Minimal p-divisible groups

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Introduction. A p-divisible group X can be seen as a tower of building blocks, each of which is isomorphic to the same finite group scheme $X[p]$. Clearly, if X_1 and X_2 are isomorphic then $X_1[p] \cong X_2[p]$; however, conversely $X_1[p] \cong X_2[p]$ does in general not imply that X_1 and X_2 are isomorphic. Can we give, over an algebraically closed field in characteristic p, a condition on the p-kernels which ensures this converse? Here are two known examples of such a condition: consider the case that X is *ordinary*, or the case that X is *superspecial* (X is the p-divisible group of a product of supersingular elliptic curves); in these cases the p-kernel uniquely determines X.

These are special cases of a surprisingly complete and simple answer:

$$\textit{if } G \textit{ is "minimal", then } X_1[p] \cong G \cong X_2[p] \textit{ implies } X_1 \cong X_2,$$

see (1.2); for a definition of "minimal" see (1.1). This is "necessary and sufficient" in the sense that for any G that is *not minimal* there exist infinitely many mutually non-isomorphic p-divisible groups with p-kernel isomorphic to G; see (4.1).

Remark (motivation). You might wonder why this is interesting.

EO In [7] we have defined a natural *stratification* of the moduli space of polarized abelian varieties in positive characteristic: moduli points are in the same stratum if and only if the corresponding p-kernels are geometrically isomorphic. Such strata are called EO-strata.

Fol In [8] we define in the same moduli spaces a *foliation* : moduli points are in the same leaf if and only if the corresponding p-divisible groups are geometrically isomorphic; in this way we obtain a foliation of every open Newton polygon stratum.

Fol $\subset$ EO The observation $X \cong Y \Rightarrow X[p] \cong Y[p]$ shows that any leaf in the second sense is contained in precisely one stratum in the first sense; the main result of this paper, "*X is minimal if and only if $X[p]$ is minimal*", shows that *a stratum* (in the first sense) *and a leaf* (in the second sense) *are equal* if we are in the minimal, principally polarized situation.

In this paper we consider p-divisible groups and finite group schemes over an *algebraically closed* field k of characteristic p.

An apology. In (2.5) and in (3.5) we fix notations, used for the proof of (2.2), respectively (3.1); according to the need, the notations in these two different cases are different. We hope this difference in notations in Section 2 versus Section 3 will not cause confusion.

Group schemes considered are supposed to be commutative. We use *covariant* Dieudonné module theory. We write $W = W_\infty(k)$ for the ring of infinite Witt vectors with coordinates in k. Finite products in the category of W-modules are denoted "$\times$" or by "$\prod$", while finite products in the category of Dieudonné modules are denoted by "$\oplus$"; for finite products of p-divisible groups we use "$\times$" or "$\prod$". We write F and V, as usual, for "Frobenius" and "Verschiebung" on commutative group schemes; we write $\mathcal{F} = \mathbb{D}(V)$ and $\mathcal{V} = \mathbb{D}(F)$, see [7], 15.3, for the corresponding operations on Dieudonné modules.

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1 Notations and the main result.

(1.1) Some definitions and notations.

$H_{m,n}$. We define the p-divisible group $H_{m,n}$ over the prime field $\mathbb{F}_p$ in case m and n are coprime non-negative integers, see [2], 5.2. This p-divisible group $H_{m,n}$ is of dimension m, its Serre-dual X^t is of dimension n, it is isosimple, and *its endomorphism ring* $\mathrm{End}(H_{m,n} \otimes \overline{\mathbb{F}_p})$ *is the maximal order in the endomorphism algebra* $\mathrm{End}^0(H_{m,n} \otimes \overline{\mathbb{F}_p})$ (and these properties characterize this p-divisible group over $\overline{\mathbb{F}_p}$). We will use the notation $H_{m,n}$ over any base S in characteristic p, i.e. we write $H_{m,n}$ instead of $H_{m,n} \times_{\mathrm{Spec}(\mathbb{F}_p)} S$, if no confusion can occur.

The ring $\mathrm{End}(H_{m,n} \otimes \mathbb{F}_p) = R'$ is commutative; write L for the field of fractions of R'. Consider integers x, y such that for the coprime positive integers m and n we have $x{\cdot}m+y{\cdot}n = 1$. In L we define the element $\pi = \mathcal{F}^y{\cdot}\mathcal{V}^x \in L$. Write $h = m+n$. Note that $\pi^h = p$ in L. Here $R' \subset L$ is the maximal order, hence R' integrally closed in L, and we conclude that $\pi \in R'$. This element π will be called the uniformizer in this endomorphism ring. In fact, $W_\infty(\mathbb{F}_p) = \mathbb{Z}_p$, and $R' \cong \mathbb{Z}_p[\pi]$. In L we have:

$$m+n =: h, \quad \pi^h = p, \quad \mathcal{F} = \pi^n, \quad \mathcal{V} = \pi^m.$$

For a further description of π, of $R = \mathrm{End}(H_{m,n} \otimes k)$ and of $D = \mathrm{End}^0(H_{m,n} \otimes k)$ see [2], 5.4; note that $\mathrm{End}^0(H_{m,n} \otimes k)$ is non-commutative if $m > 0$ and $n > 0$. Note that R is a "discrete valuation ring" (terminology sometimes also used for non-commutative rings).

Newton polygons. Let β be a Newton polygon. By definition, in the notation used here, this is a lower convex polygon in $\mathbb{R}^2$ starting at $(0,0)$, ending at (h,c) and having break points with integral coordinates; it is given by h slopes in non-decreasing order; every slope λ is a rational number, $0 \leq \lambda \leq 1$.

To each ordered pair of nonnegative integers (m,n) we assign a set of $m+n = h$ slopes equal to $n/(m+n)$; this Newton polygon ends at $(h, c = n)$.

In this way a Newton polygon corresponds with a set of ordered pairs; such a set we denote symbolically by $\sum_i (m_i, n_i)$; conversely such a set determines a Newton polygon. Usually we consider only coprime pairs (m_i, n_i); we write $H(\beta) := \times_i \, H_{m_i,n_i}$ in case $\beta = \sum_i (m_i, n_i)$. A p-divisible group X over a field of positive characteristic defines a Newton polygon where h is the height of X and c is the dimension of its Serre-dual X^t. By the Dieudonné-Manin classification, see [5], Th. 2.1 on page 32, we know: *two p-divisible groups over an algebraically closed field of positive characteristic are isogenous if and only if their Newton polygons are equal.*

Definition. *A p-divisible group X is called* minimal *if there exists a Newton polygon β and an isomorphism $X_k \cong H(\beta)_k$, where k is an algebraically field.*

Note that in every isogeny class of p-divisible groups over an algebraically closed field there is precisely one minimal p-divisible group.

Truncated p-divisible groups. A finite group scheme G (finite and flat over some base, but in this paper we will soon work over a field) is called a BT_1, see [1], page 152, if $G[F] := \mathrm{Ker} F_G = \mathrm{Im} V_G =: V(G)$ and $G[V] = F(G)$ (in particular this implies that G is annihilated by p). Such group schemes over a perfect field appear as the p-kernel of a p-divisible group, see [1], Prop. 1.7 on page 155. The abbreviation "BT_1" stand for "1-truncated Barsotti-Tate group"; the terms "p-divisible group" and "Barsotti-Tate group" indicate the same concept.

The Dieudonné module of a BT_1 over a perfect field K is called a DM_1; for $G = X[p]$ we have $\mathbb{D}(G) = \mathbb{D}(X)/p\mathbb{D}(X)$. In other terms: such a Dieudonné module $M_1 = \mathbb{D}(X[p])$ is a finite dimensional vector space over K, on which $\mathcal{F}$ and $\mathcal{V}$ operate (with the usual relations), with the property that $M_1[\mathcal{V}] = \mathcal{F}(M_1)$ and $M_1[\mathcal{F}] = \mathcal{V}(M_1)$.

Definition. *Let G be a* BT_1 *group scheme; we say that G is* minimal *if there exists a Newton polygon β such that $G_k \cong H(\beta)[p]_k$. A* DM_1 *is called* minimal *if it is the Dieudonné module of a minimal* BT_1.

(1.2) Theorem. *Let X be a p-divisible group over an algebraically closed field k of characteristic p. Let β be a Newton polygon. Then*

$$X[p] \cong H(\beta)[p] \quad \Longrightarrow \quad X \cong H(\beta).$$

In particular: if X_1 and X_2 are p-divisible groups over k, with $X_1[p] \cong G \cong X_2[p]$, where G is minimal, *then $X_1 \cong X_2$.*

Remark. We have no a priori condition on the Newton polygon of X, nor do we a priori assume that X_1 and X_2 have the same Newton polygon.

Remark. In general an isomorphism $\varphi_1 : X[p] \to H(\beta)[p]$ does not lift to an isomorphism $\varphi : X \to H(\beta)$.

(1.3) Here is another way of explaining the result of this paper. Consider the map

$$[p] : \{X \mid \text{a } p\text{-divisible group}\}/\cong_k \;\longrightarrow \{G \mid \text{a } \mathrm{BT}_1\}/\cong_k, \qquad X \mapsto X[p].$$

This map is surjective, e.g. see [1], 1.7; also see [7], 9.10.

- By results of this paper we know: *For every Newton polygon β there is an isomorphism class $X := H(\beta)$ such that the fiber of the map $[p]$ containing X consists of one element.*
- *For every X not isomorphic to some $H(\beta)$ the fiber of $[p]$ containing X is infinite;* see (4.1)

Convention. The slope $\lambda = 0$, given by the pair $(1,0)$, defines the p-divisible group $G_{1,0} = \mathbb{G}_m[p^\infty]$, and its p-kernel is μ_p. The slope $\lambda = 1$, given by the pair $(0,1)$, defines the p-divisible group $G_{0,1} = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$ and its p-kernel is $\underline{\mathbb{Z}/p\mathbb{Z}}$. These p-divisible groups and their p-kernels split off naturally over a perfect field, see [6], 2.14. The theorem is obvious for these minimal BT_1 group schemes over an algebraically closed field. Hence it suffices to prove the theorem in case all group schemes considered are of local-local type, i.e. all slopes considered are strictly between 0 and 1; from now on we make this assumption.

(1.4) We give already one explanation about notation and method of proof. Let $m, n \in \mathbb{Z}_{>0}$ be coprime. Start with $H_{m,n}$ over $\mathbb{F}_p$. Let $Q' = \mathbb{D}(H_{m,n} \otimes \mathbb{F}_p)$. In the terminology of [2], 5.6 and Section 6, a semi-module of $H_{m,n}$ equals $[0, \infty) = \mathbb{Z}_{\geq 0}$. Choose a non-zero element in $Q'/\pi Q'$, this is a one-dimensional vector space over $\mathbb{F}_p$, and lift this element to $A_0 \in Q'$. Write $A_i = \pi^i A_0$ for every $i \in \mathbb{Z}_{>0}$. Note that

$$\pi A_i = A_{i+1}, \quad \mathcal{F} A_i = A_{i+n}, \quad \mathcal{V} A_i = A_{i+m}.$$

Fix an algebraically closed field k; we write $Q = \mathbb{D}(H_{m,n} \otimes k)$. Clearly $A_i \in Q' \subset Q$, and the same relations as given above hold. Note that $\{A_i \mid i \in \mathbb{Z}_{\geq 0}\}$ generate Q as a W-module. *The fact that a semi-module of the minimal p-divisible group $H_{m,n}$ does not contain "gaps" is the essential (but sometimes hidden) argument in the proofs below.*

The set $\{A_0, \cdots, A_{m+n-1}\}$ is a W-basis for Q. If $m \geq n$ we see that $\{A_0, \cdots, A_{n-1}\}$ is a set of generators for Q as a Dieudonné module; the structure of this Dieudonné module can be described as follows; for this set of generators we consider another numbering $\{C_1, \cdots, C_n\} = \{A_0, \cdots, A_{n-1}\}$ and we define positive integers γ_i by: $C_1 = A_0$ and $\mathcal{F}^{\gamma_1} C_1 = \mathcal{V} C_2, \cdots, \mathcal{F}^{\gamma_n} C_n = \mathcal{V} C_1$ (note that we assume $m \geq n$), which gives a "cyclic" set of generators for Q/pQ in the sense of [3]. These notations will be repeated and explained more in detail in (2.5) and (3.5).

2 A slope filtration

(2.1) We consider a Newton polygon β given by $r_1(m_1, n_1), \cdots, r_t(m_t, n_t)$; here $r_1, \cdots, r_t \in \mathbb{Z}_{>0}$, and every (m_j, n_j) is an ordered pair of coprime positive integers; we write $h_j = m_j + n_j$ and we suppose the ordering is chosen in such a way that $\lambda_1 := n_1/h_1 < \cdots < \lambda_t := n_t/h_t$. Write

$$H := H(\beta) = \prod_{1 \leq j \leq t} (H_{m_j,n_j})^{r_j}; \quad G := H(\beta)[p].$$

The following proposition uses this notation; suppose that $t > 0$.

(2.2) Proposition. *Suppose X is a p-divisible group over an algebraically closed field k. Suppose that $X[p] \cong H(\beta)[p]$. Suppose that $\lambda_1 = n_1/h_1 \leq 1/2$. Then there exists a p-divisible subgroup $X_1 \subset X$ and isomorphisms*

$$X_1 \cong (H_{m_1,n_1})^{r_1} \quad \textit{and} \quad (X/X_1)[p] \cong \prod_{j>1} (H_{m_j,n_j}[p])^{r_j}.$$

(2.3) Remark. The condition that $X[p]$ is *minimal* is essential; e.g. it is easy to give an example of a p-divisible group X which is isosimple, such that $X[p]$ is decomposable; see [9].

(2.4) Corollary. *For X with $X[p] \cong H(\beta)[p]$, with β as in* (2.1), *there exists a filtration by p-divisible subgroups*

$$X_0 := 0 \subset X_1 \subset \cdots \subset X_t = X \quad \textit{such that} \quad X_j/X_{j-1} \cong (H_{m_j,n_j})^{r_j}, \quad \textit{for} \quad 1 \leq j \leq t.$$

Proof of the corollary. Assume by induction that the result has been proved for all p-divisible groups where $Y[p] = H(\beta')[p]$ is minimal such that β' has at most $t-1$ different slopes; induction starting at $t - 1 = 0$, i.e. $Y = 0$. If on the one hand the smallest slope of

X is at most $1/2$, the proposition gives $0 \subset X_1 \subset X$, and using the induction hypothesis on $Y = X/X_1$ we derive the desired filtration. If on the other hand all slopes of X are bigger than $1/2$, we apply the proposition to the Serre-dual of X, using the fact that the Serre-dual of $H_{m,n}$ is $H_{n,m}$; dualizing back we obtain $0 \subset X_{t-1} \subset X$, and using the induction hypothesis on $Y = X_{t-1}$ we derive the desired filtration. Hence we see that the proposition gives the induction step; this proves the corollary. $\square$(2.2)$\Rightarrow$(2.4)

(2.5) We use notation as in (2.1) and (2.2), and we fix further notation which will be used in the proof of (2.2).

Let $M = \mathbb{D}(X)$. *We write* $Q_j = \mathbb{D}(H_{m_j,n_j})$. *Hence*

$$M/pM \cong \bigoplus_{1 \leq j \leq t} (Q_j/pQ_j)^{r_j}.$$

Using this isomorphism we construct a map

$$v : M \longrightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\}.$$

We use notation as in (1.1) and in (1.4). Let π_j be the uniformizer of $\mathrm{End}(Q_j)$. We choose $A^{(j)}_{i,s} \in Q_j$ with $i \in \mathbb{Z}_{\geq 0}$ and $1 \leq s \leq r_j$ (which generate Q_j) such that $\pi_j \cdot A^{(j)}_{i,s} = A^{(j)}_{i+1,s}$, $\mathcal{F}\cdot A^{(j)}_{i,s} = A^{(j)}_{i+n_j,s}$ and $\mathcal{V}\cdot A^{(j)}_{i,s} = A^{(j)}_{i+m_j,s}$. We have $Q_j/pQ_j = \times_{0 \leq i < h_j} k\cdot(A^{(j)}_{i,s} \bmod pQ_j)$. We write

$$A^{(j)}_i = (A^{(j)}_{i,s} \mid 1 \leq s \leq r_j) \in (Q_j)^{r_j}$$

for the vector with coordinate $A^{(j)}_{i,s}$ in the summand on the s-th place.

For $B \in M$ we uniquely write

$$B \bmod pM = a = \sum_{j,\ 0 \leq i < h_j,\ 1 \leq s \leq r_j} b^{(j)}_{i,s}\cdot(A^{(j)}_{i,s} \bmod pQ_j), \quad b^{(j)}_{i,s} \in k;$$

if moreover $B \notin pM$ we define

$$v(B) = \min_{j,\ i,\ s,\ b^{(j)}_{i,s} \neq 0} \ \frac{i}{h_j}.$$

If $B' \in p^\beta M$ and $B' \notin p^{\beta+1}M$ we define $v(B') = \beta + v(p^{-\beta}\cdot B')$. We write $v(0) = \infty$. *This ends the construction of* $v : M \longrightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\}$.

For any $\rho \in \mathbb{Q}$ we define

$$M_\rho = \{B \mid v(B) \geq \rho\};$$

note that $pM_\rho \subset M_{\rho+1}$. Let T be the least common multiple of $h_1, \cdots,\ h_t$. Note that, in fact, $v : M - \{0\} \to \frac{1}{T}\mathbb{Z}_{\geq 0}$. Note that, by construction, $v(B) \geq d \in \mathbb{Z}$ if and only if p^d divides B in M. Hence $\cap_{\rho \to \infty} \ M_\rho = \{0\}$.

The basic assumption $X[p] \cong H(\beta)[p]$ of (1.2) is:

$$M/pM = \bigoplus_{1 \leq j \leq t,\ 1 \leq s \leq r_j} \ \prod_{0 \leq i < h_j} \ k\cdot((A^{(j)}_{i,s} \bmod pQ_j))$$

(we write this isomorphism of Dieudonné modules as an equality). For $0 \leq i < h_j$ and $1 \leq s \leq r_j$ we choose $B^{(j)}_{i,s} \in M$ such that:

$$B^{(j)}_{i,s} \bmod pM = A^{(j)}_{i,s} \bmod pQ_j^{r_j}.$$

Define $B^{(j)}_{i+\beta\cdot h_j,s} = p^{\beta}\cdot B^{(j)}_{i,s}$. By construction we have: $v(B^{(j)}_{i,s}) = i/h_j$ for all $i \geq 0$, all j and all s. Note that M_ρ is generated over $W = W_\infty(k)$ by all elements $B^{(j)}_{i,s}$ with $v(B^{(j)}_{i,s}) \geq \rho$. As a short-hand we will write

$$B^{(j)}_i \text{ for the vector } (B^{(j)}_{i,s} \mid 1 \leq s \leq r_j) \in M^{r_j}.$$

We write $P \subset M$ for the sub-W-module generated by all $B^{(j)}_{i,s}$ with $j \geq 2$ and $i < h_j$; we write $N \subset M$ for the sub-W-module generated by all $B^{(1)}_{i,s}$ with $i < h_1$. Note that $M = N \times P$, a direct sum of W-modules. Note that $M_\rho = (N \cap M_\rho) \times (P \cap M_\rho)$.

In the proof the W-submodule $P \subset M$ will be fixed; its W-complement $N \subset M$ will change eventually if it is not already a Dieudonné submodule.

We write $m_1 = m,\ \ n_1 = n,\ \ h = h_1 = m{+}n$, and $r = r_1$. Note that we assumed $0 < \lambda_1 \leq 1/2$, hence $m \geq n > 0$. For $i \geq 0$ we define integers δ_i by:

$$i\cdot h \leq \delta_i\cdot n < i\cdot m + (i+1)\cdot n = ih + n$$

and non-negative integers γ_i such that

$$\delta_0 = 0, \quad \delta_1 = \gamma_1 + 1, \cdots, \delta_i = \gamma_1 + 1 + \gamma_2 + 1 + \cdots + \gamma_i + 1, \cdots;$$

note that $\delta_n = h = m + n$; hence $\gamma_1 + \cdots + \gamma_n = m$. For $1 \leq i \leq n$ we write

$$f(i) = \delta_{i-1}\cdot n - (i-1)\cdot h;$$

this means that $0 \leq f(i) < n$ is the remainder of dividing $\delta_{i-1}n$ by h; note that $f(1) = 0$. As $\gcd(n, h) = 1$ we see that

$$f : \{1, \cdots, n\} \to \{0, \cdots, n-1\}$$

is a bijective map. The inverse map f' is given by:

$$f' : \{0, \cdots, n-1\} \to \{1, \cdots, n\}, \quad f'(x) \equiv 1 - \frac{x}{h} \pmod{n}, \quad 1 \leq f'(x) \leq n.$$

In $(Q_1)^r$ we have the vectors $A^{(1)}_i$. We choose $C'_1 := A^{(1)}_0$ and we choose $\{C'_1, \cdots, C'_n\} = \{A^{(1)}_0, \cdots, A^{(1)}_{n-1}\}$ by

$$C'_i := A^{(1)}_{f(i)}, \quad C'_{f'(x)} = A^{(1)}_x;$$

this means that:

$$\mathcal{F}^{\gamma_i} C'_i = \mathcal{V} C'_{i+1}, \ \ 1 \leq i < n, \quad \mathcal{F}^{\gamma_n} C'_n = \mathcal{V} C'_1, \quad \text{hence} \quad \mathcal{F}^{\delta_i} C'_1 = p^i\cdot C'_{i+1}, \quad 1 \leq i < n;$$

note that $\mathcal{F}^h C'_1 = p^n\cdot C'_1$. With these choices we see that

$$\{\mathcal{F}^j C'_i \mid 1 \leq i \leq n, 0 \leq j \leq \gamma_i\} = \{A^{(1)}_\ell \mid 0 \leq \ell < h\}.$$

For later reference we state:

(2.6) *Suppose Q is a nonzero Dieudonné module with an element $C \in Q$, such that there exist coprime integers n and $n+m=h$ as above such that $\mathcal{F}^h{\cdot}C = p^n{\cdot}C$ and such that Q as a W-module is generated by $\{p^{-\lfloor jn/h\rfloor}\mathcal{F}^jC \mid 0 \le j < h\}$, then $Q \cong \mathbb{D}(H_{m,n})$.*
This is proved by explicitly writing out the required isomorphism. Note that $\mathcal{F}^n$ is injective on Q, hence $\mathcal{F}^h{\cdot}C = p^n{\cdot}C$ implies $\mathcal{F}^m{\cdot}C = \mathcal{V}^n{\cdot}C$.

(2.7) Accordingly we choose $C_{i,s} := B^{(1)}_{f(i),s} \in M$ with $1 \le i \le n$. Note that

$$\{\mathcal{F}^jC_{i,s} \mid 1 \le i \le n,\ \ 0 \le j \le \gamma_i\ \ 1 \le s \le r\} \quad \text{is a} \quad W\text{-basis for} \quad N,$$

$$\mathcal{F}^{\gamma_i}C_{i,s} - \mathcal{V}C_{i+1,s} \in pM,\ \ 1 \le i < n,\ \ \mathcal{F}^{\gamma_n}C_{n,s} - \mathcal{V}C_{1,s} \in pM.$$

We write $C_i = (C_{i,s} \mid 1 \le s \le r)$. As a reminder, we sum up some of the notation constructed:

$$\begin{array}{ccc} N \subset M & & \bigoplus_j\ (Q_j)^{r_j} \\ \downarrow & & \downarrow \\ M/pM & = & \bigoplus_j\ (Q_j/pQ_j)^{r_j}, \\ & & \\ B^{(j)}_{i,s} \in M & & A^{(j)}_{i,s} \in Q_j \subset (Q_j)^{r_j} \\ C_{i,s} \in N & & C'_{i,s} \in Q_1 \subset (Q_1)^{r_1}. \end{array}$$

(2.8) **Lemma.** *Use the notation fixed up to now.*
(1) *For every $\rho \in \mathbb{Q}_{\ge 0}$ the map $p : M_\rho \to M_{\rho+1}$, multiplication by p, is surjective.*
(2) *For every $\rho \in \mathbb{Q}_{\ge 0}$ we have $\mathcal{F}M_\rho \subset M_{\rho+(n/h)}$.*
(3) *For every i and s we have $\mathcal{F}B^{(1)}_{i,s} \in M_{(i+n)/h}$; for every i and s and every $j > 1$ we have $\mathcal{F}B^{(j)}_{i,s} \in M_{(i/h_j)+(n/h)+(1/T)}$.*
(4) *For every $1 \le i \le n$ we have $\mathcal{F}^{\delta_i}C_1 - p^iB^{(1)}_{f(i+1)} \in (M_{i+(1/T)})^r$; moreover $\mathcal{F}^{\delta_n}C_1 - p^nC_1 \in (M_{n+(1/T)})^r$.*
(5) *If u is an integer with $u > Tn$, and $\xi_N \in (N \cap M_{u/T})^r$, there exists*

$$\eta_N \in N \cap (M_{(u/T)-n})^r \quad \text{such that} \quad (\mathcal{F}^h - p^n)\eta_N \equiv \xi_N \pmod{(M_{(u+1)/T})^r}.$$

Proof. We know that $M_{\rho+1}$ is generated by the elements $B^{(j)}_{i,s}$ with $i/h_j \ge \rho+1$; because $\rho \ge 0$ such elements satisfy $i \ge h_j$. Note that $p{\cdot}B^{(j)}_{i-h_j,s} = B^{(j)}_{i,s}$. This proves the first property. $\square$(1)

At first we show $\mathcal{F}M \subset M_{n/h}$. Note that for all $1 \le j \le t$ and all $\beta \in \mathbb{Z}_{\ge 0}$

$$\beta h_j \le i < \beta h_j + m_j \quad \Rightarrow \quad \mathcal{F}B^{(j)}_i = B^{(j)}_{i+n_j}, \tag{$*$}$$

and

$$\beta h_j + m_j \le i < (\beta+1)h_j \quad \Rightarrow \quad B^{(j)}_i = \mathcal{V}B^{(j)}_{i-m_j} + p^{(\beta+1)}\xi, \quad \xi \in M^{r_j}. \tag{$**$}$$

from these properties, using $n/h \le n_j/h_j$ we conclude: $\mathcal{F}M \subset M_{n/h}$.
Further we see: by $(*)$ we have

$$v(\mathcal{F}B^{(j)}_{i,s}) = v(B^{(j)}_{i+n_j,s}) = (i+n_j)/h_j,$$

and
$$\frac{i+n_j}{h_j} = \frac{i+n}{h} \quad \text{if} \quad j=1; \qquad \frac{i+n_j}{h_j} > \frac{i}{h_j} + \frac{n}{h} \quad \text{if} \quad j>1.$$

By $(**)$ it suffices to consider only $m_j \leq i < h_j$, and hence $\mathcal{F}B^{(j)}_{i,s} = pB^{(j)}_{i-m_j,s} + p\mathcal{F}\xi$; so we have
$$v(\mathcal{F}B^{(j)}_{i,s}) \geq \min\left(v(pB^{(j)}_{i-m_j,s}), v(p\mathcal{F}\xi_s)\right);$$

for $j=1$ we have $v(pB^{(1)}_{i-m_1,s}) = (i+n)/h \geq 1$ and $v(p\mathcal{F}\xi) \geq 1+(n/h) > (i/h)+(n/h)$; for $j>1$ we have $v(pB^{(j)}_{i-m_j,s}) > (i/h_j)+(n/h)$ and $(i/h_j)+(n/h) < 1+(n/h) \leq v(p\mathcal{F}\xi_s)$; hence $v(\mathcal{F}B^{(j)}_{i,s})) > (i/h_j)+(n/h)$ if $j>1$. This ends the proof of (3). Using (3) we see that (2) follows. □(2)+(3)

From $\mathcal{F}^{\gamma_i}C_i = \mathcal{V}C_{i+1} + p\xi_i$ for $i<n$ and $\mathcal{F}^{\gamma_n}C_n = \mathcal{V}C_1 + p\xi_n$, here $\xi_i \in M^r$ for $i \leq n$, we conclude:
$$\mathcal{F}^{\delta_i}C_1 = p^iC_{i+1} + \sum\nolimits_{1\leq \ell \leq i} \; p^\ell \mathcal{F}^{\delta_i-\delta_\ell}\mathcal{F}\xi_\ell, \quad i<n,$$
and the analogous fomula for $i=n$ (write $C_{n+1}=C_1$). Note that
$$ih \leq \delta_i n \quad \text{and} \quad \delta_\ell n < \ell m + (\ell+1)n = \ell h + n;$$
this shows that
$$\ell h + (\delta_i - \delta_\ell)n + n > ih;$$
using (2) we conclude (4). □(4)

Note that $h=h_1$ divides T. If ℓ is an integer such that $(\ell-1)/h < u/T < \ell/h$ then $u < u+1 \leq \ell\frac{T}{h}$; in this case we see that $N\cap M_{u/T} = N \cap M_{(u+1)/T}$. In this case we choose $\eta_N = 0$.

Suppose that ℓ is an integer with $u/T = \ell/h$. Then $N\cap M_{u/T} = N_{\ell/h} \supset N_{(\ell+1)/h} = N\cap M_{(u+1)/T}$. We consider the image of $N\cap M_{(\ell/h)-n}$ under $\mathcal{F}^h - p^n$. We see, using previous results, that this image is in $N_{\ell/h} + M_{(u+1)/T}$ (here "+" stands for the span as W-modules). We obtain a factorization and an isomorphism
$$\mathcal{F}^h - p^n : N\cap M_{(\ell/h)-n} \longrightarrow \left(N_{\ell/h} + M_{(u+1)/T}\right)/M_{(u+1)/T} \cong N_{\ell/h}/N_{(\ell+1)/h}.$$

We claim that this map is surjective. The factor space $N_{\ell/h}/N_{(\ell+1)/h}$ is a vector space over k spanned by the residue classes of the elements $B^{(1)}_{\ell,s}$. For the residue class of $y_sB^{(1)}_{\ell,s}$ we solve the equation $x_s^{p^n} - x_s = y_s$ in k; lifting these x_s to W (denoting the lifts by the same symbol), we see that $\eta_N := \sum_s x_sB^{(1)}_{\ell-nh,s}$ has the required properties. This proves the claim, and it gives a proof of part (5) of the lemma. □(5),(2.8)

(2.9) Lemma (the induction step). *Let $u\in\mathbb{Z}$ with $u \geq nT+1$. Suppose $D_1 \in M^r$ such that $D_1 \equiv C_1 \pmod{(M_{1/T})^r}$, and such that $\xi := \mathcal{F}^hD_1 - p^nD_1 \in (M_{u/T})^r$. Then there exists $\eta \in (M_{(u/T)-n})^r$ such that for $E_1 := D_1 - \eta$ we have $\mathcal{F}^hE_1 - p^nE_1 \in (M_{(u+1)/T})^r$ and $E_1 \equiv C_1 \pmod{(M_{1/T})^r}$.*

Proof. We write $\xi = \xi_N + \xi_P$ according to $M = N\times P$. We conclude that $\xi_N \in (N\cap M_{u/T})^r$ and $\xi_P \in (P\cap M_{u/T})^r$. Using (2.8), (5), we construct $\eta_N \in (N\cap M_{1/T})^r$ such that

$(\mathcal{F}^h - p^n)\eta_N \equiv \xi_N \pmod{(M_{(u+1)/T})^r}$. As $M_{u/T} \subset M_n$ we can choose $\eta_P := -p^{-n}\xi_P$; we have $\eta_P \in M^r_{(u/T)-n} \subset (M_{1/T})^r$. With $\eta := \eta_N + \eta_P$ we see that

$$(\mathcal{F}^h - p^n)\eta \equiv \xi \pmod{(M_{(u+1)/T})^r} \quad \text{and} \quad \eta \in (M_{1/T})^r.$$

Hence $(\mathcal{F}^h - p^n)(D_1 - \eta) \in (M_{(u+1)/T})^r$ and we see that $E_1 := D_1 - \eta$ has the required properties. This proves the lemma.

$\square$(2.9)

(2.10) Proof of (2.2). **(1)** *There exists* $E_1 \in M^r$ *such that* $(\mathcal{F}^h - p^n)E_1 = 0$ *and* $E_1 \equiv C_1 \pmod{(M_{1/T})^r}$.

Proof. For $u \in \mathbb{Z}_{\geq nT+1}$ we write $D_1(u) \in M^r$ for a vector such that

$$D_1(u) \equiv C_1 \pmod{(M_{1/T})} \quad \text{and} \quad \mathcal{F}^h D_1(u) - p^n D_1(u) \in (M_{u/T})^r.$$

By (2.8), (4), the vector $C_1 =: D_1(nT+1)$ satifies this condition for $u = nT+1$. Here we start induction. By repeated application of (2.9) we conclude there exists a sequence

$$\{D_1(u) \mid u \in \mathbb{Z}_{\geq nT+1}\} \quad \text{such that} \quad D_1(u) - D_1(u+1) \in (M_{(u/T)-n})^r$$

satisfying the conditions above. As $\cap_{\rho\to\infty} \quad M_\rho = \{0\}$ this sequence converges. Writing $E_1 := D_1(\infty)$ we achieve the conclusion. $\square$(1)

(2) *Choose* E_1 *as in* (1). *For every* $j \geq 0$ *we have*

$$p^{-[\frac{jn}{h}]}\mathcal{F}^j E_1 \in M; \quad \text{define} \quad N' := \prod_{1\leq j<h} W{\cdot}p^{-[\frac{jn}{h}]}\mathcal{F}^j E_1 \subset M.$$

This is a Dieudonné submodule and it is a W*-module direct summand of* M*. Moreover there is an isomorphism*

$$\mathbb{D}((H_{m,n})^r) \cong N',$$

the map $N' \prod P \to N' + P$ *is an isomorphism of* W*-modules, and* $N' + P = M$*. This constructs* $X_1 \subset X$*, with*

$$\mathbb{D}(X_1 \subset X) = (N' \subset M) \quad \textit{such that} \quad (X/X_1)[p] \cong \prod_{j>1} (M_{m_j,n_j})^{r_j}.$$

Proof. By (2.8), (2), we see that $\mathcal{F}^j E_1 \in M_{[jn/h]}$, hence the first statement follows. As $\mathcal{F}^h E_1 = p^n E_1$ it follows that $N' \subset M$ is a Dieudonné submodule; using (2.6) this shows $\mathbb{D}((H_{m,n})^r) \cong N'$.

Claim. *The images* $N' \twoheadrightarrow N' \otimes k = N'/pN' \subset M/pM$ *and* $P \twoheadrightarrow P/pP \subset M/pM$ *inside* M/pM *have zero intersection and* $N' \otimes k + P \otimes k = M/pM$*.* Here we write $- \otimes k = - \otimes_W (W/pW)$.

For $y \in \mathbb{Z}_{\geq 0}$ we write $g(y) := yn - h{\cdot}[\frac{yn}{h}]$; note that, in the notation in (2.5), we have

$$p^{-[\frac{jn}{h}]}\mathcal{F}^j C_1' = A^{(1)}_{g(j)}.$$

Suppose

$$\tau := \sum_{0\leq j<h} \beta_{j,s} p^{-[\frac{jn}{h}]}\mathcal{F}^j{\cdot}(E_{1,s} \bmod pM) \in \left(N' \otimes k \cap P \otimes k\right) \subset M/pM, \quad \beta_j \in k$$

such that $\tau \neq 0$. Let x, s be a pair of indices such that $\beta := \beta_{x,s} \neq 0$ and for every y with $g(y) < g(x)$ we have $\beta_{y,s} = 0$. Project inside M/pM on the factor N_s. Then

$$\tau_s \equiv \beta \cdot B^{(1)}_{g(x),s} \pmod{M_{\frac{g(x)}{h}+\frac{1}{T}} + P},$$

which is a contradiction with the fact that $N \cap P = 0$ and with the fact that the residue class of

$$B^{(1)}_{g(x),s} \quad \text{generates} \quad \Big((M_{\frac{g(x)}{h}} + P)/(M_{\frac{g(x)}{h}+\frac{1}{T}} + P)\Big)_s = N_{\frac{g(x)}{h},s}/N_{\frac{g(x)}{h}+\frac{1}{h},s}.$$

We see that $\tau \neq 0$ leads to a contradiction. This shows that $N' \otimes k \cap P \otimes k = 0$ and $N' \otimes k + P \otimes k = M/pM$. Hence the claim is proved.

As $(N' \cap P) \otimes k \subset N' \otimes k \cap P \otimes k = 0$ this shows $(N' \cap P) \otimes k = 0$. By Nakayama's lemma this implies $N' \cap P = 0$. The proof of the remaining statements follows, in particular we see that N' is a W-module direct summand of M. This finishes the proof of **(2)**, and it ends the proof of the proposition. $\square$(2.2)

3 Split extensions and proof of the theorem

In this section we prove a proposition on split extensions. We will see that Theorem (1.2) follows.

(3.1) Proposition. *Let (m, n) and (d, e) be ordered pairs of pairwise coprime positive integers. Suppose that $n/(m+n) < e/(d+e)$. Let*

$$0 \to Z := H_{m,n} \longrightarrow T \longrightarrow Y := H_{d,e} \to 0$$

be an exact sequence of p-divisible groups such that the induced sequence of the p-kernels spits:

$$0 \to Z[p] \overset{\leftarrow}{\longrightarrow} T[p] \overset{\leftarrow}{\longrightarrow} Y[p] \to 0.$$

Then the sequence of p-divisible groups splits: $T \cong Z \oplus Y$.

(3.2) Remark. It is easy to give examples of a non-split extension $T/Z \cong Y$ of p-divisible groups, with Z non-minimal or Y non-minimal, such that the extension $T[p]/Z[p] \cong Y[p]$ does split.

(3.3) Proof of (1.2). The theorem follows from (2.4) and (3.1). $\square$(1.2)

(3.4) *In order to show* (3.1) *it suffices to prove* (3.1) *under the extra condition that $\frac{1}{2} \leq e/(d+e)$.*
In fact, if $n/(m+n) < e/(d+e) < \frac{1}{2}$, we consider the exact sequence

$$0 \to H^t_{d,e} = H_{e,d} \longrightarrow T^t \longrightarrow H^t_{m,n} = H_{n,m} \to 0$$

with $\frac{1}{2} < d/(e+d) < m/(n+m)$. $\square$(3.4)

From now on we assume that $\frac{1}{2} \leq e/(d+e)$.

(3.5) We fix notation which will be used in the proof of (3.1). We write the Dieudonné modules as: $\mathbb{D}(Z) = N$, $\mathbb{D}(T) = M$ and $\mathbb{D}(Y) = Q$; we obtain an exact sequence of Dieudonné modules $M/N = Q$, which is a split exact sequence of W-modules, where $W = W_\infty(k)$. We write $m+n = h$ and $d+e = g$. We know that Q is generated by elements A_i, with $i \in \mathbb{Z}_{\geq 0}$ such that $\pi(A_i) = A_{i+1}$, where $\pi \in \mathrm{End}(Q)$ is the uniformizer, and $\mathcal{V}{\cdot}A_i = A_{i+d}$, $\mathcal{F}{\cdot}A_i = A_{i+e}$; we know that $\{A_i \mid 0 \leq i < g = d+e\}$ is a W-basis for Q. Because $\frac{1}{2} \leq e/(d+e)$, hence $e \geq d$ we can choose generators for the Dieudonné module Q in the following way. We choose integers δ_i by:

$$i{\cdot}g \leq \delta_i{\cdot}d < (i+1){\cdot}d + i{\cdot}e = ig + d$$

and integers γ_i such that:

$$\delta_1 = \gamma_1 + 1, \cdots, \delta_i = \gamma_1 + 1 + \gamma_2 + 1 + \cdots + \gamma_i + 1;$$

note that $\delta_d = g = d + e$. We choose $C = A_0 = C_1$ and $\{C_1, \cdots, C_d\} = \{A_0, \cdots, A_{d-1}\}$ such that:

$$\mathcal{V}^{\gamma_i} C_i = \mathcal{F} C_{i+1}, \quad 1 \leq i < d, \quad \mathcal{V}^{\gamma_d} C_d = \mathcal{F} C_1, \quad \text{hence} \quad \mathcal{V}^{\delta_i} C = p^i{\cdot}C_{i+1}, \quad 1 \leq i < d;$$

note that $\mathcal{V}^g C = p^d{\cdot}C$. With these choices we see that

$$\{p^{-[\frac{jd}{g}]}\mathcal{V}^j C \mid \ 0 \leq j < g\} = \{\mathcal{V}^j C_i \mid 1 \leq i \leq d, \ \ 0 \leq j \leq \gamma_i\} = \{A_\ell \mid 0 \leq \ell < g\}.$$

Choose an element $B = B_1 \in M$ such that

$$M \longrightarrow Q \quad \text{gives} \quad B_1 = B \mapsto (B \bmod N) = C = C_1.$$

Let π' be the uniformizer of $\mathrm{End}(N)$. Consider the filtration $N = N^{(0)} \supset \cdots \supset N^{(i)} \supset N^{(i+1)} \supset \cdots$ defined by $(\pi')^i(N^{(0)}) = N^{(i)}$. Note that $\mathcal{F}N^{(i)} = N^{(i+n)}$, and $\mathcal{V}N^{(i)} = N^{(i+m)}$, and $p^i N = N^{(i{\cdot}h)}$ for $i \geq 0$.

(3.6) Proof of (3.1).
(1) Construction of $\{B_1, \cdots, B_d\}$. *For every choice of $B = B_1 \in M$ with $(B \bmod N) = C$, and every $1 \leq i < d$ we claim that $\mathcal{V}^{\delta_i} B$ is divisible by p^i. Defining $B_{i+1} := p^{-i}\mathcal{V}^{\delta_i} B$, we see that $B_i \bmod N = C_i$ for $1 \leq i \leq d$. Moreover we claim:*

$$\mathcal{V}^g B - p^d{\cdot}B \in N^{(dh+1)}.$$

Choose $B_i'' \in M$ with $B_i'' \bmod N = C_i$. Then $\mathcal{V}^{\gamma_i} B_i'' - \mathcal{F} B_{i+1}'' =: p{\cdot}\xi_i \in pN$; hence $\mathcal{V}^{\gamma_i+1} B_i'' - p{\cdot}B_{i+1}'' = p\mathcal{V}\xi_i \in p\mathcal{V}N$. For $1 < i \leq d$ we obtain that

$$\mathcal{V}^{\delta_i} B - p^i{\cdot}B = \sum_{1 \leq j < i} \mathcal{V}^{\delta_i - \delta_j} p^j \mathcal{V}\xi_j, \quad \xi_j \in N.$$

From $n/(m+n) < e/(d+e)$ we conclude $g/d > h/m$; using $\delta_i{\cdot}d \geq ig$ and $\delta_j d < (j+1)d + je$ we see:

$$i > j \quad \text{implies} \quad \delta_i - \delta_j + 1 > (i-j)(g/d) > (i-j)(h/m);$$

hence

$$(\delta_i - \delta_j)m + j(m+n) + m > ih;$$

This shows

$$\mathcal{V}^{\delta_i - \delta_j} p^j \mathcal{V}\xi_j \in p^i N^{(1)}.$$

As $\delta_d = g$ we see that $\mathcal{V}^g B - p^d{\cdot}B \in p^d N^{(1)} = N^{(dh+1)}$. $\square$(1)

(2) The induction step. *Suppose that for a choice $B \in M$ with $(B \bmod N) = C$, there exists an integer $s \geq dh+1$ such that $\mathcal{V}^g B - p^d{\cdot}B \in N^{(s)}$; then there exists a choice $B' \in M$ such that $B' - B \in N^{(s-dh)}$ and*

$$\mathcal{V}^g B' - p^d{\cdot}B' \in N^{(s+1)}.$$

In fact, write $p^d{\cdot}B - \mathcal{V}^g B = p^d{\cdot}\xi$. Then $\xi \in N^{(s-dh)}$. Choose $B' := B - \xi$. Then:

$$\mathcal{V}^g B' - p^d{\cdot}B' = \mathcal{V}^g B - p^d{\cdot}B - \mathcal{V}^g \xi + p^d \xi = -\mathcal{V}^g \xi \in N^{(gm-dh+s)};$$

note that $gm - dh > 0$. $\square$(2)

(3) *For any integer $r \geq d+1$, and $w \geq rh$ there exists $B = B_1$ as in* (3.5) *such that $\mathcal{V}^g B - p^d B \in N^{(w)} = p^r{\cdot}N^{(w-rh)}$. This gives a homomorphism φ_{r-d}*

$$M/p^{r-d}M \longleftarrow Q/p^{r-d}Q \quad \text{extending} \quad M/pM \longleftarrow Q/pQ.$$

The induction step (2) proves the first statement, induction starting at $w = (d+1)h > dh+1$. Having chosen B_1, using (1) we construct $B_{i+1} := p^{-i}\mathcal{V}^{\delta_i} B_1$ for $1 \leq i < d$. In that case on the one hand $\mathcal{V}^{\gamma_d} B_d - \mathcal{F} B_1 = p{\cdot}\xi_d$, on the other hand $\mathcal{V}^g B - p^d B \in N^{(w)} \subset p^r N$. Hence $p^d \mathcal{V}\xi_d \in p^r N$; hence $p\xi_d \in p^{r-d}N$. This shows that the residue classes of $B_1, \cdots, B_d$ in $M/p^{r-d}M$ generate a Dieudonné module isomorphic to $Q/p^{r-d}Q$ which moreover by (3.5) extends the given isomorphism induced by the splitting. $\square$(3)

By [8], 1.6 we see that for some large r the existence of $M/p^{r-d}M \longleftarrow Q/p^{r-d}Q$ as in (3) shows that its restriction $M/pM \longleftarrow Q/pQ$ lifts to a homomorphism φ of Dieudonné modules $M \leftarrow Q$; in that case φ_1 is injective. Hence φ splits the extension $M/N \cong Q$. Taking into account (3.4) this proves the proposition. $\square$(3.1)

Remark. Instead of the last step of the proof above, we could construct an infinite sequence $\{B(u) \mid u \in \mathbb{Z}_{(d+1)h}\}$ such that $\mathcal{V}^g B(u) - p^d B \in N^{(u)}$ and $B(u+1) - B(u) \in N^{(u-dh)}$ for all $u \geq (d+1)h$. This sequence converges and its limit $B(\infty)$ can be used to define the required section.

4 Some comments

(4.1) Remark. For any G, a BT_1 over k, which is *not minimal* there exist infinitely many mutually non-isomorphic p-divisible groups X over k such that $X[p] \cong G$. Details will appear in a later publication, see [9].

(4.2) Remark. Suppose that G is a *minimal* BT_1; we can recover the Newton polygon β with the property $H(\beta)[p] \cong G$ from G. This follows from the theorem, but there are also other ways to prove this fact.

(4.3) For BT_1 group schemes we can define a Newton polygon; let G be a BT_1 group scheme over k, and let $G = \times_i\, G_i$ be a decomposition into indecomposable ones, see [3]. Let G_i be of rank p^{h_i}, and let n_i be the dimension of the tangent space of G_i^D; here G_i^D stands for the Cartier dual of G_i; define $\mathcal{N}'(G_i)$ as the isoclinic polygon consisting of h_i slopes equal to n_i/h_i; arranging the slopes in non-decreasing order, we have defined $\mathcal{N}'(G)$. For a p-divisible group X we compare $\mathcal{N}(X)$ and $\mathcal{N}'(X[p])$. These polygons have the same endpoints. If X is minimal, equivalently $X[p]$ is minimal, then $\mathcal{N}(X) = \mathcal{N}'(X[p])$. Besides this I do not see rules describing the relation between $\mathcal{N}(X)$ and $\mathcal{N}'(X[p])$. For Newton polygons β and γ with the same endpoints we write $\beta \prec \gamma$ if every point of β is on or below γ. Note:

- There exists a p-divisible group X such that $\mathcal{N}(X) \succneqq \mathcal{N}'(X[p])$; indeed, choose X isosimple, hence $\mathcal{N}(X)$ isoclinic, such that $X[p]$ is decomposable.
- There exists a p-divisible group X such that $\mathcal{N}(X) \precneqq \mathcal{N}'(X[p])$; indeed, choose X such that $\mathcal{N}(X)$ is not isoclinic, hence X not isosimple, all slopes strictly between 0 and 1 and $a(X) = 1$; then $X[p]$ is indecomposable, hence $\mathcal{N}'(X[p])$ is isoclinic.

Here we use $a(X) := \dim_k \mathrm{Hom}(\alpha_p, X)$. It could be useful to have better insight in the relation between various properties of X and $X[p]$.

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