^*(i) - \zeta(i))$ can be seen as follows. From every break point of ζ^* draw vertical lines up, and horizontal lines to the left; from every break point of ζ draw vertical lines down and horizontal line to the right. This divides the remaining space in the $h \times d$ rectangle in triangles and rectangles. Pair opposite triangles to a rectangle. In each of these take lattice points, in the interior and in the lower or right hand sides; in this way all lattice points in the large rectangle belong to precisely one of the subspaces; for each of the subspaces we have the formula that the number of such lattice points is the total length of vertical lines. This proves the desired equality for $\mathrm{cdu}(\zeta)$. The other equalities follow by a straightforward computation. $\square$

(8.3) Theorem (Dimension formula, the unpolarized case.) *Let X_0 be a p-divisible group, $D = \mathrm{Def}(X_0)$; let Y be a p-divisible group with $\mathcal{N}(Y) \succ \mathcal{N}(X_0)$;*

$$\dim(\mathcal{C}_Y(D)) \quad = \quad \mathrm{cdu}(\beta).$$

See [1], 7.10.
Example:



$\dim(\mathcal{C}_X(D)) = \#\left((\Diamond(\zeta;\zeta^*)\right)$.
$(\frac{4}{5} - \frac{1}{6}){\cdot}5{\cdot}6 = 19$,
$d_1h_2 - d_2h_1 = 4{\cdot}6 - 1{\cdot}5 = 19$;
$d_1c_2 - d_2c_1 = 4{\cdot}5 - 1{\cdot}1 = 19$.

First proof. It suffices to prove this theorem in case $Y = X_0$. Write $\mathcal{N}(Y) = \zeta$. By (2.5) it suffices to prove this lemma in case $Y = H(\zeta)$. By (4.3), see (4.4), we know that in this case $\mathcal{C}_Y(D) = S_{Y[p]}(D)$. Let $\beta = \sum \mu_i{\cdot}(m_i, n_i)$; we suppose that $i < j \Rightarrow m_i/(m_i + n_i) > m_j/(m_j + n_j)$; write $d_i = \mu_i{\cdot}m_i$ and $d = \sum d_i$; write $c_i = \mu_i{\cdot}n_i$ and $c = \sum c_i$. We know: $\dim(\mathrm{Def}(Y)) = \dim Y{\cdot}\dim Y^t = dc$. By (5.1), using (6.1) and (8.2), we conclude:

$$\dim(\mathcal{C}_Y(D)) = \dim(S_{H(\beta)[p]}(D)) = \dim(\mathrm{Def}(Y)) - \dim(\underline{\mathrm{Aut}}(H(\beta)[p])) =$$

$$= (\sum d_i)(\sum c_i) - (\sum_i d_i{\cdot}c_i) - 2{\cdot}\sum_{i<j}(c_i{\cdot}d_j) = \sum_{i<j}(d_ih_j - d_jh_i) = \mathrm{cdu}(\beta).$$

Second proof. Assume, as above, that $Y = X_0 = H_\beta$. Write $Z_{i,j} = H_{d_i,c_i} \times H_{d_j,c_j}$. A **proof** of (8.3) follows from (7.5) using (7.2):

$$\dim(\mathcal{C}_Y(D)) = \sum_{i<j} \dim(\mathcal{C}_{Z_{i,j}}(\mathrm{Def}(Z_{i,j}))) = \sum_{i<j} h_j{\cdot}h_i{\cdot}(\nu_i - \nu_j),$$

where $\nu_i = d_i/(d_i + c_i) = m_i/(m_i + n_i)$ and $h_i = d_i + c_i$. $\square$(8.3)

(8.4) Remark. A variant of the first proof can be given as follows. At first prove (8.3) in the case of two slopes, as was done above. Then conclude using (7.2).

9 The dimension of central leaves, the polarized case

(9.1) Notation. Let ξ be a symmetric Newton polygon. For convenience we adapt notation to the symmetric situation:

$$\xi = \mu_1{\cdot}(m_1, n_1) + \cdots + \mu_s{\cdot}(m_s, n_s) + r{\cdot}(1, 1) + \mu_s{\cdot}(n_s, m_s) + \cdots + \mu_1{\cdot}(n_1, m_1)$$

with:
$m_i > n_i$ and $\gcd(m_i, n_i) = 1$ for all i,
$1 \le i < j \le s \Rightarrow (m_i/(m_i + n_i)) > (m_j/(m_j + n_j))$,

$r \geq 0$ and $s \geq 0$.
We write $d_i = \mu.m_i$, and $c_i = \mu.n_i$, and $h_i = d_i + c_i$. Write $g := \left(\sum_{1 \leq i \leq s}(d_i + c_i)\right) + r$ and write $u = 2s + 1$.

We write:

$$\triangle(\xi;\xi^*) := \{(x,y) \in \mathbb{Z} \mid (x,y) \prec \xi, \quad (x,y) \gneqq \xi^*, \quad x \leq g\}; \qquad \boxed{\mathrm{cdp}(\zeta) := \#\left(\triangle(\zeta;\zeta^*)\right)};$$

"cdp" = <u>d</u>imension of <u>c</u>entral leaf, <u>p</u>olarized case.

Write $\xi = \sum_{1 \leq i \leq u} \mu_i \cdot (m_i, n_i)$, i.e. $(m_j, n_j) = (n_{u+1-j}, m_{u+1-j})$ for $s < j \leq u$ and $r(1,1) = \mu_{s+1}(m_{s+1}, n_{s+1})$. Write $\nu_i = m_i/(m_i + n_i)$ for $1 \leq i \leq u$; hence $\nu_i = 1 - \nu_{u+1-i}$ for all i.

(9.2) Combinatorial Lemma, the polarized case. *The following numbers are equal*

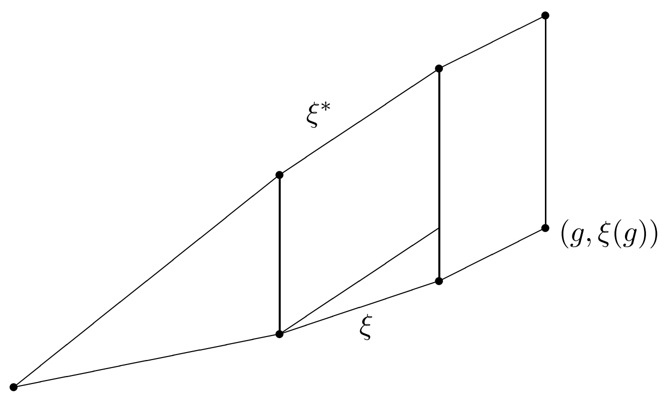
$$\#\left(\triangle(\xi;\xi^*)\right) =: \mathrm{cdp}(\xi) \quad = \quad \frac{1}{2}\mathrm{cdu}(\xi) + \frac{1}{2}(\xi^*(g) - \xi(g)) \quad = \quad \sum_{1 \leq j \leq g} (\xi^*(j) - \xi(j)) =$$

$$= \sum_{1 \leq i \leq s} \left(\frac{1}{2} \cdot d_i(d_i+1) - \frac{1}{2} \cdot c_i(c_i+1)\right) + \sum_{1 \leq i < j}^{j \leq s} (d_i - c_i)h_j + \left(\sum_{i=1}^{i=s}(d_i - c_i)\right) \cdot r =$$

$$= \frac{1}{2} \sum_{1 \leq i \leq s} (2\nu_i - 1)h_i(h_i+1) + \frac{1}{2} \sum_{1 \leq i < j \neq u+1-i} (\nu_i - \nu_j)h_i h_j.$$

Compare [1], Proposition 7.12.

Example:



A **proof** of this lemma is not difficult. The first equalities follow from the unpolarized lemma, and from the definitions of cdu and cdp. For a proof of the one but last equality draw vertical lines connecting breakpoints, and then draw lines from the breakpoints of ξ with slopes and lengths as in ξ^*; this divides $\triangle(\xi;\xi^*)$ into subspaces, where lattice points are considered in the interior, and on lower and right hand sides of the triangles and parallelograms created. Counting points in each of these give all summands of the right hand side of the last equality.

For the last equality:

$$\frac{1}{2} \cdot d_i(d_i+1) - \frac{1}{2} \cdot c_i(c_i+1) = \frac{1}{2}(d_i - c_i)(d_i + c_i + 1) = \frac{1}{2}(2\nu_i - 1)h_i(h_i+1);$$

for $1 \leq i \leq s$:

$$(d_i - c_i) \cdot r = \frac{1}{2}\left((\nu_i - \frac{1}{2}) + (\frac{1}{2} - \nu_{u+1-i})\right) \cdot h_i \cdot \frac{1}{2} = (\nu_i - \nu_{s+1})h_i h_{s+1} + (\nu_{s+1} - \nu_{u+i-1})h_{s+1}h_i;$$

for $1 \leq i < j \leq s$ we have:

$$2(d_i - c_i)h_j = 2((d_i c_j - c_i d_j) + (d_i d_j - c_i c_j)) =$$

$$= (\nu_i - \nu_j)h_i h_j + (\nu_i - \nu_{u+1-j})h_i h_j + (\nu_j - \nu_{u+1-i})h_i h_j + (\nu_{u+1-j} - \nu_{u+1-i})h_i h_j;$$

this shows

$$\sum_{1 \leq i < j}^{j \leq s} (d_i - c_i)h_j = \frac{1}{2} \sum_{1 \leq i < j \neq u+1-i, \;\; i \neq s+1, \;\; j \neq s+1} \; (\nu_i - \nu_j)h_i h_j.$$

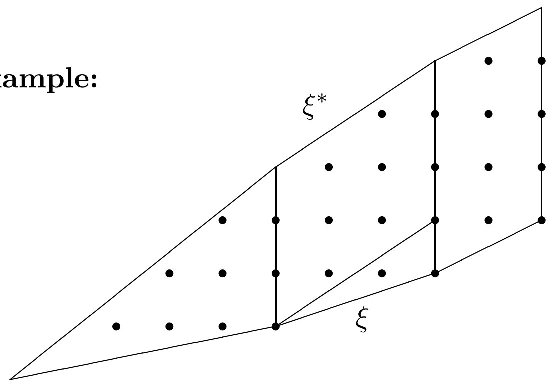
Hence the last equality is proved. $\square$

(9.3) Theorem (Dimension formula, the polarized case.) *Let (A, λ) be a polarized abelian variety. Let $(X, \lambda) = (A, \lambda)[p^\infty]$; write $\xi = \mathcal{N}(A)$; then*

$$\dim \left(\mathcal{C}_{(X,\lambda)}(\mathcal{A} \otimes \mathbb{F}_p) \right) \quad = \quad \mathrm{cdp}(\xi).$$

See [1], Proposition 7.12.

Example:



$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) = \sum_{0 < i \leq g} (\xi^*(i) - \xi(i)).$

slopes $1/5, 4/5, h = 5$: $\quad \frac{1}{2}4{\cdot}5 - \frac{1}{2}1{\cdot}2 = 9,$

slopes $1/3, 2/3, h = 3$: $\quad \frac{1}{2}2{\cdot}3 - \frac{1}{2}1{\cdot}2 = 2,$

$(d_1 - c_1)h_2 = 3{\cdot}3 = 9,$

$(d_1 + d_2 - c_1 - c_2)r = 4{\cdot}2 = 8,$

$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) = \#\left(\triangle(\zeta; \zeta^*)\right) = 28.$

(9.4) Notation used in the proof of (9.3). Using (10.8) and (2.7) we only need to prove Theorem (9.3) in case λ is a principal polarization on

$$A[p^\infty] = H(\xi) =: Z = Z_1 \times \cdots \times Z_s \times Z_{s+1} \times Z_{s+2} \times \cdots \times Z_u,$$

$$Y_i := Z_i = H_{d_i,c_i}, \quad Z_{s+1} = S_{s+1} = (H_{1,1})^r, \quad X_i := Z_{u+1-i} = H_{c_i,d_i} \quad 1 \leq i \leq s+1.$$

Write $S_i = H_{d_i,c_i} \times H_{c_i,d_i}$ for $i \leq s$, and we write λ for the induced quasi-polarization on S_i for $1 \leq i \leq s+1$; note that $r \geq 0$. We have:

$$Z = Y_1 \times \cdots \times Y_s \times Z_{s+1} \times X_s \times \cdots \times X_1.$$

First proof. We have:

$$\dim(\underline{\mathrm{Aut}}((Z,\lambda)[p])) = \sum_{i \leq s+1} \dim(\underline{\mathrm{Aut}}((S_i,\lambda)[p])) + \tfrac{1}{2} \cdot \sum_{i \neq j \neq u+1-i} \dim(\underline{\mathrm{Hom}}(Z_i, Z_j)) =$$

$$= \sum_{i \leq s+1} \dim(\underline{\mathrm{Aut}}((S_i,\lambda)[p])) \quad + \quad \sum_{1 \leq i < j \neq u+1-i} \dim(\underline{\mathrm{Hom}}(Z_i, Z_j)).$$

Using (6.1) and (6.2) and using the notation introduced a computation we show:

$$\mathrm{cdp}(\xi) \quad + \quad \dim(\underline{\mathrm{Aut}}((Z,\lambda)[p])) = \frac{1}{2} \cdot g(g+1);$$

indeed, write

$$I = \sum_{1\leq i\leq s} \left(\frac{1}{2}\cdot d_i(d_i+1) - \frac{1}{2}\cdot c_i(c_i+1)\right),$$

$$II = \sum_{1\leq i<j}^{j\leq s} (d_i - c_i)h_j,$$

$$III = \left(\sum_{i=1}^{i=s}(d_i - c_i)\right)\cdot r;$$

note that:

$$1 \leq i < j \leq s: \quad \dim(\underline{\mathrm{Hom}}(Y_i, Y_j)) = c_i\cdot d_j,$$

$$1 \leq i < j = s+1: \quad \dim(\underline{\mathrm{Hom}}(Y_i, Z_{s+1})) = c_i\cdot r,$$

$$1 \leq i < s < j: \quad \dim(\underline{\mathrm{Hom}}(Y_i, Z_j)) = c_i\cdot c_{u+1-j}, \quad Z_j = X_{u+1-j},$$

$$i = s+1 < j: \quad \dim(\underline{\mathrm{Hom}}(Z_{s+1}, Z_j)) = r\cdot c_{u+1-j},$$

$$s < i < j: \quad \dim(\underline{\mathrm{Hom}}(Z_i, Z_j)) = d_{u+1-i}\cdot c_{u+1-j};$$

direct verification gives:

$$I + II + III + \sum_{i\leq s}(d_ic_i + c_i(c_i+1)) \quad + \quad \frac{1}{2}\cdot r(r+1) \quad + \frac{1}{2}\cdot \sum_{i\neq j\neq u+1-i} \dim(\underline{\mathrm{Hom}}(Z_i, Z_j)) =$$

$$= (d_1 + \cdots + d_s + r + c_s + \cdots + c_1)(d_1 + \cdots + d_s + r + c_s + \cdots + c_1 + 1)/2.$$

First we suppose $p > 2$, and prove the theorem in this case. Indeed, using (4.5) and (5.2) we see:

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) = \dim(\mathcal{A}\otimes\mathbb{F}_p) - \dim(\underline{\mathrm{Aut}}((Z,\lambda)[p])) = \mathrm{cdp}(\xi).$$

Hence Theorem (9.3) is proved in case $p > 2$.

Let p and q be prime numbers, let ξ be a symmetric Newton polygon, and let $H^{(p)}(\xi)$ respectively $H^{(q)}(\xi)$ be this minimal p-divisible group in characteristic p, respectively in characteristic q, both with a principal quasi-polarization λ. Their elementary sequences as defined in [23] are equal:

Claim.

$$\varphi((H^{(p)}(\xi), \lambda)[p]) = \varphi((H^{(q)}(\xi), \lambda)[q]).$$

The proof is a direct verification: in the process of constructing the canonical filtration, the characteristic plays no role. □Claim

Next we assume Let $p > 2$ and $q = 2$, and we prove the theorem in characteristic $q = 2$. We have seen that the theorem holds in the case $p > 2$. In that case we know, using (4.5) and [23], Theorem 1.2, that

$$\mathrm{cdp}(\xi) = \dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) = \dim\left(S_{(X,\lambda)[p]}(\mathcal{A}\otimes\mathbb{F}_p)\right) = \varphi((H^{(p)}(\xi), \lambda)[p]).$$

Hence

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_q)\right) = \varphi((H^{(q)}(\xi), \lambda)[q]) = \varphi((H^{(p)}(\xi), \lambda)[p]) = \mathrm{cdp}(\xi).$$

This ends the first proof of Theorem (9.3).

(9.5) Lemma (a proof of (9.3) in the case of two slopes). *Suppose* $\xi = (d,c) + (c,d)$ *with* $d > c$, *i.e.* $s = 1$ *and* $r = 0$ *in the notation used above,* i.e. the case of a symmetric Newton polygon with only two different slopes. Write $g = d + c$. *In this case*

$$\mathrm{cdp}(\xi) \quad = \quad \frac{1}{2}d(d+1) - \frac{1}{2}c(c+1) = \frac{1}{2}\cdot(d-c)(d+c+1) = (1+\cdots+g)(\frac{d}{d+c} - \frac{c}{d+c}).$$

We choose $X = H_{d,c} + H_{c,d}$, *and* $G = X[p]$; *let* λ *be the principal quasi-polarization on* X *over* k. Note this is unique up to isomorphism, see [25], Proposition 3.7. *Let* $\varphi(G) = \mathrm{ES}(G)$ *be the elementary sequence of* G, in the notation and terminology as in [23]. *Then*

$$\varphi = \{0, \cdots, \varphi(c) = 0, 1, 2, \cdots, \varphi(d) = d-c, d-c, \cdots, d-c\}.$$

Hence in this case

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) = c\cdot 0 + (1+\cdots+d-c) + c\cdot(d-c) \quad = \quad \mathrm{cdp}(\xi).$$

Proof. In order to write down a final sequence for $(H_{m,n} \times H_{n,m})^{\mu}$ it suffice to know a canonical filtration for $Z = (H_{m,n} \times H_{n,m})$. Write $\mathbb{D}(H_{m,n}) = M_{m,n}$, the covariant Dieudonné module; there is a W-basis $M_{m,n} = W\cdot e_0 \oplus \cdots \oplus e_{m+n-1}$, and $\mathcal{F}(e_i) = e_{i+m}$, and $\mathcal{V}(e_i) = e_{i+n}$, with the convention $e_{j+m+n} = pe_j$. Also we have $\mathbb{D}(H_{n,m}) = M_{n,m} = W\cdot f_0 \oplus \cdots \oplus W\cdot f_{m+n-1}$, and $\mathcal{F}(f_j) = f_{j+n}$ and $\mathcal{V}(f_j) = f_{i+m}$; the quasi-polarization can be given by $< e_i, f_j >= \delta_{i,m+n-1-j}$. Consider the k-basis for $\mathcal{V}(\mathbb{D}(Z[p]))$ given by

$$\{x_1 = e_{m+n-1}, \cdots, x_n = e_m, x_{n+1} = e_{m-1}, \cdots, x_m = e_n, x_{m+1} = f_{m+n-1}, \cdots, x_{m+n} = f_m\};$$

this can be completed to a symplectic basis for $\mathbb{D}(Z[p])$; write $N_j = k\cdot x_1 \oplus \cdots \oplus k\cdot x_j$ for $1 \leq j \leq m+n$. Direct verification shows that $0 \subset N_1 \subset \cdots \subset N_{m+n}$ plus the symplectic dual filtration is a final filtration of $\mathbb{D}(Z[p])$. From this we compute φ as indicated, and the result for $H_{m,n} \times H_{n,m}$ follows. This proves the lemma. $\square$

Remark. It seems attractive to prove (9.3) in the general case along these lines by computing $\mid \varphi \mid$. There is an algorithm for determining the canonical filtration in general, but I do not have a closed formula for computing $\mid \varphi \mid$ knowing ξ. Therefore, in the previous proof of (9.3) we made a detour via (5.1).

(9.6) Lemma. *Let* $(Z = Y \times X, \lambda)$ *be a principally quasi-polarized p-divisible group, where* X *is isoclinic of slope* ν_X, *height* h_X, *and* Y *isoclinic of slope* ν_Y *and height* h_Y. *Suppose* $1 \geq \nu_Y > \frac{1}{2} > \nu_X = 1 - \nu_Y \geq 0$. *Write* $d_x = h_X\cdot\nu_X$, *and* $\nu_X = d_X/c_X$; *analogous notation for* d_Y *and* c_Y; *write* $d = d_Y = c_X$, *and* $c = c_Y = d_X$ *and* $g = d + c$. *Then:*

$$\dim\left(\mathcal{C}_{(Z,\lambda)}(\mathrm{Def}(Z,\lambda))\right) \quad = \quad \frac{1}{2}(\nu_Y - \nu_X)\cdot h_X\cdot(h_X+1) = \frac{1}{2}d(d+1) - \frac{1}{2}c(c+1).$$

First proof. By (2.7) it suffices to prove this lemma in case $X = H_{c,d}$ and $Y = H_{d,c}$. By (9.5) the result follows.
Second proof. The result follows by (7.5). $\square$(9.6)

(9.7) Second proof. The **proof** of (9.3) follows from (7.3) using Lemma (8.3) and Lemma (9.6). $\square$(9.3)

(9.8) Remark. In contrast with (9.3), in general $\mathrm{sdim}(\xi) \neq \sum_{0<i\leq g}(i - \xi(i))$.

10 Appendix: Some results used.

(10.1) Serre - Tate. Let A_0 be an abelian variety, and $X_0 = A_0[p^\infty]$. We obtain a natural morphism

$$\mathrm{Def}(A_0) \xrightarrow{\sim} \mathrm{Def}(X_0), \quad A \mapsto A[p^\infty];$$

a very basic theorem of Serre and Tate says that this is an *isomorphism.* An analogous statement holds for (polarized abelian variety) $\mapsto$ (quasi-polarized p-divisible group).

(10.2) Chow - Grothendieck. Using the Chow-Grothendieck algebraization method for polarized formal schemes, see [7], III^1.5.4, it follows that a polarized formal abelian scheme is algebraizable. Hence for $[(A, \lambda, \gamma)] = x \in \mathcal{A}_{g,d,n}$ and $(X, \lambda) = (A, \lambda)[p^\infty]$ we obtain a canonical isomorphism $(\mathcal{A}_{g,d,n})^{/x} = \mathrm{Def}(X, \lambda)$.

(10.3) Deformations of p-divisible groups. For p-divisible groups there exists a solution to the universal deformation problem in equi-characteristic: given a p-divisible group X_0 over a field K, there exists a formal p-divisible group $\mathcal{G} \to \mathrm{Spf}(R)$ which is universal for this problem, and $R \cong K[[t_1, \cdots, t_{dc}]]$; here $d = \dim(X_0)$ and $c = \dim(X_0^t)$. As finite group schemes are "algebraizable", the same holds for certain limits, and this results in a universal family $\mathcal{G} \to \mathrm{Spec}(A)$, see [10], 2.4.4. We use the passage from formal p-divisible groups over $\mathrm{Spf}(R)$ to p-divisible groups over $\mathrm{Spec}(R)$ without further comments.

(10.4) Grothendieck - Katz. A theorem by Grothendieck and Katz, see [12], Th. 2.3.1 on page 143 says that for any $\mathcal{X} \to S$ and for any Newton polygon ζ

$$\mathcal{W}(S) \subset S \text{ is a } \textit{closed} \text{ set.}$$

(10.5) Definition. *Let S be a scheme, and let $X \to S$ be a p-divisible group. We say that X/S is* geometrically fiberwise constant, *abbreviated* gfc *if there exist a field K, a p-divisible group X_0 over K, a morphism $S \to \mathrm{Spec}(K)$, and for every $s \in S$ an algebraically closed field $k \supset \kappa(s) \supset K$ containing the residue class field of s and an isomorphism $X_0 \otimes k \cong_k X_s \otimes k$.* The analogous terminology will be used for quasi-polarized p-divisible groups and for (polarized) abelian schemes.

See [25], 1.1.

(10.6) Theorem (T. Zink & FO). *Let S be an integral,* normal *Noetherian scheme. Let $\mathcal{X} \to S$ be a p-divisible group with* constant Newton polygon. *Then there exists a p-divisible $\mathcal{Y} \to S$ and an S-isogeny $\varphi : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}/S$ is* gfc.

See [33], [31], 2.1, and [25], 1.8.

(10.7) Theorem. *Let S be a scheme which is integral, and such that the normalization $S' \to S$ gives a noetherian scheme. Let $\mathcal{X} \to S$ be a p-divisible group; let $n \in \mathbb{Z}_{\geq 0}$. Suppose that $\mathcal{X} \to S$ is* gfc. *Then there exists a finite surjective morphism $T_n = T \to S$, such that $\mathcal{X}[p^n] \times_S T$ is constant over T.*

See [25], 1.3.

(10.8) Lemma. *Let (X, λ') be a quasi-polarized p-divisible group over k, with $\mathcal{N}(X) = \xi$. There exists an isogeny between (X, ξ') and $(H(\xi), \lambda)$, where λ is a principal quasi-polarization.*

See [25], 3.7.

(10.9) Let ξ_1 be a symmetric Newton polygon "below" ξ_2. We write $W^0_{\xi_1}$ for the open Newton polygon stratum inside $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ (note we work with principal polarizations here) of moduli points with Newton polygon equal to ξ_1; we write W_{ξ_2} for the open Newton polygon stratum inside $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ of moduli points with Newton polygon equal to or above ξ_2. Then

$$W_{\xi_2} \quad \subset \quad (W^0_{\xi_1})^{\mathrm{Zariski-closure}}, \quad \xi_2 \prec \xi_1.$$

However for non-principally polarized abelian varieties the analogous statement is not correct.

11 A question and some related results

(11.1) In (1.8) we have seen how to compute the dimension of a Newton polygon stratum in the *principally* polarized case. that formula is not the right one in case of arbitrary degree of polarization. In [24], 5.8 we find an expectation for the dimension of Newton polygon strata. We expect that For a symmetric Newton polygon ξ, corresponding with p-rank equal to f, we know by [19], Th. 4.1, that an irreducible component of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ has dimension at most $g(g+1)/2 \; - (g-f) = g(g-1)/2 + f$, and

we expect that indeed there is a component of $\mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ which has dimension equal to $g(g-1)/2 + f$.

(11.2) We know that $\mathcal{W}_\sigma(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)$ has "many irreducible components", where σ is the supersingular Newton polygon (all slopes are equal to 1/2). However for $\xi \neq \sigma$ we know that $\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)$ is geometrically irreducible, see [26], Theorem 1.5.

(11.3) For (A, λ) such that A is *not supersingular*, and λ is a principal polarization, the leaf $\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_{g,1}) \otimes \mathbb{F}_p$ is geometrically irreducible; this will be published in [3], see [28].

(11.4) For a principally polarized abelian variety (A, λ) one can consider the isomorphism class of $(A, \lambda)[p]$. This gives rise to EO-strata, see [23]; such an isomorphism class over an algebraically closed field is determined by an "elementary sequence". We can decide which EO-strata are contained in the supersingular locus:

Lemma (FO, unpublished). *We have:*

$$S_\varphi \subset \mathcal{W}_\sigma(\mathcal{A}_{g,1} \otimes \mathbb{F}_p) \quad \Longleftrightarrow \quad \varphi([(g+1)/2]) = 0.$$

In [23], 14.1 we find the conjecture that an EO-stratum not contained in the supersingular locus is geometrically irreducible. In [6] this has been proved:
Theorem (T. Ekedahl & G. van der Geer). *If* $\varphi([(g+1)/2]) \neq 0$, i.e. S_φ not contained in the supersingular locus, *then $S_\varphi \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is geometrically irreducible.*

Moreover I expected that for $\varphi([(g+1)/2]) = 0$ and p large, the locus S_φ is *geometrically reducible.* This seems to be true, see S. Harashita, [9].

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P.S.
After my talk in Düsseldorf Torsten Wedhorn told me about a result contained in his manuscript:
T. Wedhorn – *Specializations of F-Zips.* Ms. 22 pp., 20-VI-2005.

In this manuscript we find Theorem (6.4) which allows us to extend results mentioned above in Section 2. Proving flatness of a morphism from $\mathcal{A}_g$ to a space of isomorphism classes of F-zips Wedhorn shows that Theorem (5.2) above also holds in characteristic 2.
Conclusion. The first proof of (9.3) can be simplified: "Indeed, using (4.5) and (5.2) we see:

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right)=\dim(\mathcal{A}\otimes\mathbb{F}_p)-\dim(\underline{\mathrm{Aut}}((Z,\lambda)[p]))=\mathrm{cdp}(\xi)."$$

This argument in the proof of (9.3) works in all characteristics by the generalized Theorem (5.2); QED for (9.3); the last 18 lines of the proof of (9.3) can be deleted.

P.S.
Theorem B of the paper

Eva Mierendorff – *Moduli spaces of p-divisible groups.*

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computes the dimension of an isogeny leaf (in the unpolaraized case). Using (1.6), and the product struture as in [25], a direct computation shows that (8.3) follows; hence this result by Mierendorff gives a new proof of our dimension formula in the unpolarized case.

It seems plausible that analogous considerations directly give a proof for the dimension formula of an isogeny leaf in the principally polzarized case; this would give another proof of (9.3).

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